Schur processes and dimer models

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Based on joint works with Dan Betea, Cédric Boutillier, Guillaume Chapuy, Sylvie Corteel, Sanjay Ramassamy and Mirjana Vuletić

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Les Aléas de la science

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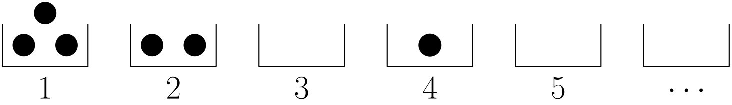
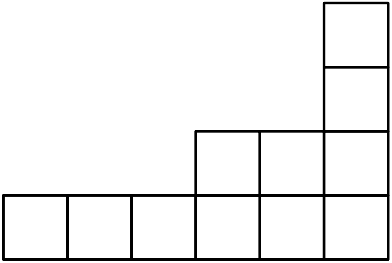
- 2015+: extension aux cas cylindrique (périodique) et pfaffien (bords libres) (en cours avec D. Betea, P. Nejjar et M. Vuletić), etc.

Outline

Outline

Bosons, Young diagrams, integer partitions...



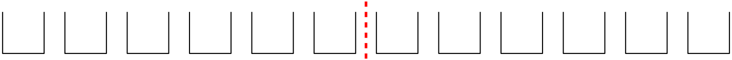


$$1 + 1 + 1 + 2 + 2 + 4 = 11$$

Integer partition: $\lambda = (4, 2, 2, 1, 1, 1) = (4, 2, 2, 1, 1, 1, 0, 0, \dots)$, $|\lambda| = 11$.

Fermions and Maya diagrams

Boxes labeled by half-integers ("energy levels", positive or negative):



$$\dots \quad \frac{-9}{2} \quad \frac{-7}{2} \quad \frac{-5}{2} \quad \frac{-3}{2} \quad \frac{-1}{2} \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{5}{2} \quad \frac{7}{2} \quad \frac{9}{2} \quad \dots$$

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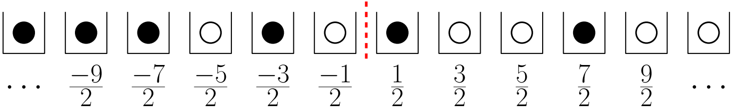

Each box may contain at most one particle ($\bullet$). No particle = "hole" ($\circ$).

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Maya diagram: there are finitely many particles on the positive side and holes on the negative side.

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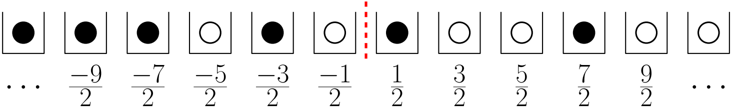

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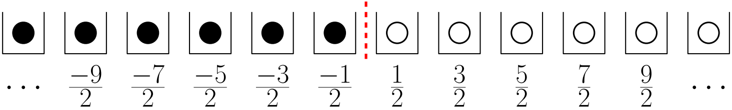

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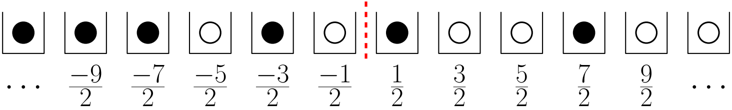

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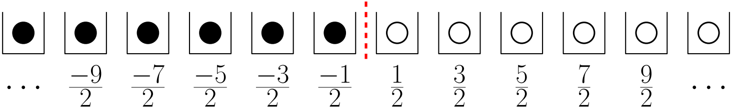

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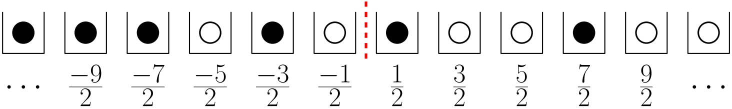

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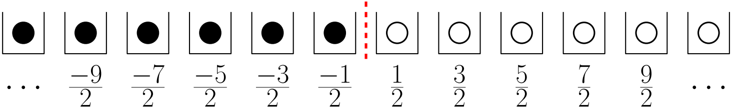

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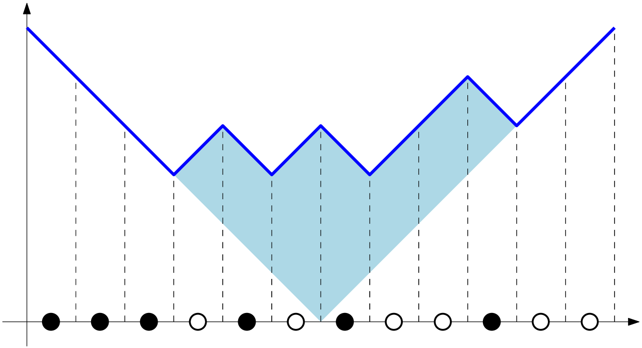
(total energy increases in both cases!)

Boson-fermion correspondence: combinatorial version



Maya diagrams are in bijection with pairs (λ, c) with λ a partition and c an integer (the charge). Here $\lambda = (4, 2, 1)$.

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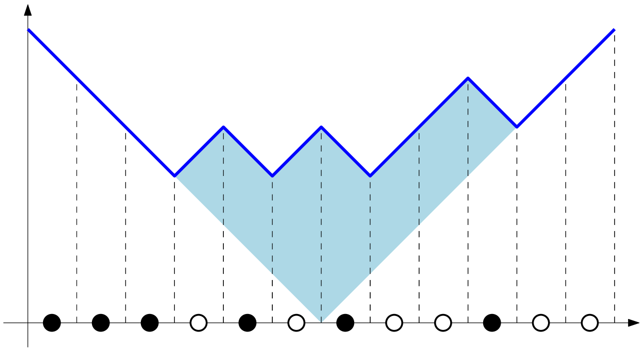

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For $i \geq 1$, the i-th rightmost particle is at position $\lambda_i - i + c + 1/2$ and the i-th leftmost hole is at position $-\lambda'_i + i + c - 1/2$ (λ': conjugate partition).

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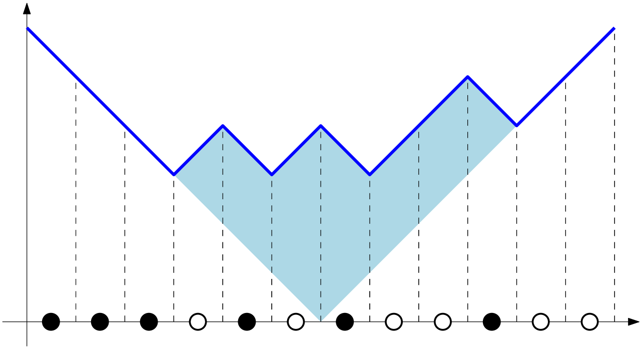

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Total energy is $|\lambda| + c^2/2$.

Outline

Schur processes [Okounkov-Reshetikhin]

A Schur process is a probability measure over sequences of integer partitions

$$\emptyset = \mu^{(0)} \subset \lambda^{(1)} \supset \mu^{(1)} \subset \lambda^{(2)} \supset \cdots \supset \mu^{(n-1)} \subset \lambda^{(n)} \supset \mu^{(n)} = \emptyset$$

where to each such sequence we associate a weight proportional to

$$\prod_{i=1}^{n} s_{\lambda^{(i)}/\mu^{(i-1)}}(\rho_i^+) s_{\lambda^{(i)}/\mu^{(i)}}(\rho_i^-)$$

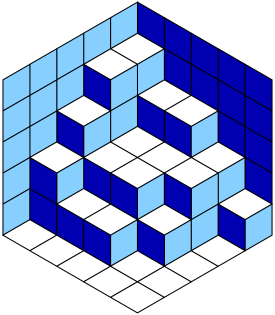
where $s_{\lambda/\mu}(\rho)$ is the skew Schur function

$$s_{\lambda/\mu}(\rho) = \det_{1 \leq i,j \leq N} h_{\lambda_i - i - \mu_j + j}(\rho)$$

with $h_k(\rho)$ a totally nonnegative sequence (with $h_0(\rho) = 1$ and $h_k(\rho) = 0$ for $k < 0$), depending on some dummy parameter ρ (called specialization). The $(h_k(\rho_i^{\pm}))_{k \geq 0, 1 \leq i \leq n}$ are the parameters of the Schur process.

Example: plane partitions [Okounkov-Reshetikhin]

Lozenge tilings



Plane partition

4	3	3	2	1
4	2	2	2	
3	2	2	1	
2	1	1		

Schur process

$$2 \prec \begin{matrix} 3 \\ 1 \end{matrix} \prec \begin{matrix} 4 \\ 2 \\ 1 \end{matrix} \prec \begin{matrix} 4 \\ 2 \\ 2 \end{matrix} \succ \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \succ \begin{matrix} 3 \\ 2 \end{matrix} \succ 2 \succ 1$$

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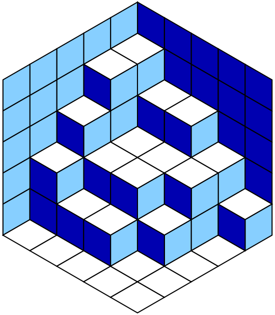

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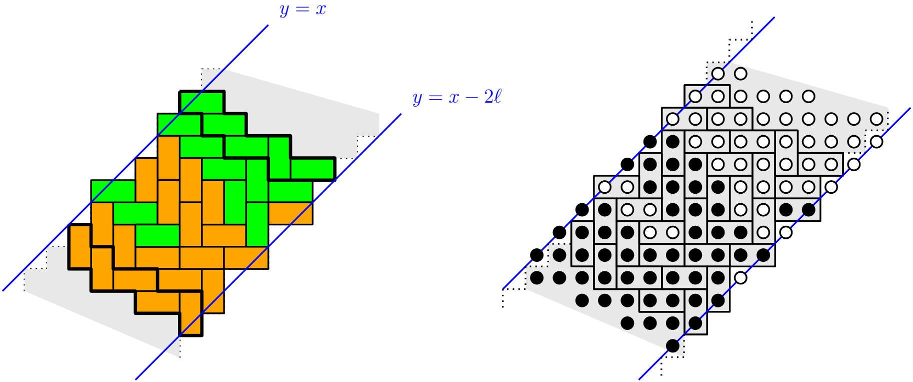
Obtained by taking specific parameters $h_k(\rho_i^{\pm}) = (z_i^{\pm})^k$ for all i.

Exemple: steep tilings [B.-Chapuy-Corteel]

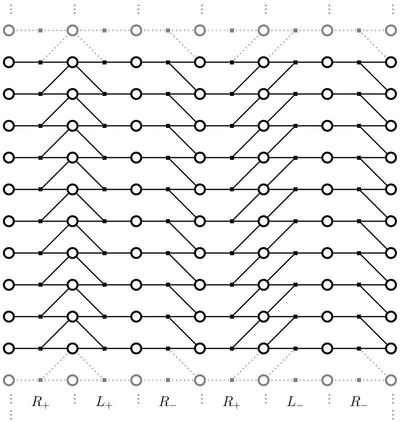


Contains domino tilings of the Aztec diamond and pyramid partitions as special cases. Can there be other cases?

Rail yard graphs

Theorem (Boutillier, B., Chapuy, Corteel, Ramassamy)

Every Schur process is equivalent to a dimer model on a certain graph called *rail yard graph* (or a limiting case thereof).



Proof idea

The weight associated to a sequence of partitions must be nonnegative, which is ensured by asking that $s_{\lambda/\mu}(\rho_i^{\pm}) \geq 0$ for all partitions λ, μ. This constrains the parameters $h_k(\rho_i^{\pm})$ (total nonnegativity).

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We can reduce to the case where each sequence $(h_k(\rho))_{k\geq 0}$ is such that

$$\sum_{k\geq 0} h_k(\rho) z^k = \frac{1}{1-\alpha z} \quad \text{or} \quad 1+\beta z \qquad \text{for some } \alpha \text{ or } \beta.$$

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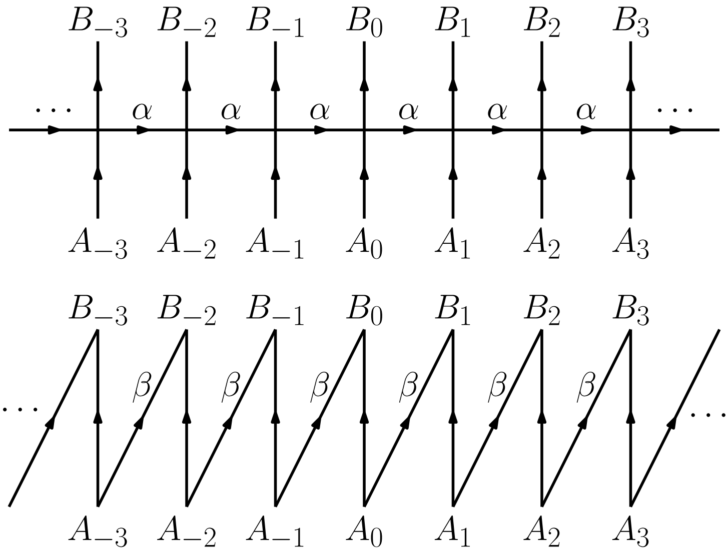
In both cases we may interpret the skew Schur function $s_{\lambda/\mu}(\rho) = \det_{1\leq i,j\leq N} h_{\lambda_i - i - \mu_j + j}(\rho)$ as a LGV determinant counting nonintersecting paths connecting the "sources" $A_{\mu_j - j}$ to the "sinks" $B_{\lambda_i - i}$ in some suitable graph (recall that the $\lambda_i - i$'s correspond to the fermion positions in the Maya diagram associated to λ !).

Proof idea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$\cdots$ α α α α α α $\cdots$
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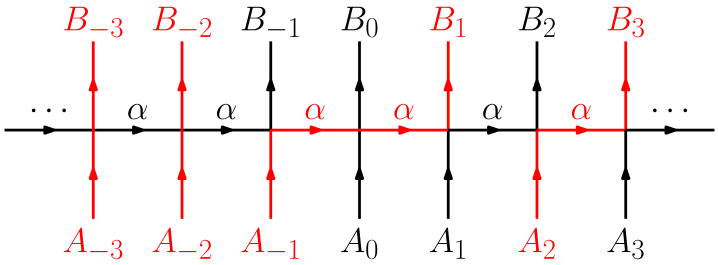

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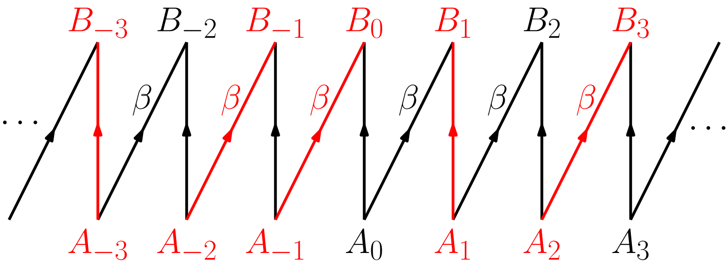

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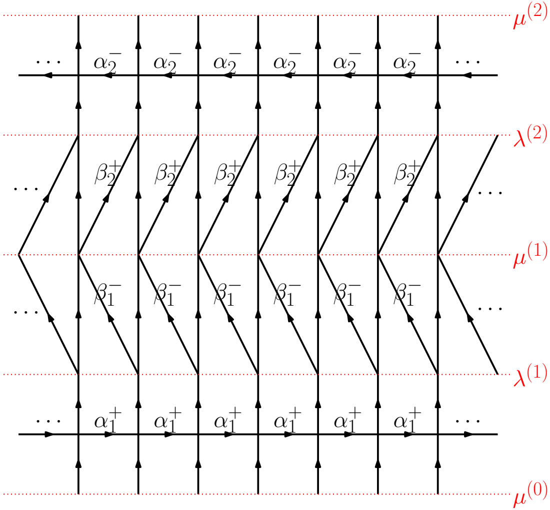
Proof idea

The Schur process

$$\emptyset = \mu^{(0)} \subset \lambda^{(0)} \supset \mu^{(1)} \subset \lambda^{(1)} \supset \cdots \supset \mu^{(n)} \subset \lambda^{(n)} \supset \mu^{(n+1)} = \emptyset$$

corresponds to nonintersecting paths on a LGV graph obtained by "stacking" the previous elementary graphs (and their reflections) together.

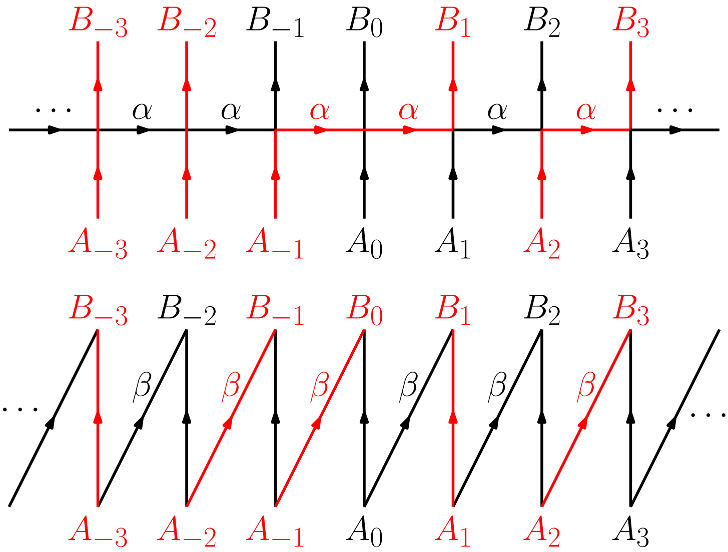
Proof idea



Proof idea

The last step is a bijection between nonintersecting paths on the LGV graph and dimer configurations on a related graph (the rail yard graph). We proceed independently within each elementary graph.

Proof idea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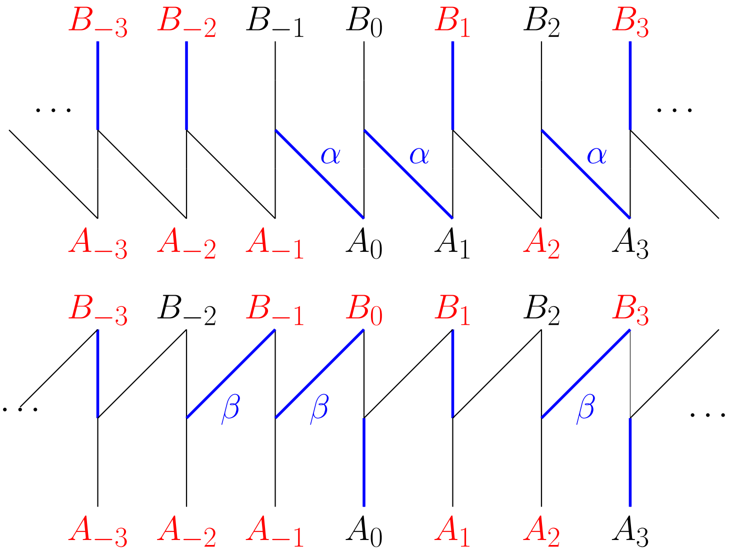

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α α α
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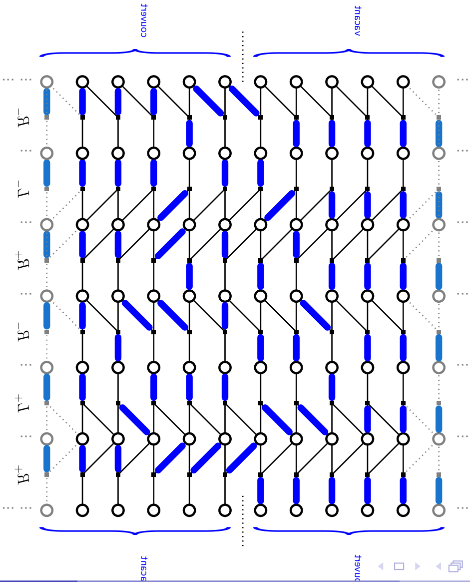

Outline

Preliminary: the transfer matrix method

Suppose we have a model where the configurations are sequences of symbols $a = (a_0, a_1, a_2, \ldots, a_n)$ ($a_i \in \mathcal{A}$) and the weight associated to such a sequence is of the form $w(a) = t^{(1)}_{a_0,a_1} t^{(2)}_{a_1,a_2} \cdots t^{(n)}_{a_{n-1},a_n}$.

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We are interested in the partition function

$$Z_{a_0,a_n} = \sum_a w(a)$$

where the sum runs over all sequences with given first and last elements. Then obviously Z_{a_0,a_n} is an entry of the matrix product

$$T^{(1)} T^{(2)} \dots T^{(n)}$$

where $T^{(i)}$ is the matrix with rows and columns indexed by $\mathcal{A}$ such that $(T^{(i)})_{a,b} = t^{(i)}_{a,b}$.

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For a given i, suppose we want to compute a restricted sum

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$$\Pi_{a,b} = \begin{cases} 1 & \text{if } a = b \in \mathcal{A}', \\ 0 & \text{otherwise.} \end{cases}$$

and then we have

$$Z^{(i)}_{a_0,a_n} = \left(T^{(1)} \cdots T^{(i)} \Pi T^{(i+1)} \cdots T^{(n)} \right)_{a_0,a_n}.$$

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This is readily generalized to the case where we want to constrain several a_i's, etc.

Application to the Schur process

Recall that, in the Schur process, the weight associated to a sequence

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We may apply the transfer matrix method! We introduce the transfer matrices $\Gamma_\pm(\rho)$, with rows and columns indexed by partitions, defined by

$$(\Gamma_-(\rho))_{\lambda,\mu} = (\Gamma_+(\rho))_{\mu,\lambda} = s_{\lambda/\mu}(\rho).$$

Then, the partition function of the Schur process reads

$$Z = \left(\Gamma_+(\rho_1^+)\Gamma_-(\rho_1^-) \cdots \Gamma_+(\rho_n^+)\Gamma_-(\rho_n^-)\right)_{\emptyset,\emptyset}.$$

Evaluation of the partition function

We may evaluate the matrix product thanks to the (quasi-)commutation relation:

$$\Gamma_+(\rho)\Gamma_-(\rho') = H(\rho;\rho')\Gamma_-(\rho')\Gamma_+(\rho)$$

where

$$H(\rho;\rho') = \begin{cases} \frac{1}{1-\alpha\alpha'} & \text{if } \rho,\rho' \text{ of } \alpha\text{-type}, \\ \frac{1}{1-\beta\beta'} & \text{if } \rho,\rho' \text{ of } \beta\text{-type}, \\ 1+\alpha\beta' & \text{if } \rho \text{ of } \alpha\text{-type}, \rho' \text{ of } \beta\text{-type}, \\ 1+\beta\alpha' & \text{if } \rho \text{ of } \beta\text{-type}, \rho' \text{ of } \alpha\text{-type}. \end{cases}$$

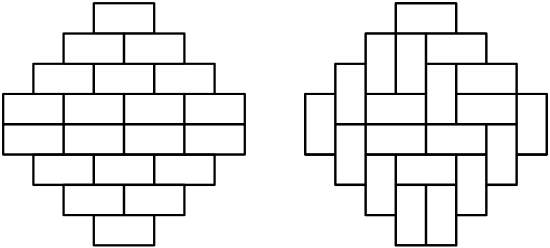
Then, noting that $(\Gamma_+(\rho))_{\mu,\emptyset} = (\Gamma_-(\rho))_{\emptyset,\mu} = 1$ for $\mu = \emptyset$ and 0 otherwise, we obtain

Proposition [Okounkov-Reshetikhin, Borodin]

The partition function of the Schur process reads

$$Z = \prod_{1\leq i\leq j\leq n} H(\rho_i^+;\rho_j^-).$$

Example: domino tilings of the Aztec diamond



(a) (b)

Domino tilings of the Aztec diamond of size n correspond to the case where each ρ_i^+ is of α-type and each ρ_i^- of β-type. The partition function then reads

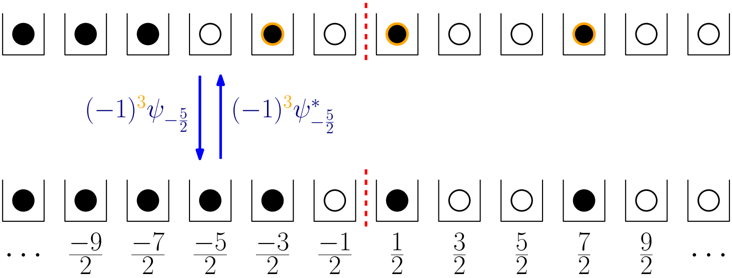
$$Z = \prod_{1 \leq i \leq j \leq n} (1 + \alpha_i^+ \beta_j^-)$$

which is equivalent to a multivariate formula due to Stanley. By taking $\alpha_i^+ = \beta_i^- = 1$ for all i we recover the well-known enumeration $2^{n(n+1)/2}$.

Statistics

We may also study statistics of the Schur process or of the related dimer model (probability of having dimers at certain positions). This can be done by introducing fermionic operators acting on Maya diagrams:

$$(\psi_k)_{m,m'} = (\psi_k^*)_{m',m} = \begin{cases} \pm 1 & \text{if } m \text{ has one particle more than } m' \text{ in box } k, \\ 0 & \text{otherwise.} \end{cases}$$



$\psi_k\psi_k^*$ is the diagonal matrix which "tests" where box k is occupied, hence this combination can be used to apply the transfer matrix method.

Statistics

Fermionic operators are useful because they enjoy nice commutation with the transfer matrices (seen as acting on Maya diagrams): for $\psi(z) = \sum_k \psi_k z^k$ we have for instance

$$\Gamma_+(\rho)\psi(z) = \begin{cases} \frac{1}{1-\alpha z}\psi(z)\Gamma_+(\rho) & \text{if } \rho \text{ of } \alpha\text{-type,} \\ (1+\beta z)\psi(z)\Gamma_+(\rho) & \text{if } \rho \text{ of } \beta\text{-type.} \end{cases}$$

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This can be used to eliminate the transfer matrices as in the computation of the partition function, and then apply Wick's theorem:

$$\left(\psi_{k_1}\psi^*_{\ell_1}\cdots\psi_{k_r}\psi^*_{\ell_r}\right)_{\emptyset,\emptyset} = \det_{1\le i,j\le r} M_{i,j}, \quad M_{i,j} = \begin{cases} 1 & \text{if } i<j \text{ and } k_i = \ell_j < 0, \\ 1 & \text{if } i>j \text{ and } k_i = \ell_j > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We deduce that the correlation function are determinantal, and we compute the inverse of the Kasteleyn matrix for general rail yard graphs.

Application: local statistics in the Aztec diamond

We consider again domino tilings of the Aztec diamond of size n under the uniform measure. We denote by $P(n, x, y)$ the probability that $(x - 1/2, y)$ is the center of a vertical domino (the origin being at the center of the diamond).

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Theorem [Du-Gessel-Ionescu-Propp?, Helfgott, BBCCR]

We have

$$\sum_{\substack{n,x,y \\ n+x+y \text{ odd}}} P(n, x, y) t^n u^x v^y = \frac{t}{(1 - t/u)\left(2(1 + t^2) - t(u + u^{-1} + v + v^{-1})\right)}$$

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Remarks:

- can be generalized to the biased case (weight λ per vertical domino)
- can be used to provide an analytic combinatorics proof of the arctic circle theorem [Baryshnikov-Pemantle] (but we also have a direct saddle-point proof from a contour integral expression for $P(n,x,y)$).
- connection with diagonals of bivariate rational Laurent series?

Outline

Bijective enumeration ?

The basic idea is that our enumeration using transfer matrices can be reformulated as a bijection between realizations of the Schur process (sequences of integer partitions) and arrays of integers (unconstrained).

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Bijective enumeration ?

The basic idea is that our enumeration using transfer matrices can be reformulated as a bijection between realizations of the Schur process (sequences of integer partitions) and arrays of integers (unconstrained).These arrays can easily be sampled since they involve independant random variables with Bernoulli or geometric distributions. By applying the bijection we obtain a sample of the Schur process. Actually our bijection is "well-known":

- plane partitions: Robinson-Schensted-Knuth algorithm, formulated in terms of growth diagrams à la Fomin
- domino tilings of the Aztec diamond: domino shuffling algorithm
- general case: RSK-type algorithm for "oscillating tableaux" (Gessel, Pak-Postnikov, Krattenthaler...).

Local rule

The starting point is a bijective interpretation of the quasi-commutation relation

$$\Gamma_+(\rho)\Gamma_-(\rho') = H(\rho;\rho')\Gamma_-(\rho')\Gamma_+(\rho)$$

i.e.

$$\sum_{\nu} s_{\nu/\lambda}(\rho) s_{\nu/\mu}(\rho') = H(\rho;\rho') \sum_{\kappa} s_{\lambda/\kappa}(\rho') s_{\mu/\kappa}(\rho)$$

for α- or β-type specializations.

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for α- or β-type specializations.

If ρ,ρ' are both α-type, $H(\rho;\rho') = 1/(1-\alpha\alpha')$, there is a bijection proving

$$\sum_{\nu\in S^+_{\lambda,\mu}} \alpha^{|\nu|-|\lambda|}(\alpha')^{|\nu|-|\mu|} = \sum_{k=0}^{\infty}(\alpha\alpha')^k \sum_{\kappa\in S^-_{\lambda,\mu}} (\alpha')^{|\lambda|-|\kappa|}\alpha^{|\mu|-|\kappa|}.$$

Sampling application: suppose that, conditionally on λ,μ, κ is distributed as $s_{\lambda/\kappa}(\rho')s_{\mu/\kappa}(\rho)$. By drawing $k\sim\mathrm{Geom}(\alpha\alpha')$ and applying the bijection, get ν distributed as $s_{\nu/\lambda}(\rho)s_{\nu/\mu}(\rho')$.

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If ρ of α-type and ρ' of β-type, $H(\rho;\rho') = 1+\alpha\beta'$, have bijection for

$$\sum_{\nu\in\tilde{S}^+_{\lambda,\mu}} \alpha^{|\nu|-|\lambda|}(\beta')^{|\nu|-|\mu|} = \sum_{k=0,1}(\alpha\beta')^k \sum_{\kappa\in\tilde{S}^-_{\lambda,\mu}} (\beta')^{|\lambda|-|\kappa|}\alpha^{|\mu|-|\kappa|}.$$

Sampling application: suppose that, conditionally on λ,μ, κ is distributed as $s_{\lambda/\kappa}(\rho')s_{\mu/\kappa}(\rho)$. By drawing $k\sim\mathrm{Bernoulli}(\alpha\beta')$ and applying the bijection, get ν distributed as $s_{\nu/\lambda}(\rho)s_{\nu/\mu}(\rho')$.

Local rule

The mixed α- and β-type correspond to the domino shuffling algorithm!