Heaps of segments, parallelogram polyominoes and 321-avoiding (affine) permutations

Frédéric Jouhet (ICJ, University Lyon 1)

Collaborations with R. Biagioli, M. Bousquet-Mélou and P. Nadeau

ALEA, Luminy, March 2016

321-avoiding permutations

A permutation $\sigma \in S_n$ is 321-avoiding if no integers $i < j < k$ are such that $\sigma(i) > \sigma(j) > \sigma(k)$

In S_6, $\sigma = 513624$ is not 321-avoiding while $\sigma = 231564$ is

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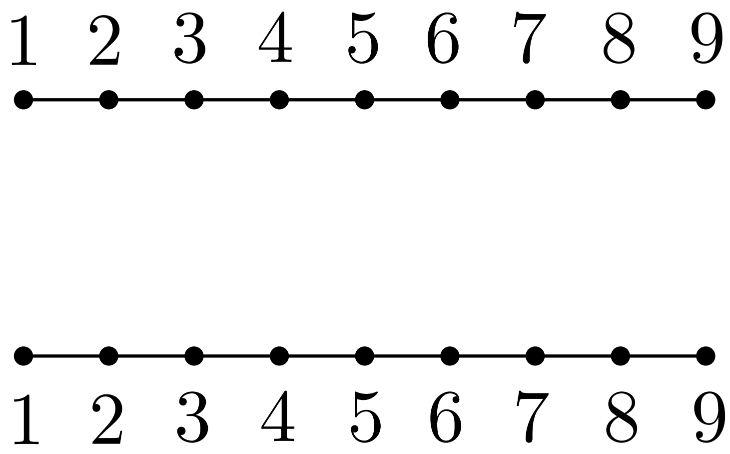
Counted by Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$

The inversion number $inv(\sigma)$ is the number of descents of the permutation σ

In S_6, $inv(231564) = 1 + 1 + 1 + 1 = 4$

Connection with alternating diagrams

Take a 321-avoiding permutation $\sigma = 461279358 \in S_9$



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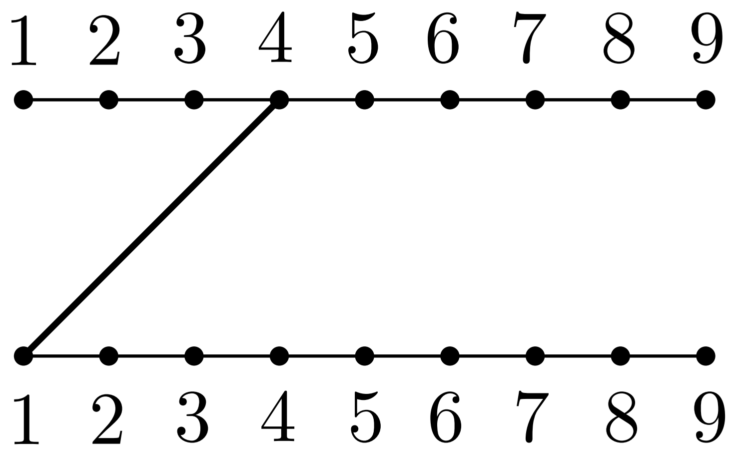

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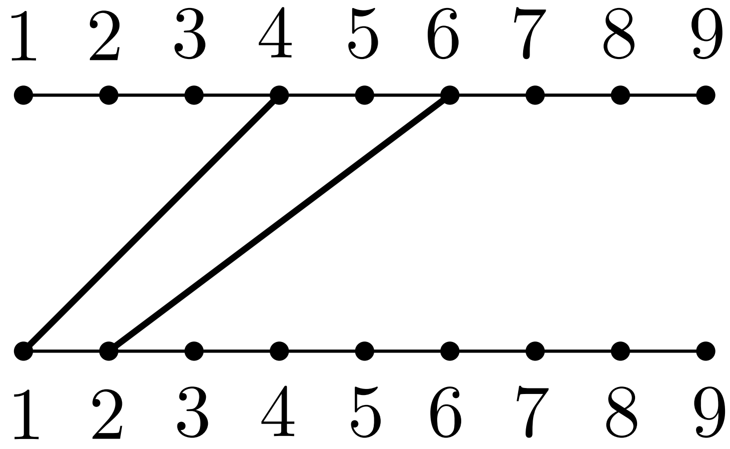

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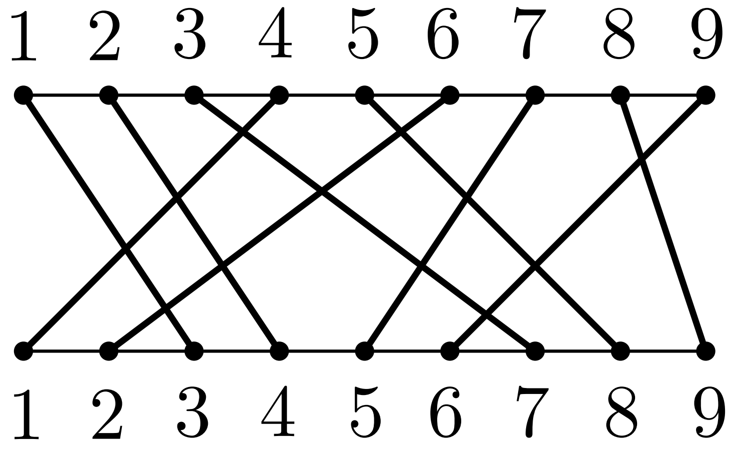

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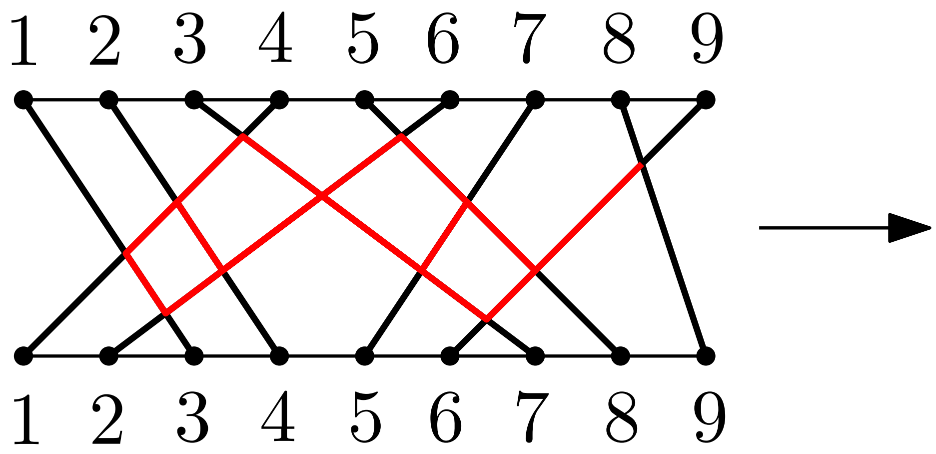

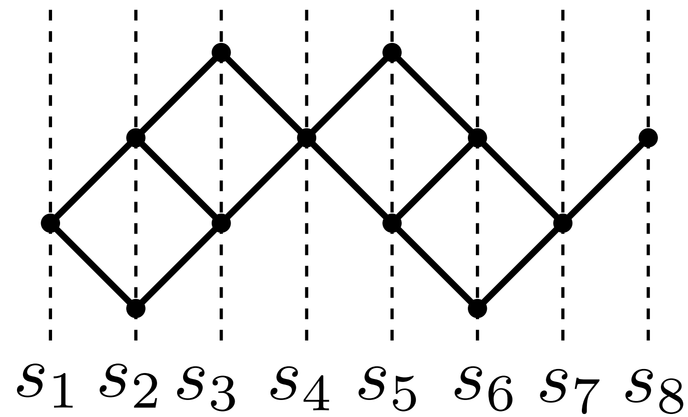

$inv(\sigma)$
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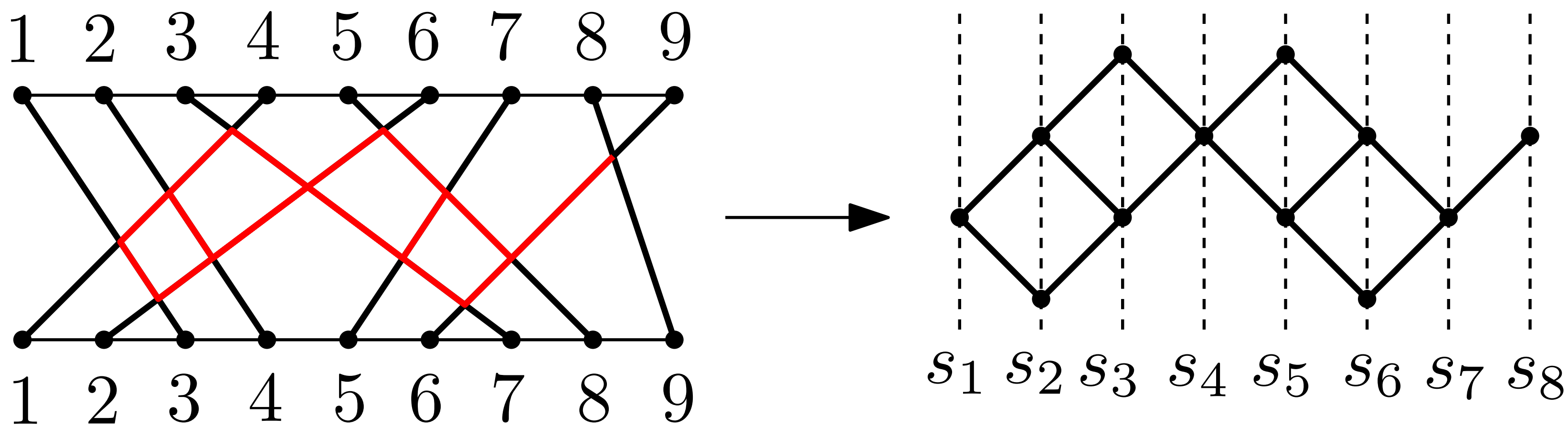

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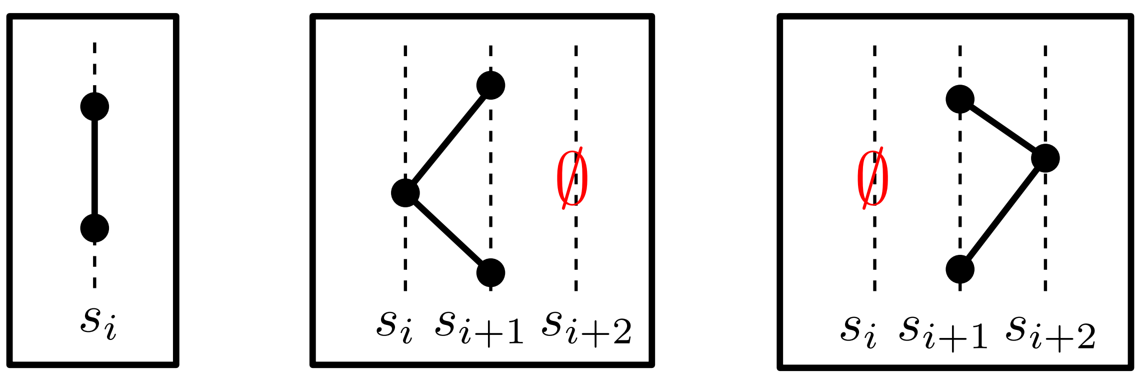

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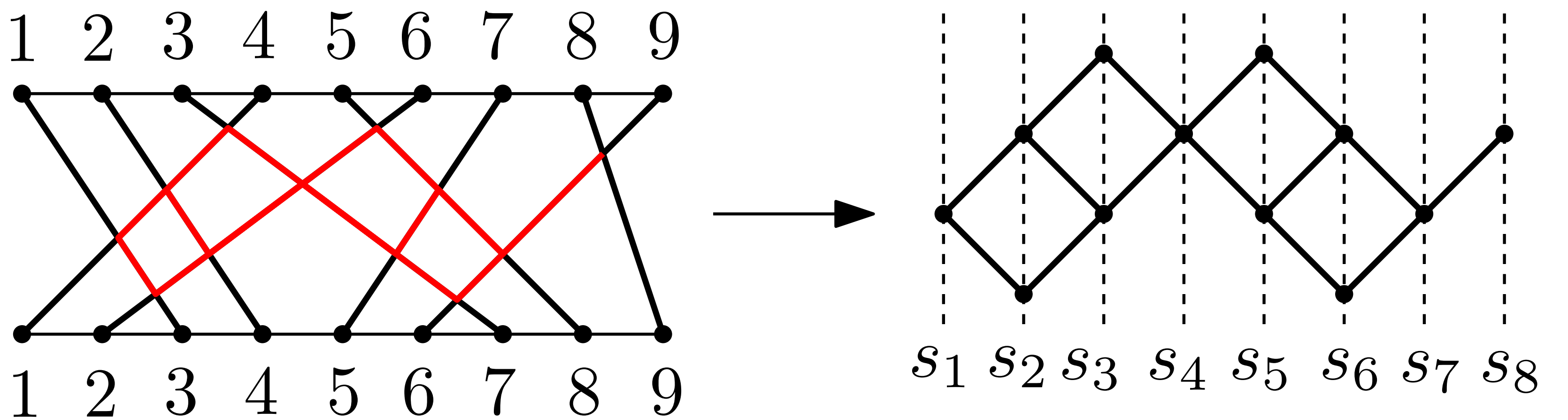

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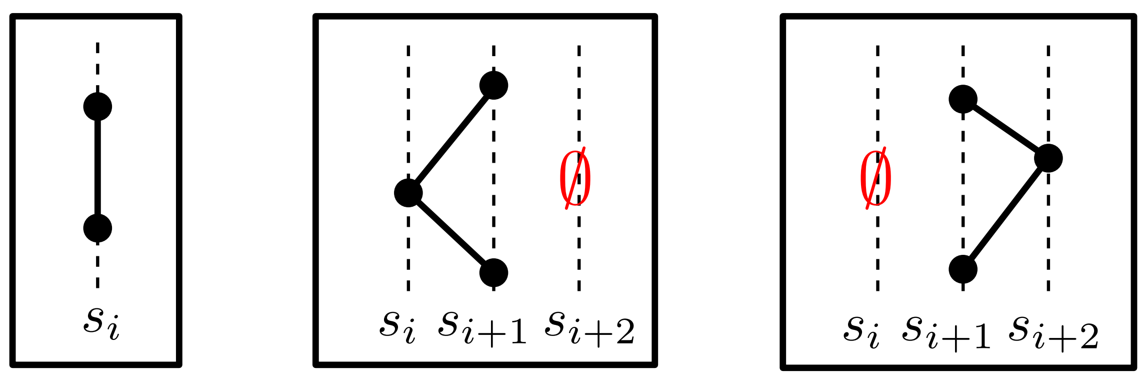

Proposition Alternating diagrams of S_n are characterized by:
(a) At most one occurrence of s_1 (*resp.* s_{n-1})
(b) $\forall i$, elements with labels s_i, s_{i+1} form an alternating chain

321-avoiding affine permutations

The group $\widetilde{S}_n$ is the set of permutations σ of $\mathbb{Z}$ satisfying $\sigma(i+n) = \sigma(i) + n$, and $\sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i$

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$$\sigma(1)\sigma(2)\sigma(3)\sigma(4)$$

Here $\sigma \in \widetilde{S}_4(321)$ and $inv(\sigma) = 5 + 4 = 9$

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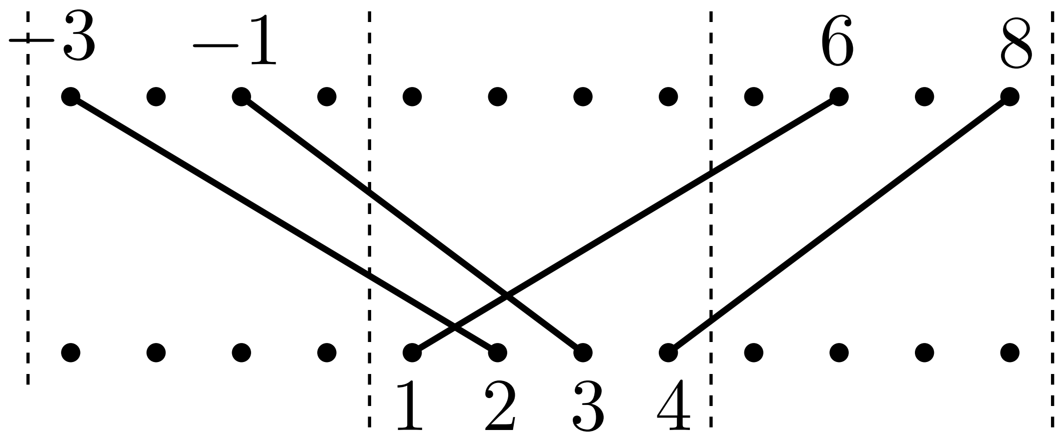

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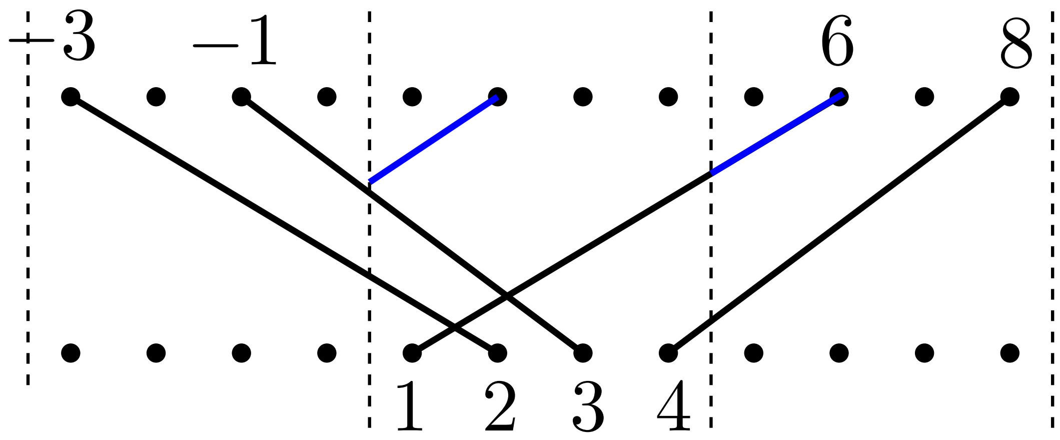

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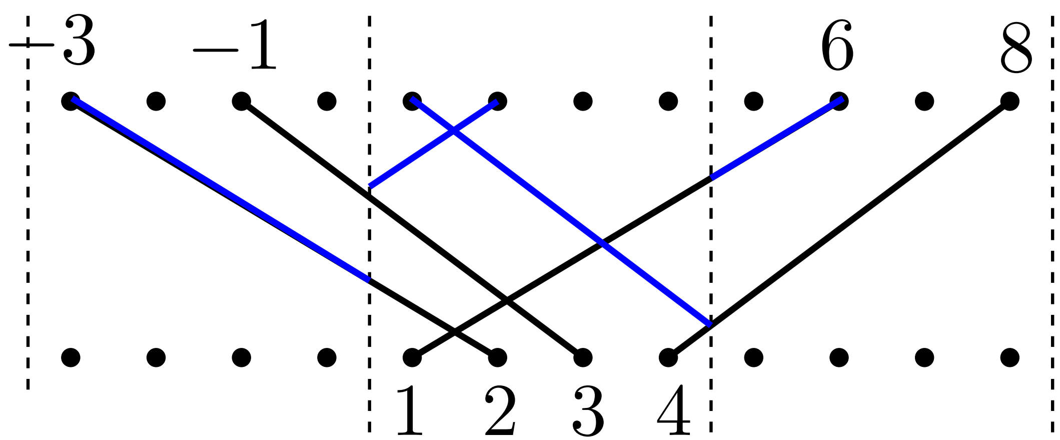

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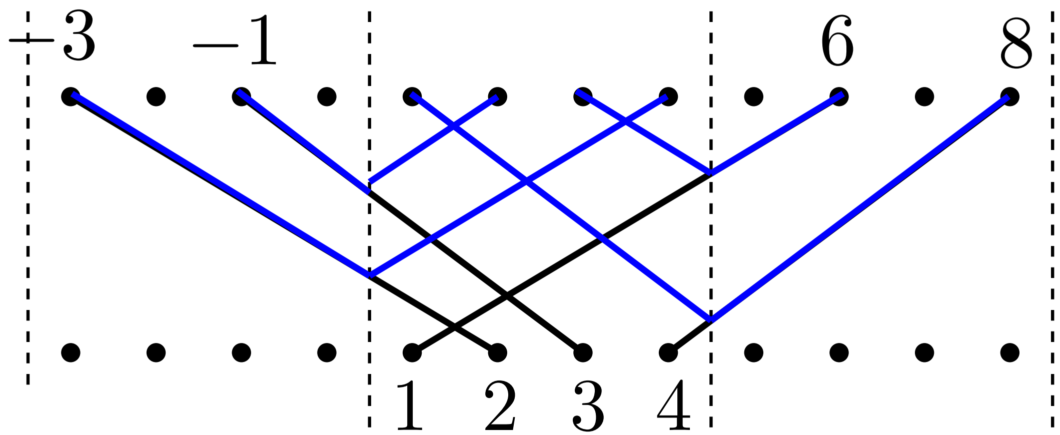

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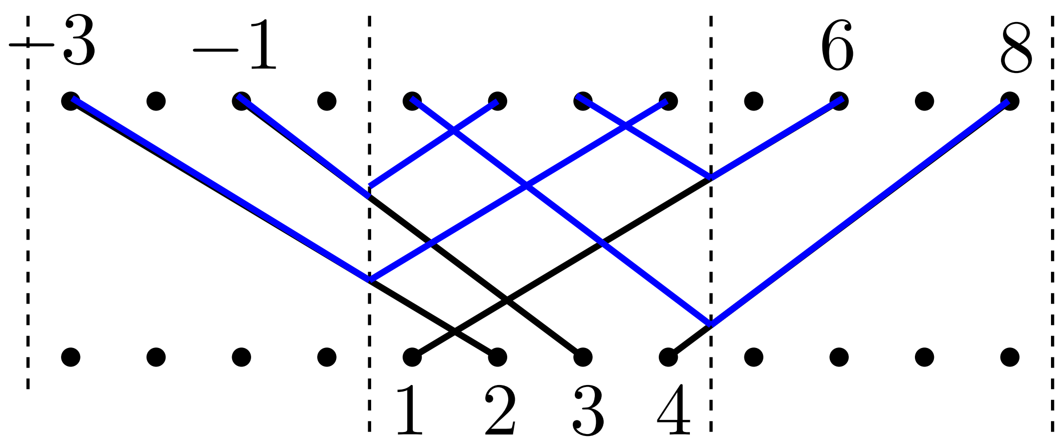
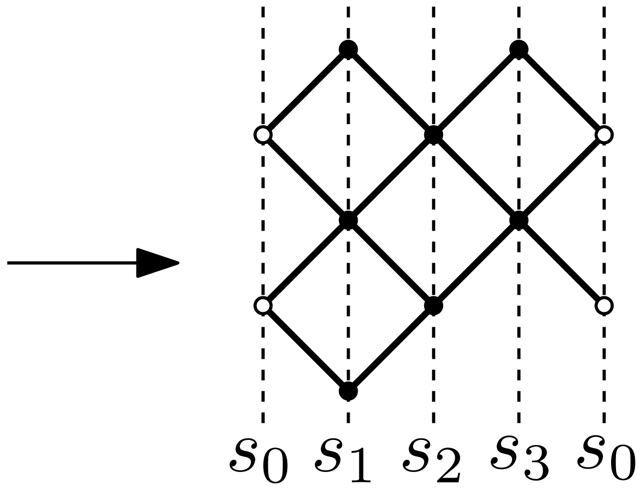
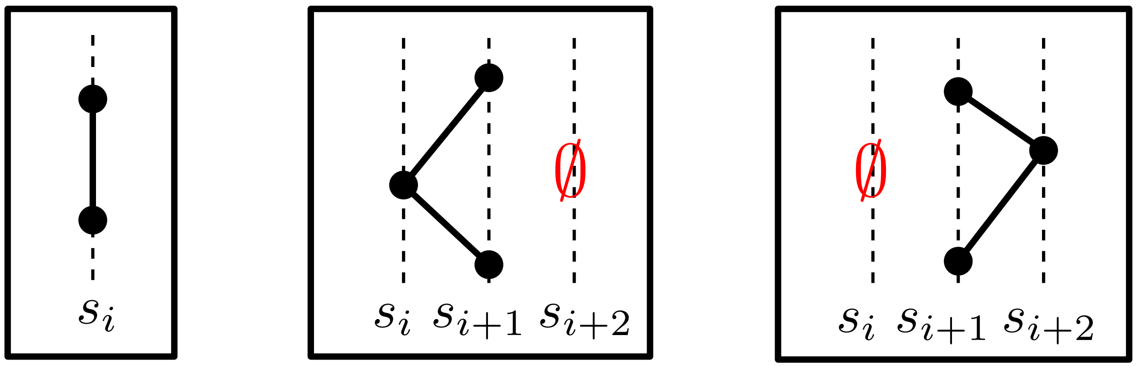




affine alternating diagram of size 9

Affine alternating diagrams

Same local conditions as for S_n: must avoid the shapes



(here $s_n := s_0$)

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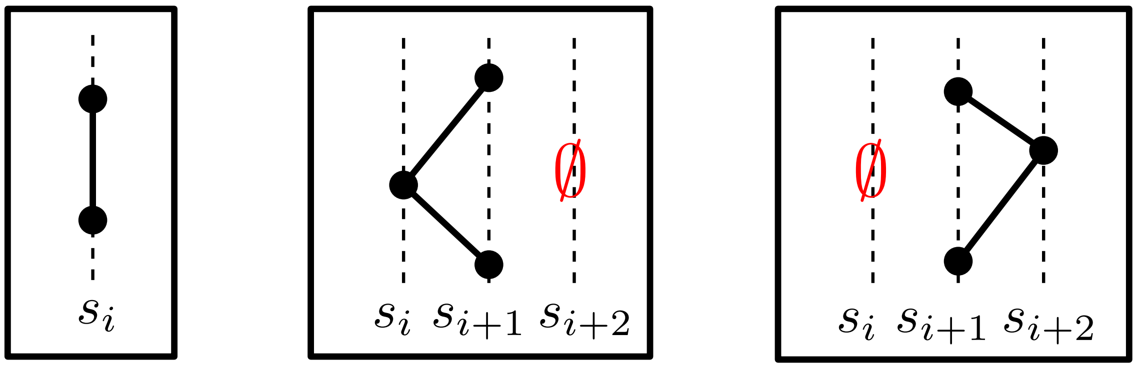

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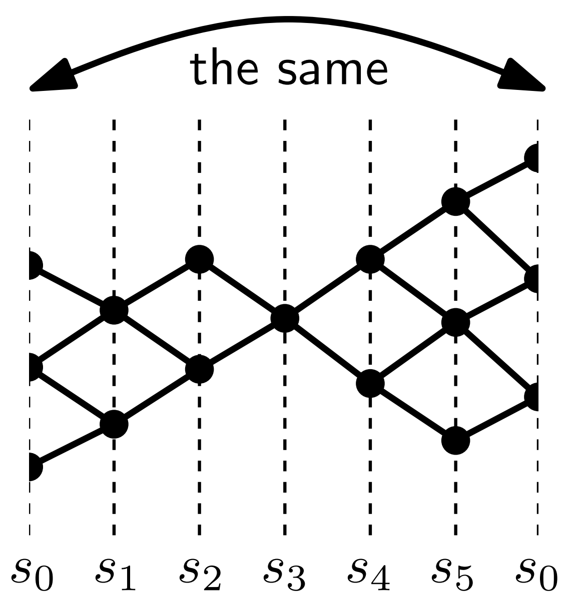
Difference: the labels above must be taken with index modulo n; the posets must be thought of as "drawn on a cylinder"



affine alternating diagram

of size $13 = inv(\sigma)$

Previous work

• [Billey–Jockush–Stanley (1993)] 321-avoiding permutations in $S_n(321)$ correspond to fully commutative elements of type A_{n-1}

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• [Green (2001)] 321-avoiding affine permutations in $\widetilde{S}_n(321)$ correspond to fully commutative elements of type $\widetilde{A}_{n-1}$

• [Hanusa–Jones (2010)] enumeration of $\widetilde{S}_n(321)$ with respect to the inversions (or Coxeter length): nice periodicity properties but very complicated expression

• [Biagioli–J–Nadeau (2014)] enumeration of $S_n(321)$ and $\widetilde{S}_n(321)$ with respect to the inversions using alternating diagrams and Motzkin-type lattice walks: recursive GF

Generating function for permutations in $S_n(321)$

$$A_{n-1}(q) := \sum_{\sigma \in S_n(321)} q^{inv(\sigma)} \quad \text{and} \quad A(x) := 1 + \sum_{n \geq 1} A_{n-1}(q) x^n$$

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Theorem: [Barcucci et al. (2001)]

We have $A(x) = \dfrac{1}{1-xq} \times \dfrac{J(xq)}{J(x)}$, where

$$J(x) := \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n}$$

Here $(a)_0 := 1$

and $(a)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),\ n \geq 1$

Generating function for affine permutations in $\widetilde{S}_n(321)$

$$\widetilde{A}_{n-1}(q) := \sum_{\sigma \in \widetilde{S}_n(321)} q^{inv(\sigma)} \text{ and } \widetilde{A}(x) := \sum_{n \geq 1} \widetilde{A}_{n-1}(q) x^n$$

Theorem: [Biagioli, Bousquet-Mélou, J, Nadeau (2015)]
We have

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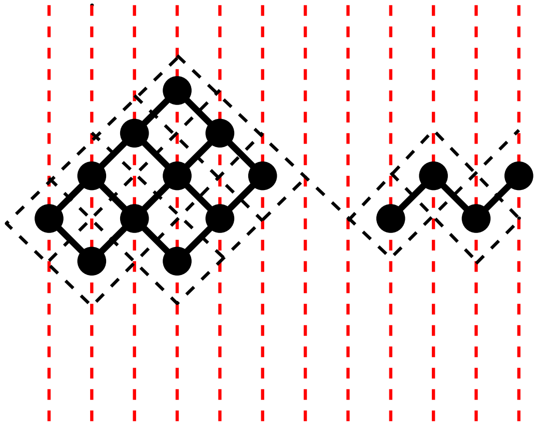
Our strategy: encode 321-avoiding (affine) permutations by alternating (affine) diagrams, then (periodic) parallelogram polyominoes, and finally heaps of segments

From alternating diagrams to parallelogram polyominoes

Classical case (Viennot)



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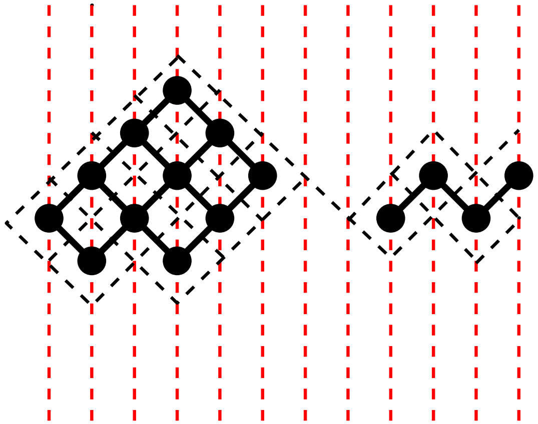

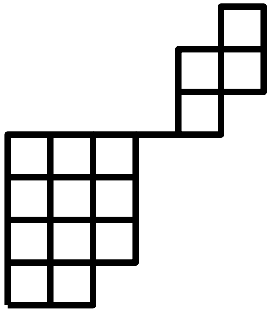

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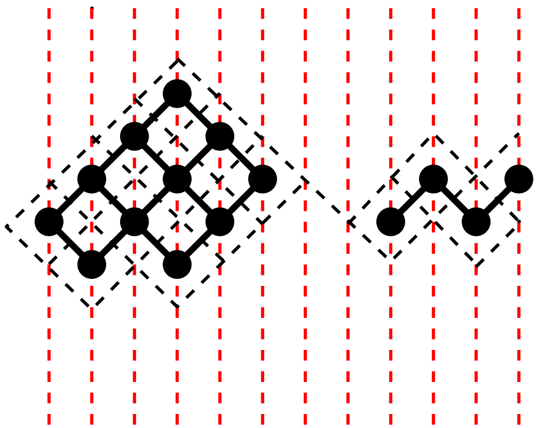
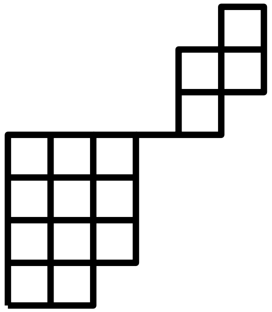
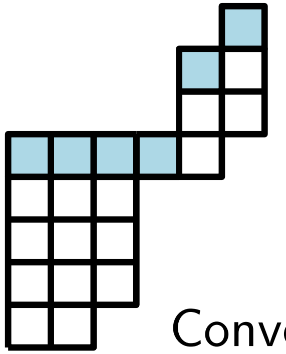




$(1,5)\ (5,5)\ (4,4)\ (1,1)\ (1,3)\ (2,3)$



Convention: $a_1 = 1$

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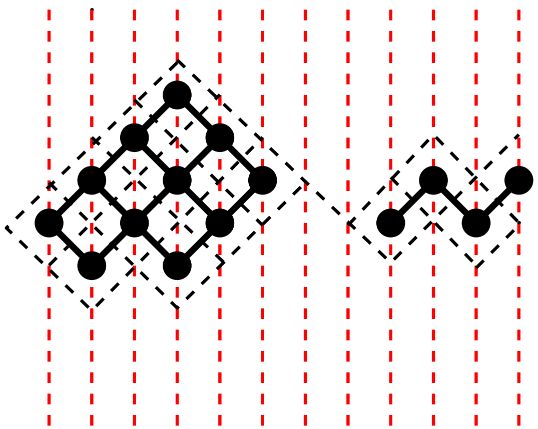

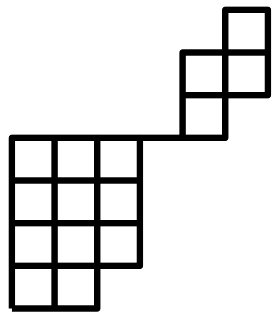

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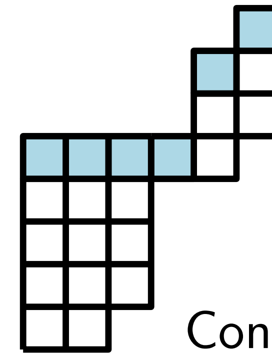

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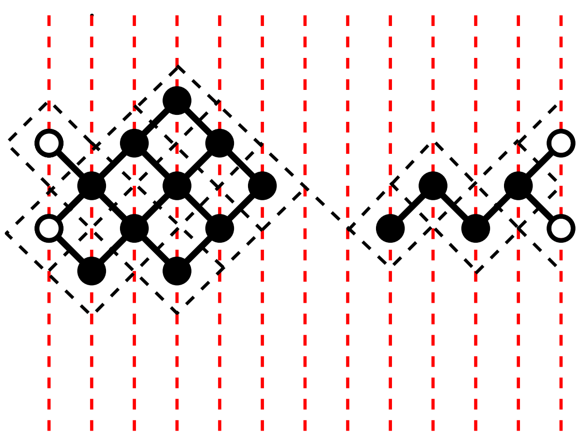
Affine case



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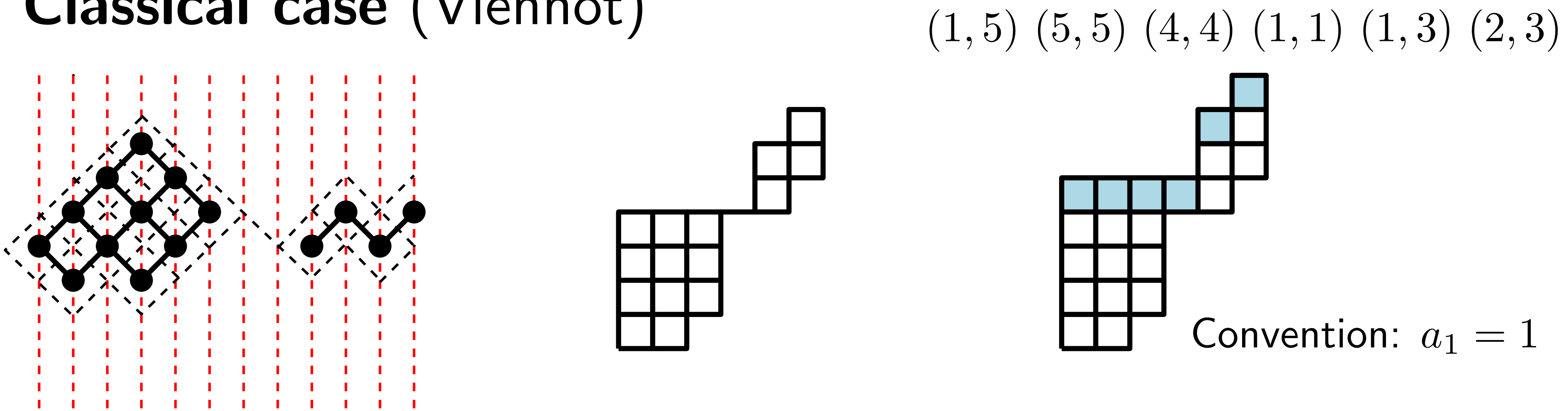

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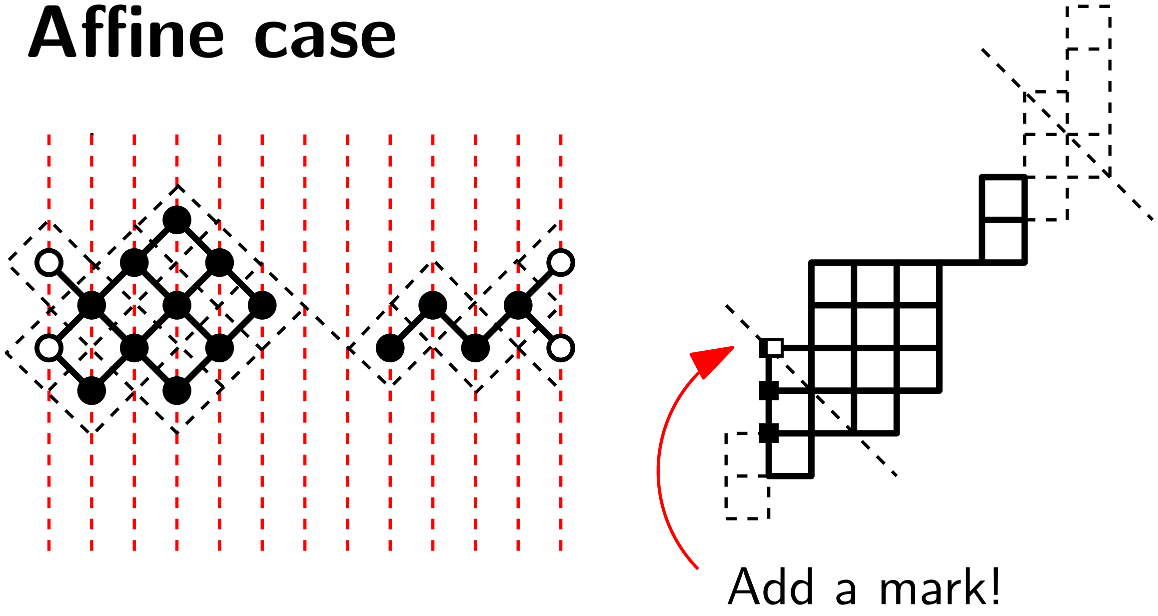

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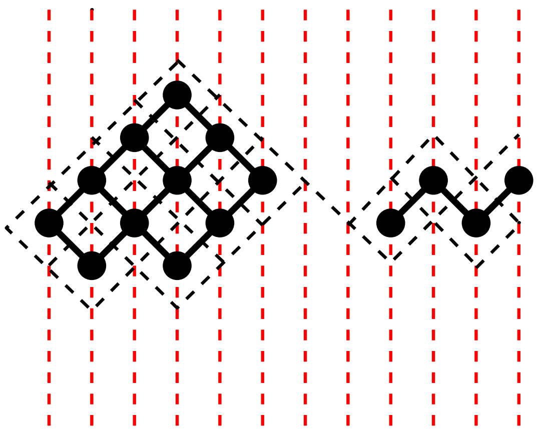

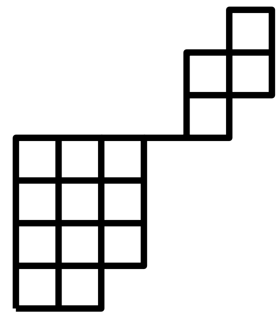

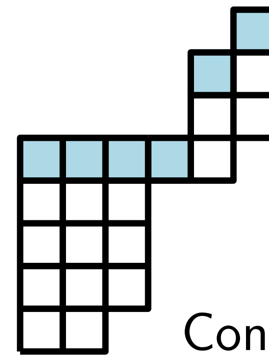

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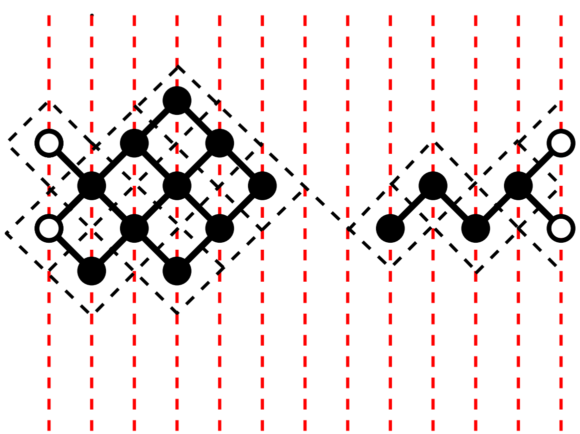
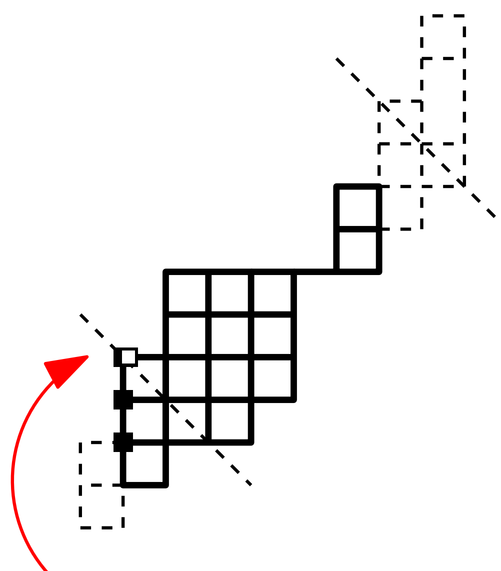
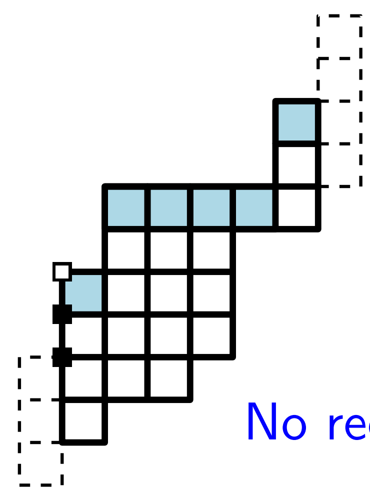




Add a mark!



No rectangle!

A mark in $[a_1, b_1]$ needed

Periodicity: $1 \leq a_1 \leq b_n$

From parallelogram polyominoes to heaps of segments

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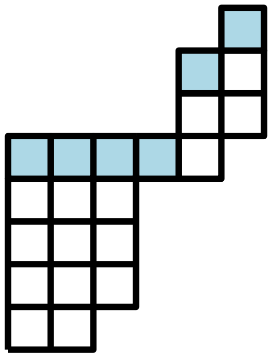

$(1,5)\ (5,5)\ (4,4)\ (1,1)\ (1,3)\ (2,3)$

Parallelogram polyominoes

$(a_i, b_i)_{1\le i\le n}$ with

$$1 = a_1 \le b_1 \ge a_2 \le b_2 \ge \cdots \ge a_n \le b_n$$

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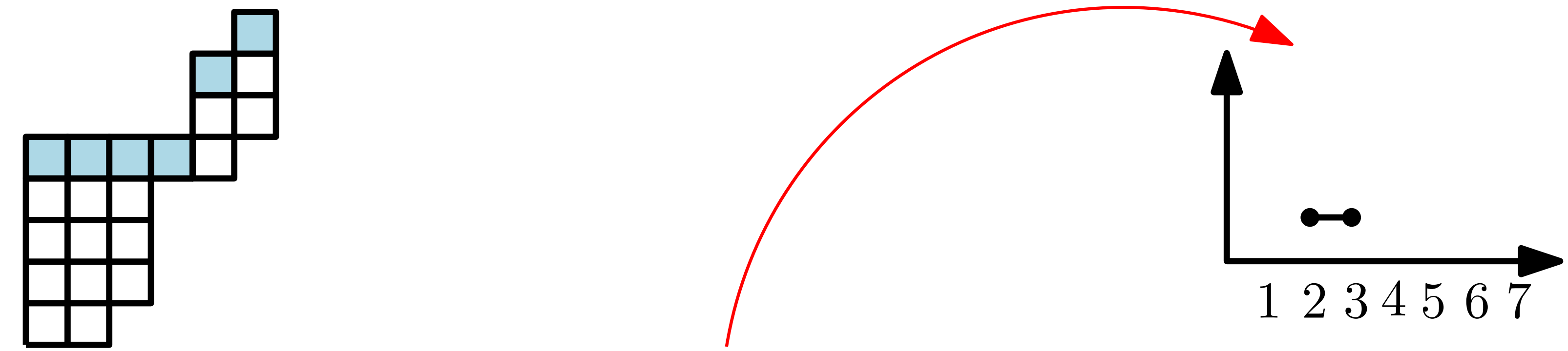

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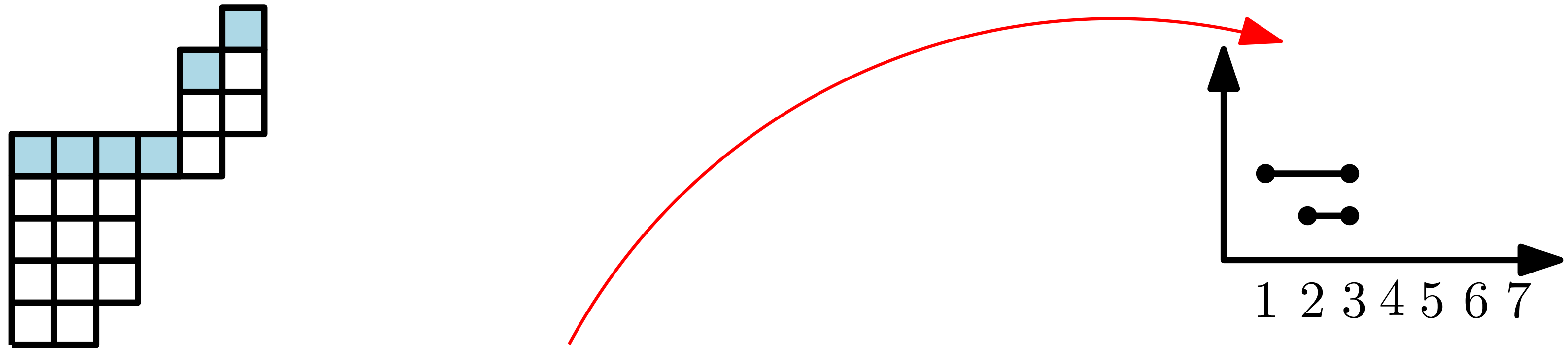

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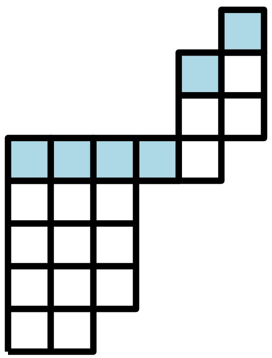

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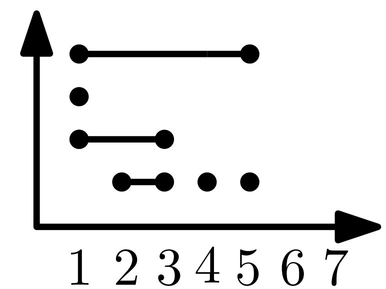
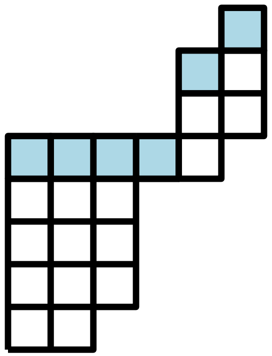
Bijection →



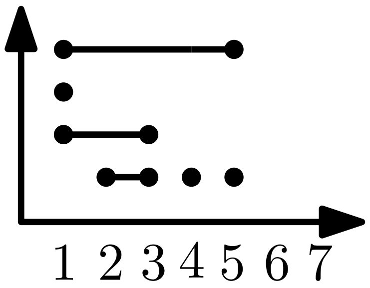
Semi-Pyramid SP: a unique max $[1, b]$

From parallelogram polyominoes to heaps of segments

Classical case (Bousquet-Mélou, Viennot)



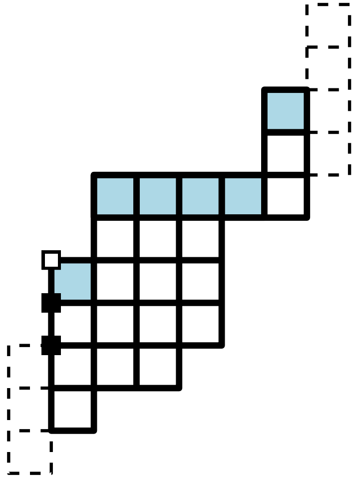
Bijection $\longrightarrow$



$(1,5)$ $(5,5)$ $(4,4)$ $(1,1)$ $(1,3)$ $(2,3)$

Semi-Pyramid SP: a unique max $[1, b]$

Affine case



Periodic parallelogram polyominoes

$b_n \geq a_1 \leq b_1 \geq a_2 \leq b_2 \geq \cdots \geq a_n \leq b_n$

marked in their first column, between a_1 and b_1

$(2,4)$ $(3,5)$ $(5,5)$ $(4,4)$ $(1,1)$ $(1,3)$

From parallelogram polyominoes to heaps of segments

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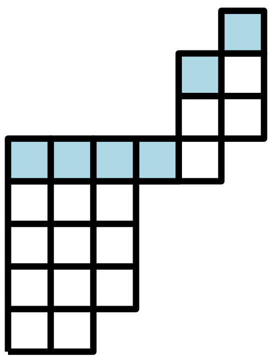
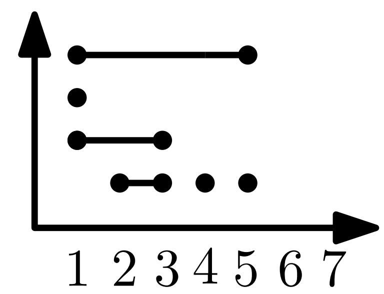


Bijection



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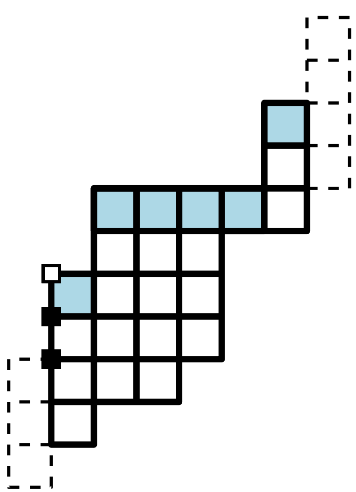
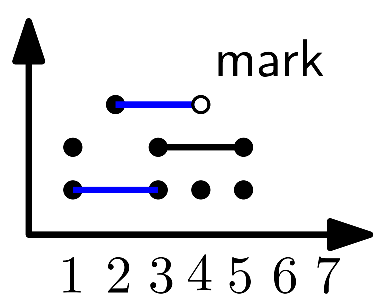
Semi-Pyramid SP: a unique max $[1,b]$

Affine case

Marked heaps $\mathcal{H}_I^*$



Bijection



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Condition I on the leftmost min

and the rightmost max

Generating functions for heaps of segments

Use the weights $v(H) = x^{\ell(H)} y^{|H|} q^{e(H)}$

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Proposition[Viennot (1985)] We have

$$H(x,y,q) = \frac{1}{T(x,y,q)} \text{ and } SP(x,y,q) = \frac{T^c(x,y,q)}{T(x,y,q)}$$

where T (resp. T^c) is the signed GF for trivial heaps (resp. not touching abscissa 1)

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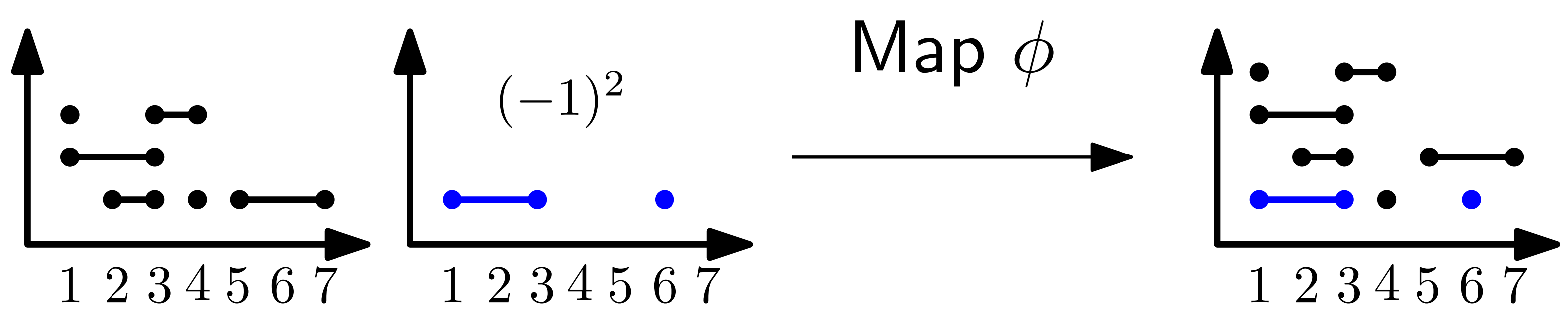

$$H \times T = \sum_{F\in\mathcal{H}} v(F) \sum_{U\subset min(F)} (-1)^{|U|}$$

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Lemma[Bousquet-Mélou, Viennot (1992)]

$$T = \sum_{n\geq 0} \frac{(-xy)^n q^{\binom{n+1}{2}}}{(q)_n (xq)_n} \text{ and } T^c = \sum_{n\geq 1} \frac{(-xy)^n q^{\binom{n+1}{2}}}{(q)_{n-1} (xq)_n}$$

This gives back the result of Barcucci et al by setting $y = 1/q$

Adaptation to our special heaps of segments in $\mathcal{H}_I^{(*)}$

$$\sum_{H \in \mathcal{H}_I} v(H) \times \sum_{T \in \mathcal{T}} (-1)^{|T|} v(T) = \sum_{(F,U) \in \phi(\mathcal{H}_I \times \mathcal{T})} v(F)(-1)^{|U|}$$

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$$= \sum_{F \in \mathcal{T}} |F| (-1)^{|F|-1} v(F)$$

$$+ \sum_{F, F \setminus U_1(F) \in \mathcal{H}_I} v(F) \times \sum_{U_1(F) \subseteq U \subseteq U_2(F)} (-1)^{|U|}$$

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this is $-y\partial_y T$

this is 0 if $U_1(F) \neq U_2(F)$

Adaptation to our special heaps of segments in $\mathcal{H}_I^{(*)}$

$$\sum_{H \in \mathcal{H}_I^*} v(H) \times \sum_{T \in \mathcal{T}} (-1)^{|T|} v(T) = \sum_{(F,U) \in \phi(\mathcal{H}_I^* \times \mathcal{T})} v(F)(-1)^{|U|}$$

$$= \sum_{F \in \mathcal{T}} \ell(F)(-1)^{|F|-1} v(F)$$

$$+ \sum_{F, F \setminus U_1(F) \in \mathcal{H}_I^*} v(F) \times \sum_{U_1(F) \subseteq U \subseteq U_2(F)} (-1)^{|U|}$$

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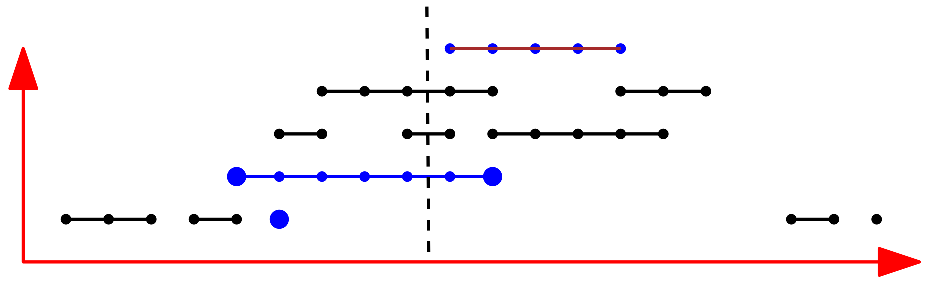
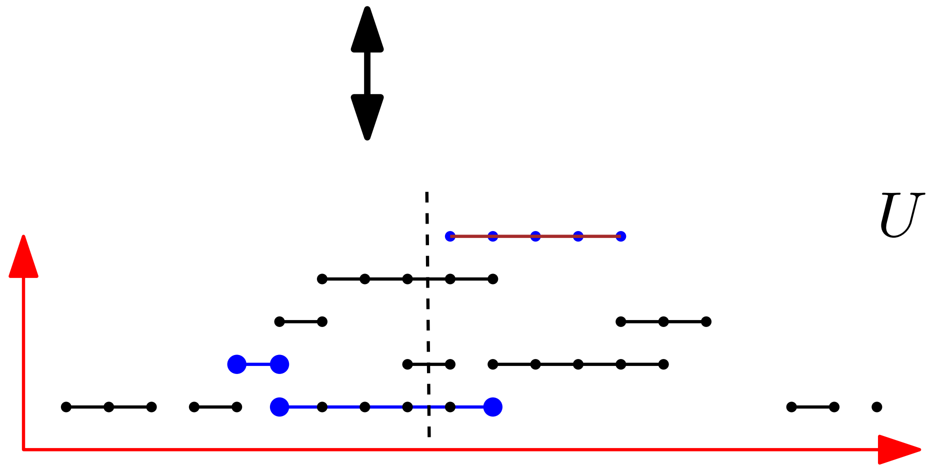
A sign-reversing involution on our heaps of segments



$$U = U_1(F) = U_2(F) = \min(F)$$

$$F \setminus U_1(F) \in \mathcal{H}_I^{(*)}$$



$$U = U_1(F) = U_2(F) = \min(F) \setminus \{S_0\}$$

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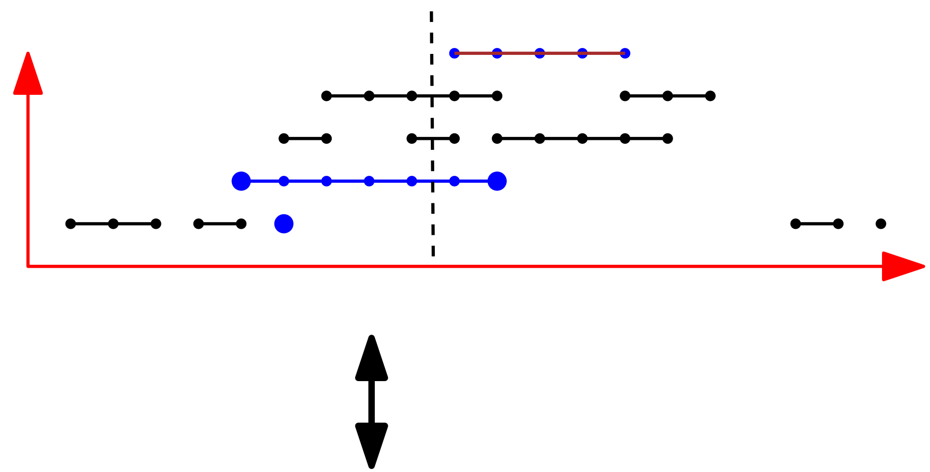
A sign-reversing involution on our heaps of segments



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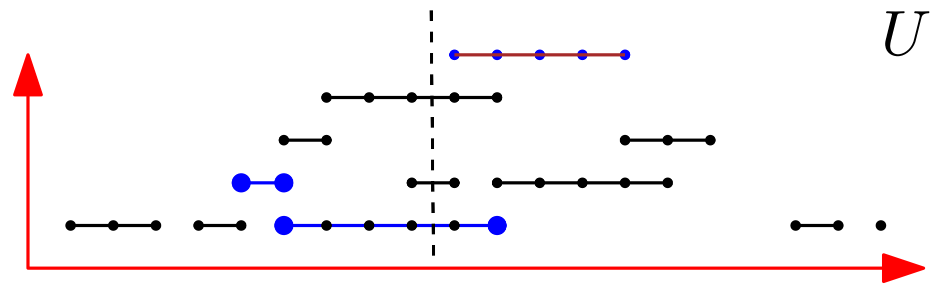

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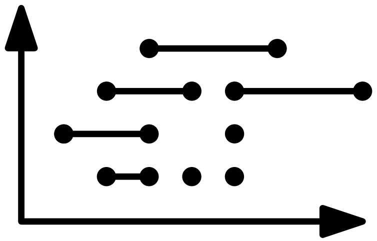
Theorem[Biagioli, Bousquet-Mélou, J, Nadeau (2015)]
The GF for periodic parallelogram polyominoes and marked ones, weighted with $x^{half-perimeter} y^{\#columns} q^{area}$ are respectively given by

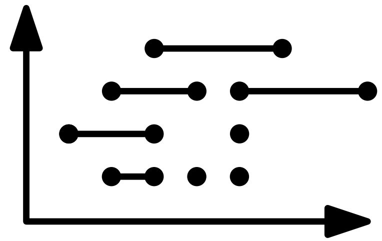
$$-y\frac{\partial_y T}{T} \text{ and } -x\frac{\partial_x T}{T}$$

Connection to pyramids



Pyramid Π: a unique max

Connection to pyramids



Pyramid Π: a unique max

Using the bijection ϕ we find

$$\sum_{H \in \Pi} v(H) = -y\frac{\partial_y T}{T} = \sum_{H \in \mathcal{H}_I} v(H)$$

A bijection between the sets $\mathcal{H}_I$ and Π would be nice

As would be a direct way of encoding periodic parallelogram polyominoes as pyramids