Asymptotics of pyramid partitions via the Schur process

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(Work in progress, with **C. Boutillier, M. Vuletić**)

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Outline

- Pyramid partitions
- Interlude into partitions and the Schur process
- Asymptotics of pyramid partitions
- Random sampling
- Related stuff

Let them eat cake!

"S'ils n'ont pas de pain, qu'ils mangent de la brioche!"
–Marie Antoinette d'Autriche (1755–1793)

Pyramid partitions

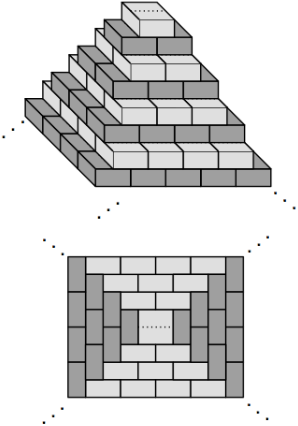


Figure: Piles of $2 \times 2 \times 1$ boxes, each viewed as a pair of dominoes in the 2D projection looking downwards. On the left, the *empty* pyramid partition.

More pictures

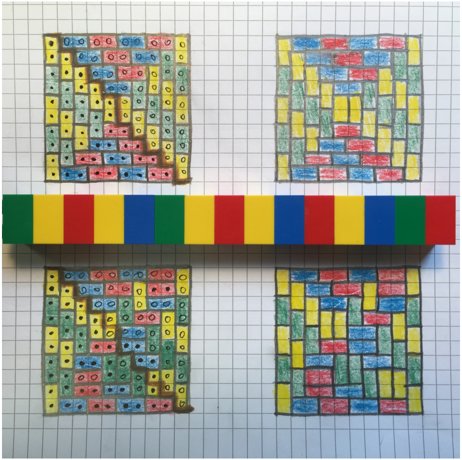


Figure: Pyramid partitions in 2D, LEGO coloring.

LEGO

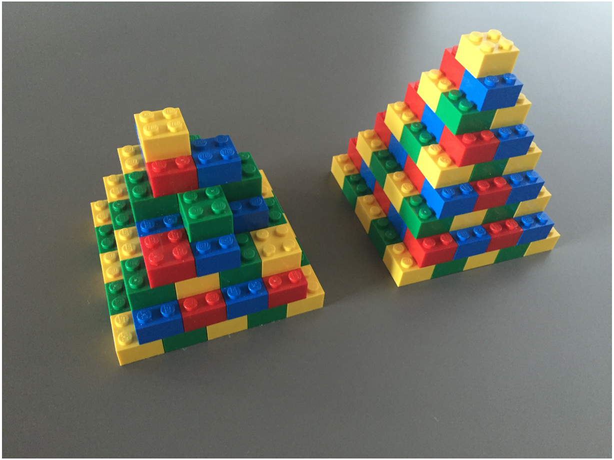


Figure: Pyramid partitions in 3D, natural light coloring.

Flips and the volume

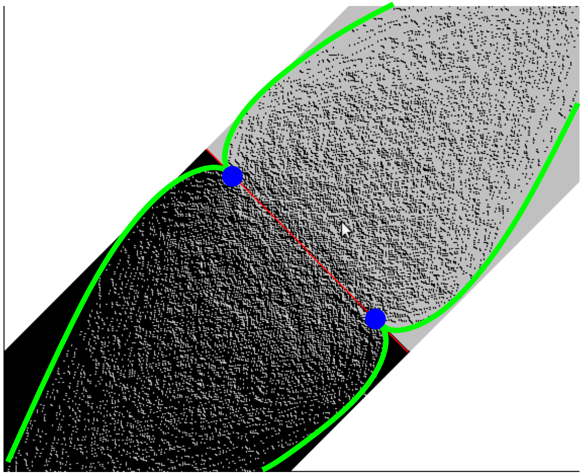
- pyramid partition = what's left after a finite number of box removals from the empty configuration (introduced by Kenyon and Szendröi)
- removal = flip (adjacent vertical dominoes $\leftrightarrow$ adjacent horizontal dominoes)
- Volume = Number of flips

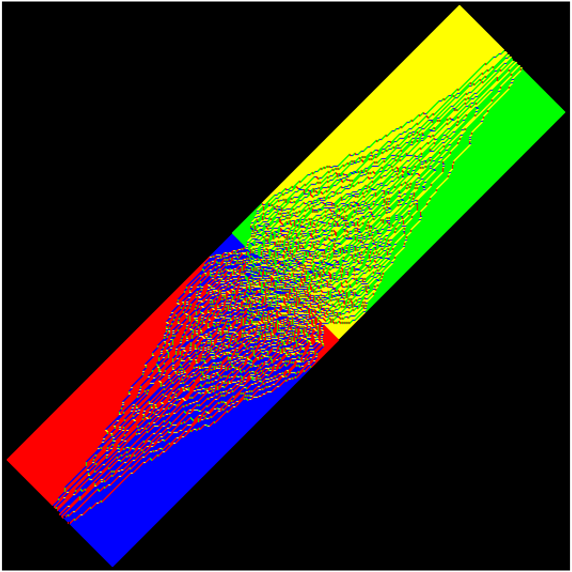
Theorem (Young 2010)

$$\sum_{\Lambda} q^{Volume(\Lambda)} = \prod_{n \geq 1} \frac{(1+q^{2n-1})^{2n-1}}{(1-q^{2n})^{2n}}.$$

This is a consequence of RSK for supersymmetric Schur functions (BCC 2014).

How do large pyramid partitions look like?





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Partitions

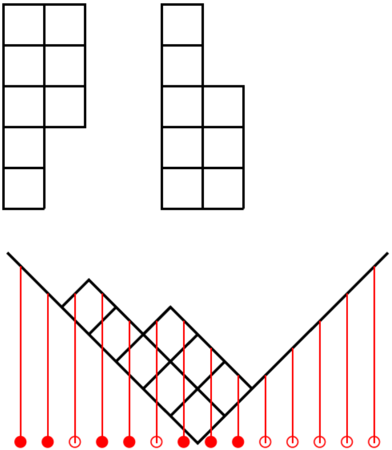

Figure: Partition $(2, 2, 2, 1, 1)$ in English, French and Russian notation, with associated Maya diagram (particle-hole representation).

Horizontal and vertical strips

Given partitions $\mu \subseteq \lambda$, we can form skew diagram λ/μ, which we call a

- horizontal strip, and write $\mu \prec \lambda$ if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \dots$$

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********      /      *****         -----***
*****        /       ***           ---**
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- vertical strip, and write $\mu \prec' \lambda$, if $\lambda' \prec \mu'$ ($' =$ conjugate) or

$$\lambda_i - \mu_i \in \{0, 1\}$$

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********         /      *******         -------*
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The Schur process

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \{\prec, \succ, \prec', \succ'\}^n$ be a word. We say a sequence of partitions $\Lambda = (\emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n) = \emptyset)$ is *ω-interlaced* if $\lambda(i-1)\, \omega_i\, \lambda(i)$, for $i = 1, \ldots, n$. The *Schur process* of word ω with parameters $Z = (z_1, \ldots, z_n)$ is the measure on the set of ω-interlaced sequences of partitions

$$\Lambda = (\emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n) = \emptyset)$$

given by

$$Prob(\Lambda) \propto \prod_{i=1}^{n} z_i^{||\lambda(i)| - |\lambda(i-1)||}.$$

Remark

For a more general definition, see the original work of Okounkov–Reshetikhin 2003, or Borodin–Rains 2006.

The Schur process is a determinantal point process

Theorem (OR 2003; BR 2006)

$$Prob(\lambda(i_s) \text{ contains a particle at position } k_s, 1 \leq s \leq n) = \det_{1 \leq u,v \leq n} K(i_u, k_u; i_v, k_v)$$

where

$$K(i,k;i',k') = \begin{cases} \left[\frac{z^k}{w^{k'}}\right] \frac{\Phi(z;Z,\omega;i)}{\Phi(w;Z,\omega;i')} \frac{\sqrt{zw}}{z-w}, & i \leq i', \\ -\left[\frac{z^k}{w^{k'}}\right] \frac{\Phi(z;Z,\omega;i')}{\Phi(w;Z,\omega;i)} \frac{\sqrt{zw}}{w-z}, & i > i' \end{cases}$$

with

$$\Phi(z;Z,\omega;i) = \prod_{\substack{j:\ j \leq i,\ \omega_j \in \{\prec, \prec'\} \\ \epsilon_j = \begin{cases} 1, & \omega_j = \prec', \\ -1, & \omega_j = \prec \end{cases}}} (1 + \epsilon_j z_j z)^{\epsilon_j} \prod_{\substack{j:\ j > i,\ \omega_j \in \{\succ, \succ'\} \\ \epsilon_j = \begin{cases} 1, & \omega_j = \succ', \\ -1, & \omega_j = \succ \end{cases}}} \left(1 + \epsilon_j \frac{z_j}{z}\right)^{-\epsilon_j}$$

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Pyramid partitions as Schur processes, pictorially

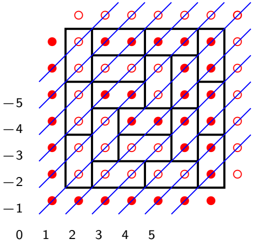


Figure: A pyramid partition of width 5 corresponding to the sequence
$\emptyset \prec (1) \prec' (2) \prec (2,2) \prec' (3,3) \prec (3,3,2) \succ' (2,2,1) \succ (2,1) \succ' (1,1) \succ (1) \succ' \emptyset.$

Pyramid partitions as Schur processes II

Let $n = 2n_0$ be an even integer. A pyramid partition is (bijectively) a sequence of $2n + 1$ partitions

$$\Lambda = (\emptyset = \lambda(-n) \prec \lambda(-n{+}1) \prec' \lambda(-n{+}2) \prec \cdots \prec' \lambda(0) \succ \lambda(1) \succ' \lambda(2) \succ \cdots \succ' \lambda(n) = \emptyset).$$

It is this a Schur process for the word $\omega_{pyr} = (\prec, \prec')^{n_0}(\succ, \succ')^{n_0}$ and parameters $Z = (z_{-n}, \ldots, z_{-1}, z_1, \ldots, z_n)$.

Remark

For volume weighting, $z_{-i} = z_i = q^{i-\frac{1}{2}},\ 1 \leq i \leq n$.

YMP

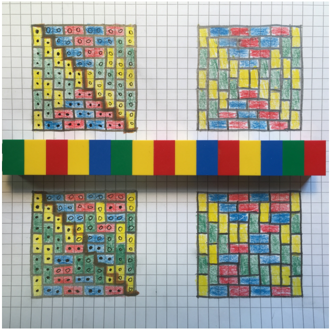


Figure: Pyramid partitions in 2D (LEGO coloring), with partitions as Maya diagrams on left: empty process top and $\cdots \prec' 0 \prec (3) \prec' (4,1,1) \succ (2) \succ' (1) \succ (1) \succ' (1) \succ (1) \succ' 0 \succ \ldots$ process bottom.

A simple word on asymptotics

Everything we'd like to know about asymptotics of large pyramid partitions can be translated into asymptotics of large particle–hole systems associated to the corresponding Schur process.

How to compute the limit shape

Let $t = 2t_0 < n$, $k \in \mathbb{Z} + \frac{1}{2}$. A weak Wick lemma shows that:

Lemma (db–Boutillier–Vuletić 2015)

$$
\begin{aligned}
&\mathit{Prob}(\lambda(-t) \text{ contains a particle at position } k) = \\
&= \left[\frac{z^k}{w^k}\right] \frac{J(z;t_0)}{J(w;t_0)} \frac{\sqrt{zw}}{z-w} \\
&= \int\int \frac{J(z;t_0)}{J(w;t_0)} \frac{1}{z^{k-\frac{1}{2}} w^{-k-\frac{1}{2}}} \frac{1}{z-w} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w}
\end{aligned}
$$

where (with $(u;q)_m = \prod_{i=0}^{m-1}(1-q^i u)$*)*

$$
J(z;t_0) = \frac{(-q^{2t_0+\frac{1}{2}} z; q^2)_{n_0-t_0} (\frac{q^{\frac{1}{2}}}{z}; q^2)_{n_0}}{(q^{2t_0+\frac{3}{2}} z; q^2)_{n_0-t_0} (-\frac{q^{\frac{3}{2}}}{z}; q^2)_{n_0}}.
$$

Asymptotics regime

We let the size of the partition grow with $q \to 1$ as $\epsilon \to 0$ like so:

$$\begin{aligned} q(\epsilon) &= \exp(-\gamma\epsilon), \\ n(\epsilon) &= a/\epsilon, \\ t(\epsilon) &= x/\epsilon, \\ k(\epsilon) &= y/\epsilon. \end{aligned}$$

A few limit formulas

If $q = \exp(-r)$ and $r \to 0+$, we have

$$\log(z;q)_\infty \sim -\frac{Li_2(z)}{r}$$

and furthermore,

$$\log(z;q)_{\frac{A}{r}} \sim \frac{1}{r}(Li_2(e^{-A}z) - Li_2(z))$$

where

$$Li_2(z) = \sum_{n\geq 1} \frac{z^2}{n^2}, \ |z| < 1$$

with analytic continuation given by

$$Li_2(z) = -\int_0^z \frac{\log(1-u)}{u} du, \quad z \in \mathbb{C} \backslash [1, \infty).$$

Asymptotics of the kernel

Lemma (db–Boutillier–Vuletić 2015)

In the limit (x is rescaled t, y is rescaled k, a is rescaled system size n), for $t < 0$

$$Prob(\lambda(t) \text{ contains a particle at position } k) \sim \int\int e^{\frac{1}{\epsilon}(S(z;x,y)-S(w;x,y))}\frac{d\mathbb{T}}{z-w}$$

where

$$S(z;x,y) = \frac{1}{2\gamma}\left(Li_2(-\frac{z}{A}) - Li_2(-Xz) + Li_2(\frac{1}{Az}) - Li_2(\frac{1}{z}) + \right.$$
$$\left. + Li_2(Xz) - Li_2(\frac{z}{A}) + Li_2(-\frac{1}{z}) - Li_2(-\frac{-1}{Az})\right) - y\log z$$

and $X = \exp(\gamma x), A = \exp(\gamma a)$.

The arctic curve

To compute the arctic curve, one solves for (x, y) (or $X = \exp(\gamma x)$, $Y = \exp(2\gamma y)$) corresponding to double critial points of $S(z; x, y)$. That is,

Theorem (db–Boutillier–Vuletić 2015)

The arctic curve is the locus (x, y) *satisfying:*

$$f(z; X) = Y,$$
$$f'(z; X) = 0$$

where $f(z; X) = \frac{(z+1)(z-A)(z-1/A)(z+1/X)}{(z-1)(z+A)(z+1/A)(z-1/X)}$.

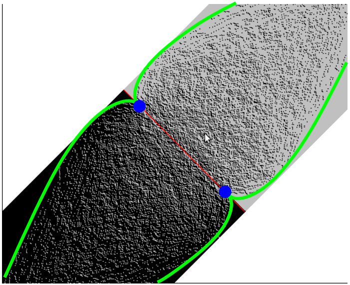
Remark

Alternatively, it can be seen as given by the *algebraic* equation

$$\Delta\left[(z+1)(z-A)(z-1/A)(z+1/X) - Y(z-1)(z+A)(z+1/A)(z-1/X)\right] = 0$$

where Δ represents taking the discriminant.

The arctic curve, pictorially



Notice the cusps (which correspond to the *double* critical points of S at $z = 0, \infty$).

Arctic curve in the infinite regime

What happens when $a_0 \to \infty$, or equivalently, $A \to 0$?

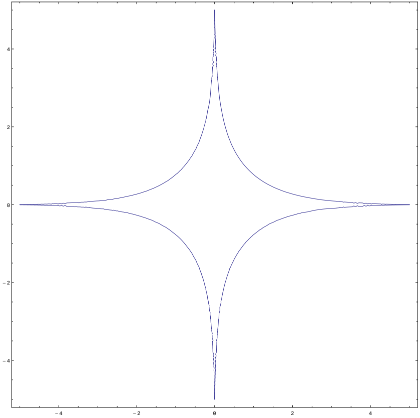
The cusps move to ∞ and the arctic curve becomes

$$(1+Z+W-ZW)(1+Z-W+ZW)(1-Z+W+ZW)(1-Z-W-ZW)=0$$

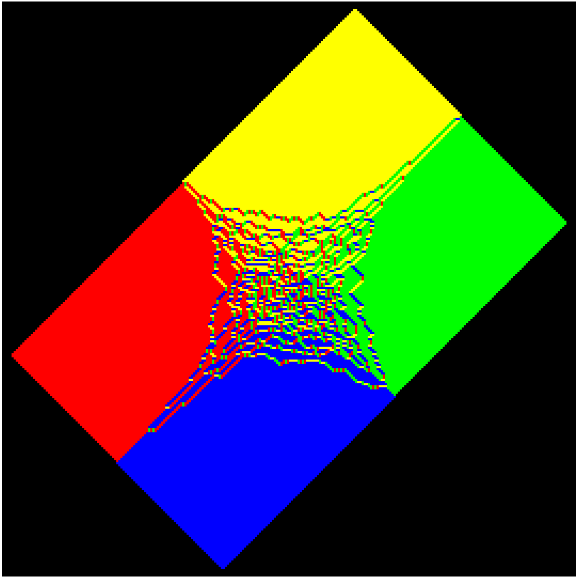
where $(Z,W)=(\sqrt{X},\sqrt{Y})$ which is the boundary of the amoeba of the (square lattice determined) polynomial

$$P(Z,W)=1+Z+W-ZW.$$

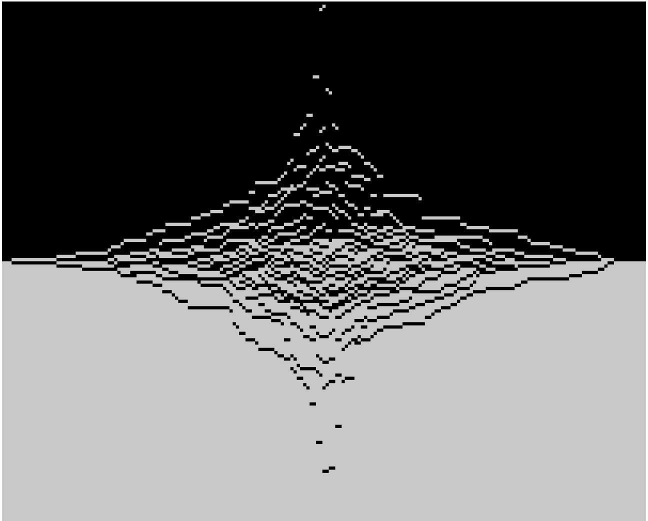
Arctic curve in the infinite regime, pictorially



A large sample in the infinite regime



A large sample in the infinite regime, particles and rotated



A word on what happens on the arctic curve: fluctuations

Everywhere but at the cusps and tangency points, fluctuations are of Airy type (cf., for example, Okounkov–Reshetikhin 2006).

At the turning (tangency) points, one has the GUE minors processes.

At the cusps, one would conjecture and expect Pearcey process fluctuations. Alas, in the absence of a triple critical point and due to additional constraints, one gets the cusp Airy process of Okounkov–Reshetikhin (2006) and Duse–Johansson–Metcalfe (2015) with kernel:

$$K(x, k; x', k') = \int \int \frac{w^{k'}}{z^k} \frac{e^{\frac{z^3}{3} + xz}}{e^{\frac{w^3}{3} + x' w}} \frac{1}{z - w} d\mathbb{T}$$

where x is the continuous time direction, and k the *discrete* space direction.

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How to sample pyramid partitions – RSK

$$\begin{pmatrix} G(q^4) & B(q^3) & G(q^2) & B(q) \\ B(q^5) & G(q^4) & B(q^3) & G(q^2) \\ G(q^6) & B(q^5) & G(q^4) & B(q^3) \\ B(q^7) & G(q^6) & B(q^5) & G(q^4) \end{pmatrix} \mapsto (P, Q)$$

where P, Q are supersymmetric semi-standard Young tableau of the same shape on $1 < 1' < 2 < 2' < \dots$ (in P non-primed integers come in horizontal strips, primed come in vertical strips; in Q the other way around). (P, Q) is the required pyramid partition (view both as sequences of partitions).

$B(x)$ and $G(x)$ stand for Bernoulli on $\{0, 1\}$ and geometric (on $\mathbb{N}$) random variables where $Prob(1)$ (resp. $Prob(k)$) is proportional to x (resp. x^k).

This bijection is what some people call supersymmetric RSK (PP 1996, K 2006, BCC 2014, db–BBCCV 2015).

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Other stuff: “skew pyramid partitions” (Cusp Airy in the middle)

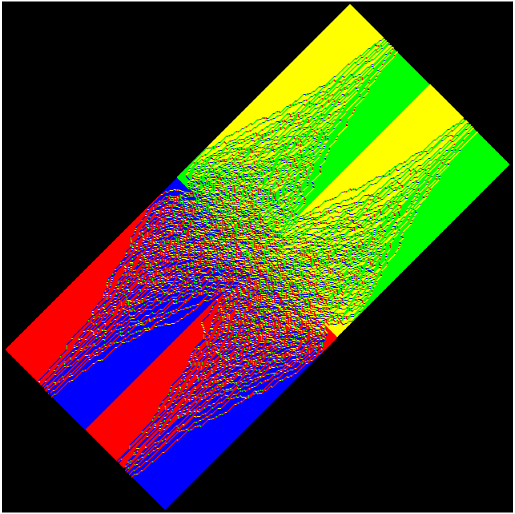


Figure: Skew pyramid partitions of symmetric word $(\prec, \prec')^{100}(\succ, \succ')^{100}(\prec, \prec')^{100}(\succ, \succ')^{100}$ (compare skew plane partitions OR 2006).

Other stuff: "skew pyramid partitions", particles

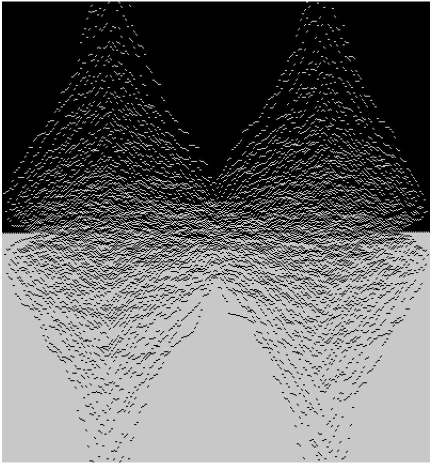


Figure: Skew pyramid partitions of symmetric word $(\prec, \prec')^{100}(\succ, \succ')^{100}(\prec, \prec')^{100}(\succ, \succ')^{100}$.

Other stuff: REALLY "skew pyramid partitions"

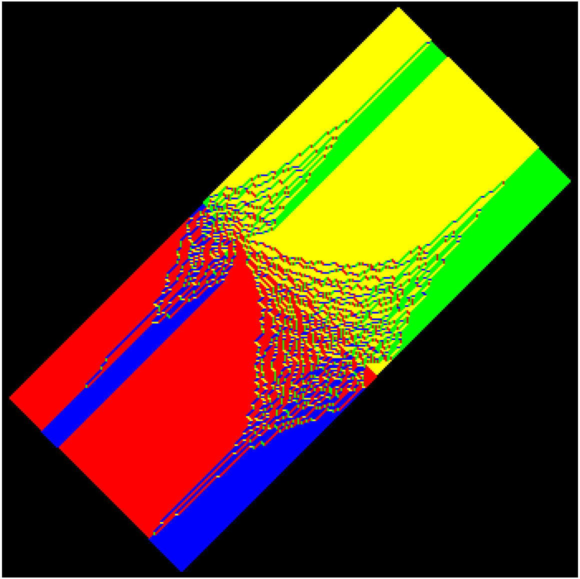


Figure: Skew pyramid partitions of nonsymmetric word $(\prec\prec')^{40}(\succ\succ')^{20}(\prec\prec')^{110}(\succ\succ')^{40}$.

Other stuff: REALLY “skew pyramid partitions”, particles

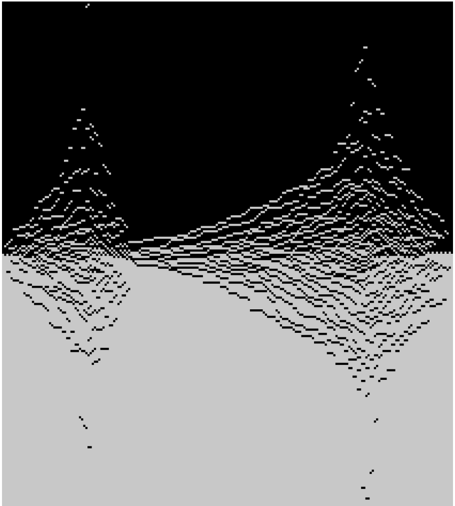


Figure: Skew pyramid partitions of nonsymmetric word $(\prec\prec')^{40}(\succ\succ')^{20}(\prec\prec')^{110}(\succ\succ')^{40}$.

Other stuff: steep tilings, Ramassamy words

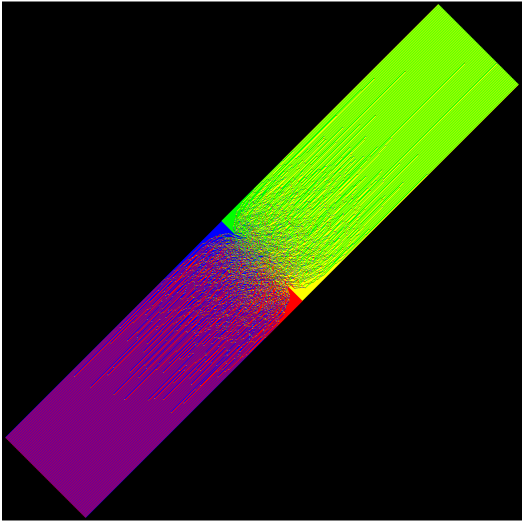


Figure: Skew pyramid partitions of word $(\prec\,\succ'\,\succ\,\prec')^{100}(\prec\,\succ')$, a domino equivalent of BMRT 2012.

Other stuff: steep tilings, Ramassamy words

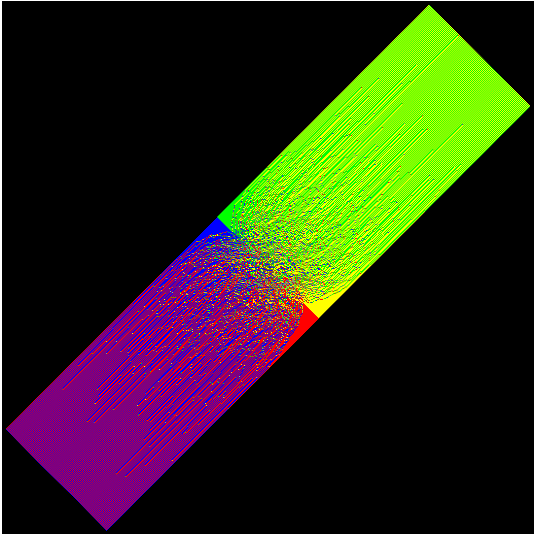
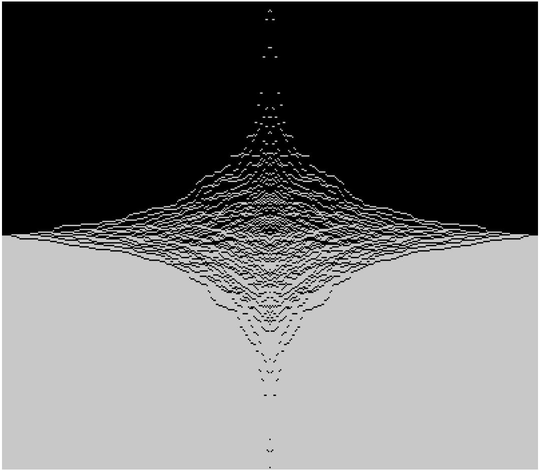
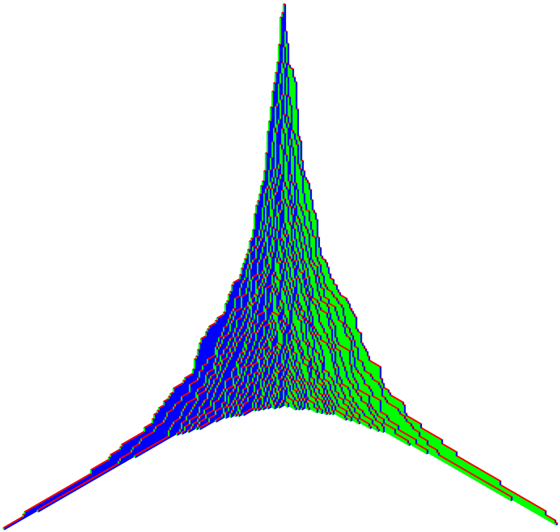


Figure: Skew pyramid partitions of word $(\prec\,\prec'\,\succ\,\succ')^{100}$ (pairs of supersymmetric oscillating tableaux). Same behavior as above (universality).

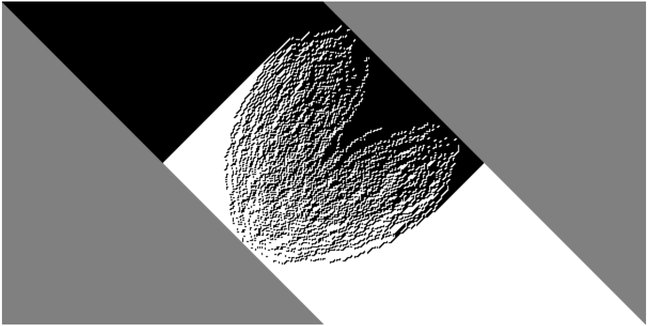
Other stuff: symmetric "pyramid partitions"



Symmetric “pyramid partitions” as plane overpartitions



This limit shape seems to be the same that Vuletić 2009 analyzed in the context of strict plane partitions and Pfaffian processes.



Thank you!