Function spaces on quantum tori

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Quantum tori

Let $\theta \in \mathbb{R}$. Let U and V be two unitary operators on a Hilbert space H satisfying the following commutation relation:

$$UV = e^{2\pi \mathrm{i}\theta} VU.$$

Example. $H = L_2(\mathbb{T})$ with $\mathbb{T}$ the unit circle; U and V are given:

$$Uf(z) = zf(z) \quad \text{and} \quad Vf(z) = f(e^{-2\pi \mathrm{i}\theta} z), \quad f \in L_2(\mathbb{T}),\ z \in \mathbb{T}.$$

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Let $\mathcal{A}_\theta$ be the universal C*-subalgebra generated by U and V. This is the quantum (or noncommutative) 2-torus associated to θ. If θ is irrational, $\mathcal{A}_\theta$ is an irrational rotation C*-algebra. These algebras are fundamental examples in operator algebra theory and noncommutative geometry.

Objective: The algebraic and geometric structures of these objects have been well understood. Our objective is to study their analytic aspect in view to applications to noncom geometry and PDE.

More generally, let $d \geq 2$ and $\theta = (\theta_{kj})$ be a $d \times d$ real skew-symmetric matrix, i.e., $\theta^{\mathrm{t}} = -\theta$. Let $U_1, \cdots, U_d$ be d unitary operators on H satisfying

$$U_k U_j = e^{2\pi \mathrm{i} \theta_{kj}} U_j U_k, \ j, k = 1, \cdots, d.$$

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Notation. Elements of $\mathbb{Z}^d$ are denoted by $m = (m_1, \cdots, m_d)$. We denote the usual d-torus by

$$\mathbb{T}^d = \{(z_1, \ldots, z_d) \ : \ |z_j| = 1, z_j \in \mathbb{C}\},$$

equipped with normalized Haar measure. For $m \in \mathbb{Z}^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{T}^d$ let

$$z^m = z_1^{m_1} \cdots z_d^{m_d} \quad \text{and} \quad U^m = U_1^{m_1} \cdots U_d^{m_d}.$$

Trace and noncommutative L_p-spaces

A polynomial in $U = (U_1, \cdots, U_d)$ is a finite sum

$$x = \sum_{m \in \mathbb{Z}^d} \alpha_m U^m \in \mathcal{A}_\theta \quad \text{with} \quad \alpha_m \in \mathbb{C}.$$

Let $\mathcal{P}_\theta$ denote the subalgebra of polynomials. The functional τ on $\mathcal{P}_\theta$ defined by $x \mapsto \alpha_0$ extends to a faithful tracial state on $\mathcal{A}_\theta$.

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Let $\mathbb{T}^d_\theta$ be the w*-closure of $\mathcal{A}_\theta$ in the GNS representation of τ. Then τ becomes a normal faithful tracial state on $\mathbb{T}^d_\theta$. Thus $(\mathbb{T}^d_\theta, \tau)$ is a tracial noncommutative probability space.

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Noncommutative L_p. For $1 \le p < \infty$ and $x \in \mathbb{T}_\theta^d$ let

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}} \quad \text{with} \quad |x| = (x^* x)^{\frac{1}{2}}.$$

This is a norm on $\mathbb{T}_\theta^d$. The corresponding completion is denoted by $L_p(\mathbb{T}_\theta^d)$, the noncommutative L_p-space associated to $(\mathbb{T}_\theta^d, \tau)$.

Fourier coefficients

The trace τ extends to a contractive functional on $L_1(\mathbb{T}_\theta^d)$. Thus for any $x \in L_p(\mathbb{T}_\theta^d)$ we can define

$$\hat{x}(m) = \tau((U^m)^* x) = \alpha_m, \quad m \in \mathbb{Z}^d.$$

These are the Fourier coefficients of x. Like in the classical case we write formally:

$$x \sim \sum_{m \in \mathbb{Z}^d} \hat{x}(m) U^m.$$

x is uniquely determined by its Fourier series.

In a previous work joint with Zeqian Chen and Zhi Yin, we have studied Fourier analysis on $\mathbb{T}_\theta^d$. We now consider function spaces. In this talk, we will discuss only Sobolev and Besov spaces.

Distributions

Let

$$\mathcal{S}(\mathbb{T}^d_\theta) = \Big\{ \sum_m \alpha_m U^m : \alpha_m \text{ rapidly decreasing} \Big\}.$$

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Define the following partial derivations on $\mathcal{S}(\mathbb{T}_\theta^d)$:

$$\partial_j(U_j) = 2\pi \mathrm{i} U_j, \quad \text{and} \quad \partial_k(U_j) = 0, \ \text{for } k \neq j.$$

More generally, for $m = (m_1, \cdots, m_d) \in \mathbb{N}_0^d$, let

$$D^m x = \prod_{1 \le j \le d} \partial_j^{m_j} x; \quad \text{order of } D^m \text{ is } |m|_1 = m_1 + \cdots + m_d.$$

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$\mathcal{S}(\mathbb{T}_\theta^d)$ carries a natural locally convex space topology. The resulting dual space $\mathcal{S}'(\mathbb{T}_\theta^d)$ is the space of distributions on $\mathbb{T}_\theta^d$. As in the classical case, the derivations and Fourier transform extend to distributions too.

Sobolev spaces

Definition. Let $1 \leq p \leq \infty$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$.

- The Sobolev space $W_p^k(\mathbb{T}_\theta^d)$ is

$$W_p^k(\mathbb{T}_\theta^d) = \Big\{ x \in \mathcal{S}'(\mathbb{T}_\theta^d) : D^m x \in L_p(\mathbb{T}_\theta^d),\ \forall m \in \mathbb{N}_0^d, |m|_1 \leq k \Big\}$$

  equipped with the norm

$$\|x\|_{W_p^k} = \Big(\sum_{0 \leq |m|_1 \leq k} \|D^m x\|_p^p \Big)^{\frac{1}{p}}.$$

- The potential Sobolev space $H_p^\alpha(\mathbb{T}_\theta^d)$ is

$$H_p^\alpha(\mathbb{T}_\theta^d) = \Big\{ x \in \mathcal{S}'(\mathbb{T}_\theta^d) : \|(I + \Delta)^{\frac{\alpha}{2}} x\|_p < \infty \Big\}$$

  equipped with the norm

$$\|x\|_{H_p^\alpha} = \|(I + \Delta)^{\frac{\alpha}{2}} x\|_p.$$

  Here $\Delta = -\sum_{1 \leq j \leq d} \partial_j^2$, the Laplacian.

Besov spaces

Let $\varphi \in C^\infty(\mathbb{R}^d)$ be a nonnegative function such that

$$\mathrm{supp}\varphi = \{\xi \in \mathbb{R}^d : 2^{-1} \le |\xi| \le 2\}, \quad \sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1,\ \xi \neq 0.$$

Define φ_k by: $\widehat{\varphi}_k(\xi) = \varphi(2^{-k}\xi)$, $k \ge 0$. For $x \in \mathcal{S}'(\mathbb{T}_\theta^d)$ define

$$\varphi_k * x = \sum_{m\in\mathbb{Z}^d} \widehat{\varphi}_k(m)\widehat{x}(m)U^m.$$

Definition. Let $1 \le p, q \le \infty, \alpha \in \mathbb{R}$. The Besov space on $\mathbb{T}_\theta^d$ is defined by

$$B_{p,q}^\alpha(\mathbb{T}_\theta^d) = \left\{x \in \mathcal{S}'(\mathbb{T}_\theta^d) : \|x\|_{B_{p,q}^\alpha} < \infty\right\},$$

where

$$\|x\|_{B_{p,q}^\alpha} = \Big(|\widehat{x}(0)|^q + \sum_{k\ge 0} 2^{qk\alpha}\|\varphi_k * x\|_p^q\Big)^{\frac{1}{q}}.$$

Properties

The previous spaces share many properties with their classical counterparts. Below is a short list of examples:

- $W_p^k(\mathbb{T}_\theta^d) = H_p^k(\mathbb{T}_\theta^d)$ for $1 < p < \infty$.
- Poincaré inequality: $\|x - \widehat{x}(0)\|_p \lesssim \|x\|_{W_p^1}$, $1 \leq p \leq \infty$.
- $B^\alpha_{p,\min(2,p)}(\mathbb{T}_\theta^d) \subset H_p^\alpha(\mathbb{T}_\theta^d) \subset B^\alpha_{p,\max(2,p)}(\mathbb{T}_\theta^d)$.
- Embedding:
  - $W_p^k(\mathbb{T}_\theta^d) \subset W_{p_1}^{k_1}(\mathbb{T}_\theta^d)$ for $k, k_1 \in \mathbb{N}, 1 < p \leq p_1 < \infty, k - \frac{d}{p} = k_1 - \frac{d}{p_1}$;
  - $B_{p,q}^\alpha(\mathbb{T}_\theta^d) \subset B_{p_1,q_1}^{\alpha_1}(\mathbb{T}_\theta^d)$ for $\alpha, \alpha_1 \in \mathbb{R}, 1 \leq p \leq p_1 < \infty, 1 \leq q \leq q_1 < \infty, \alpha - \frac{d}{p} = \alpha_1 - \frac{d}{p_1}$.

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- $B^\alpha_{p,\min(2,p)}(\mathbb{T}_\theta^d) \subset H_p^\alpha(\mathbb{T}_\theta^d) \subset B^\alpha_{p,\max(2,p)}(\mathbb{T}_\theta^d)$.
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  - $B^\alpha_{p,q}(\mathbb{T}_\theta^d) \subset B^{\alpha_1}_{p_1,q_1}(\mathbb{T}_\theta^d)$ for $\alpha, \alpha_1 \in \mathbb{R}, 1 \leq p \leq p_1 < \infty, 1 \leq q \leq q_1 < \infty, \alpha - \frac{d}{p} = \alpha_1 - \frac{d}{p_1}$.

A word on proof. Our proof of the embedding inequalities is based on Varopolous' famous semigroup approach which has been transferred to the noncommutative case by Marius Junge and Tao Mei in their study of Riesz transform on quantum Markovian semigroups.

Characterization by Poisson semigroup

Given a distribution x on $\mathbb{T}_\theta^d$ and $k \in \mathbb{Z}$, let

$$\mathbb{P}_r(x) = \sum_{m\in\mathbb{Z}^d} \widehat{x}(m) r^{|m|} U^m \,,$$ circular Poisson integral of x,

and

$$\mathcal{J}_r^k \, \mathbb{P}_r(x) = \sum_{m\in\mathbb{Z}^d} C_{m,k} \widehat{x}(m) r^{|m|-k} U^m, \quad 0 \le r < 1\,,$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^d$ and

$$C_{m,k} = \begin{cases} |m| \cdots (|m| - k + 1) & \text{if } \ k \ge 0, \\ \dfrac{1}{(|m|+1) \cdots (|m|-k)} & \text{if } \ k < 0. \end{cases}$$

Remark. $\mathcal{J}_r^k$ is the kth derivation operator relative to r if $k \ge 0$, and the $(-k)$th integration operator if $k < 0$.

Theorem. Let $1 \le p, q \le \infty$ and $\alpha \in \mathbb{R}, k \in \mathbb{Z}$ with $k > \alpha$. Then $x \in B^{\alpha}_{p,q}(\mathbb{T}^d_\theta)$ iff

$$\Big(\max_{|m|<k} |\widehat{x}(m)|^q + \int_0^1 (1-r)^{(k-\alpha)q} \big\|\mathcal{J}_r^k\, \mathbb{P}_r(x_k)\big\|_p^q \,\frac{dr}{1-r}\Big)^{\frac{1}{q}} < \infty.$$

where $x_k = x - \sum_{|m|<k} \widehat{x}(m) U^m$.

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where $x_k = x - \sum_{|m|<k} \widehat{x}(m) U^m$.

Remark. 1) The use of the integration operator (corresponding to negative k) in the above statement is completely new even in the case $\theta = 0$ (the commutative case).

2) A similar result holds for Triebel-Lizorkin spaces too. For the latter spaces, another improvement of our characterization over the classical one lies on the assumption on k:

- $k > d + \max(\alpha, 0)$ in the classical case;
- $k > \alpha$ in our case.

Characterization by differences

Difference operator: $\Delta_u x = \pi_z(x) - x$ associated to $u \in \mathbb{R}^d$, where $z = (e^{2\pi \mathrm{i} u_1}, \cdots, e^{2\pi \mathrm{i} u_d})$ and π_z is the automorphism of $\mathbb{T}_\theta^d$ determined by $U_j \mapsto z_j U_j$ for $1 \leq j \leq d$. For $k \in \mathbb{N}$, Δ_u^k is the kth difference operator associated to u.

kth Modulus of L_p-smoothness: $\omega_p^k(x, \varepsilon) = \sup_{0<|u|\leq\varepsilon} \left\| \Delta_u^k x \right\|_p$ for $x \in L_p((\mathbb{T}_\theta^d)$.

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Theorem. Let $1 \le p, q \le \infty, \alpha \in \mathbb{R}, k \in \mathbb{N}, 0 < \alpha < k$. Then

$$x \in B_{p,q}^\alpha(\mathbb{T}_\theta^d) \Leftrightarrow \|x\|_{B_{p,q}^{\alpha,\omega}} = \Big(\int_0^1 \varepsilon^{-\alpha q} \omega_p^k(x, \varepsilon)^q \frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{q}} < \infty .$$

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Corollary. For $\alpha > 0$, $B_{\infty,\infty}^\alpha(\mathbb{T}_\theta^d)$ is the quantum analogue of the classical Hölder class.
The case $0 < \alpha < 1$ was already studied by Weaver (1998).

A quantum analogue of Bourgain-Brézis-Mironescu

Another consequence of the previous characterization by differences is the following:

Theorem. Let $1 \leq p \leq \infty$, $1 \leq q < \infty, k \in \mathbb{N}, 0 < \alpha < k$. Then

$$\lim_{\alpha \to k} (k - \alpha)^{\frac{1}{q}} \|x\|_{B^{\alpha,\omega}_{p,q}} \approx q^{-\frac{1}{q}} \sum_{m \in \mathbb{N}_0^d,\, |m|_1 = k} \|D^m x\|_p \,,$$

$$\lim_{\alpha \to 0} \alpha^{\frac{1}{q}} \|x\|_{B^{\alpha,\omega}_{p,q}} \approx q^{-\frac{1}{q}} \|x\|_p \,.$$

Remark. In the commutative case (and for $k = 1$), the first equivalence is due to Bourgain-Brézis-Mironescu (2002), and the second due to Maz'ya-Shaposhnikova (2002).

Interpolation

Theorem. The K-functional of the couple $(L_p(\mathbb{T}^d_\theta), W^k_p(\mathbb{T}^d_\theta))$ is given by:

$$K(x, \varepsilon^k; L_p(\mathbb{T}^d_\theta), W^k_p(\mathbb{T}^d_\theta)) \approx \varepsilon^k |\widehat{x}(0)| + \omega^k_p(x, \varepsilon), \quad 0 < \varepsilon \leq 1.$$

Remark. This is the extension to $\mathbb{T}^d_\theta$ of Johnen-Scherer's classical theorem.

Corollary. Suppose $1 \leq p, q \leq \infty$ and $k \in \mathbb{N}$. Then

$$\big(L_p(\mathbb{T}^d_\theta), W^k_p(\mathbb{T}^d_\theta)\big)_{\eta,q} = B^{k\eta}_{p,q}(\mathbb{T}^d_\theta),\ 0 < \eta < 1.$$

Remark. Several problems remain open for the interpolation of quantum Sobolev spaces.