Operator-valued local Hardy spaces

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Background

- **Motivation**: Pseudo-differential operators are not bounded on Hardy space $H_1(\mathbb{R}^d)$. So we want to introduce a larger space on which pseudo-differential operators are bounded.

Let Γ be the truncated cone $\left\{(t,\varepsilon)\in\mathbb{R}_+^{d+1} : |t|<\varepsilon<1\right\}$, define

$$
\begin{aligned}
s(f)(s) &= \left(\int_\Gamma \left|\nabla P_\varepsilon * f(s+t,\varepsilon)\right|^2 \frac{dtd\varepsilon}{\varepsilon^{d-1}}\right)^{\frac{1}{2}} \\
g(f)(s) &= \left(\int_0^1 \varepsilon\left|\nabla P_\varepsilon * f(s,\varepsilon)\right|^2 d\varepsilon\right)^{\frac{1}{2}},
\end{aligned}
$$

where P_ε is the Poisson kernel on the strip $\{(s,\varepsilon) : s\in\mathbb{R}^d, \varepsilon\in(0,1)\}$.

- **D. Goldberg (1979)**: If $p>(n-1)/n$ then $f\in h_p$ **if only if** $s(f)\in L_p$.
  And if $\psi\in\mathcal{S}$ with $\int\psi\neq 0$, then the L_p-norms of the following functions are equivalent: $s(f)$, $g(f)$, $\sup_{(t,\varepsilon)\in\Gamma(s)}|\psi_\varepsilon * f(t)|$ and $\sup_{0<\varepsilon<1}|\psi_\varepsilon * f(s)|$.

Move to the noncommutative setting:

- Motivated by Goldberg, we want to define the operator-valued local Hardy spaces, and also get the boundedness of Pseudo-differential operators on them.
- Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ.

  For $1 \leq p \leq \infty$, let $L_p(\mathcal{M})$ be the noncommutative L_p-space associated to $(\mathcal{M}, \tau)$. The norm of $L_p(\mathcal{M})$, $1 \leq p < \infty$ is given by

  $$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} \text{ with } |x| = (x^*x)^{1/2}.$$

  Set $L_\infty(\mathcal{M}) = \mathcal{M}$.

Lusin area square function and Littlewood-Paley g-function

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Operator-valued Hardy spaces

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$$S^c(f)(s) = \left(\int_\Gamma \left|\frac{\partial}{\partial\varepsilon} P_\varepsilon(f)(s+t)\right|^2 \frac{dtd\varepsilon}{\varepsilon^{d-1}}\right)^{\frac{1}{2}}$$

$$S^r(f)(s) = \left(\int_\Gamma \left|\frac{\partial}{\partial\varepsilon} P_\varepsilon(f^*)(s+t)\right|^2 \frac{dtd\varepsilon}{\varepsilon^{d-1}}\right)^{\frac{1}{2}}$$

$$\left(\Gamma = \left\{(t,\varepsilon) \in \mathbb{R}_+^{d+1} : |t| < \varepsilon\right\}\right)$$

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Operator-valued local Hardy spaces

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$$G^c(f)(s) = \left(\int_0^\infty \varepsilon \left| \frac{\partial}{\partial \varepsilon} P_\varepsilon(f)(s) \right|^2 d\varepsilon \right)^{\frac{1}{2}}$$

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Operator-valued local Hardy spaces

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$$g^c(f)(s) = \left(\int_0^1 \varepsilon \left| \frac{\partial}{\partial \varepsilon} P_\varepsilon(f)(s) \right|^2 d\varepsilon \right)^{\frac{1}{2}}$$

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where $P_\varepsilon(f)$ denotes $P_\varepsilon * f$.

Operator-valued local Hardy spaces

For any $f \in L_1\left(\mathcal{M}; L_c^2\left(\mathbb{R}^d, \frac{dt}{1+|t|^{d+1}}\right)\right) + L_\infty\left(\mathcal{M}; L_c^2\left(\mathbb{R}^d, \frac{dt}{1+|t|^{d+1}}\right)\right)$, let $\varphi \in \mathcal{S}$ with $\int \varphi = 1 (\widehat{\varphi}(0) = 1)$, and $1 \leq p < \infty$, we define

$$\|f\|_{h_p^c(\mathbb{R}^d;\mathcal{M})} = \|s^c(f)\|_{L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})} + \|\varphi * f\|_{L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})},$$

the **column local Hardy spaces** are defined by

$$h_p^c\left(\mathbb{R}^d;\mathcal{M}\right) = \left\{f : \|f\|_{h_p^c} < \infty\right\},$$

and the **row local Hardy spaces** are defined by

$$h_p^r\left(\mathbb{R}^d;\mathcal{M}\right) = \left\{f : \|f^*\|_{h_p^c} < \infty\right\},$$

equipped with the norm $\|f\|_{h_p^r} = \|f^*\|_{h_p^c}$.

$$\begin{aligned} h_p\left(\mathbb{R}^d;\mathcal{M}\right) &= h_p^c\left(\mathbb{R}^d;\mathcal{M}\right) + h_p^r\left(\mathbb{R}^d;\mathcal{M}\right) \text{ for } 1 \leq p \leq 2, \\ h_p\left(\mathbb{R}^d;\mathcal{M}\right) &= h_p^c\left(\mathbb{R}^d;\mathcal{M}\right) \cap h_p^r\left(\mathbb{R}^d;\mathcal{M}\right) \text{ for } 2 < p < \infty. \end{aligned}$$

Lemma

Let $\varphi \in \mathcal{S}$, $\int \varphi = 1$, *then, for* $1 \leq p < \infty$,

$$\|f - \varphi * f\|_{h_p^c(\mathbb{R}^d;\mathcal{M})} \approx \|f - \varphi * f\|_{H_p^c(\mathbb{R}^d;\mathcal{M})}.$$

The Fourier transform of $f - \varphi * f$ vanishes in a neighborhood of the origin.

the case when $1 < p < \infty$

Theorem

For $1 < p < \infty$, we have the equivalence

$$\begin{aligned} h_p^c\left(\mathbb{R}^d;\mathcal{M}\right) &\approx H_p^c\left(\mathbb{R}^d;\mathcal{M}\right), \\ h_p^r\left(\mathbb{R}^d;\mathcal{M}\right) &\approx H_p^r\left(\mathbb{R}^d;\mathcal{M}\right). \end{aligned}$$

Theorem (Mei(2005))

Let $1 < p < \infty$, then with equivalent norms

$$H_p\left(\mathbb{R}^d;\mathcal{M}\right) = L_p\left(L_\infty\left(\mathbb{R}^d\right)\overline{\otimes}\mathcal{M}\right).$$

Corollary

Let $1 < p < \infty$, then with equivalent norms

$$h_p\left(\mathbb{R}^d;\mathcal{M}\right) = L_p\left(L_\infty\left(\mathbb{R}^d\right)\overline{\otimes}\mathcal{M}\right).$$

Operator-valued *bmo* spaces

Let $\varphi \in L^\infty\left(\mathcal{M}; L_c^2\left(\mathbb{R}^d, \frac{dt}{1+|t|^{d+1}}\right)\right)$, the mean value of φ over cube Q is denoted by $\varphi_Q := \frac{1}{|Q|}\int_Q \varphi(s)\, ds$. Set

$$\begin{aligned}\|\varphi\|_{bmo^c} \quad = \quad & \max\left\{\sup_{|Q|<1}\left\|\left(\frac{1}{|Q|}\int_Q |\varphi - \varphi_Q|^2\, d\mu\right)^{\frac{1}{2}}\right\|_{\mathcal{M}},\right. \\ & \left.\sup_{|Q|>1}\left\|\left(\frac{1}{|Q|}\int_Q |\varphi|^2\, d\mu\right)^{\frac{1}{2}}\right\|_{\mathcal{M}}\right\}.\end{aligned}$$

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$$\begin{aligned} \|\varphi\|_{bmo^c} &= \max\left\{ \sup_{|Q|<1} \left\| \left(\frac{1}{|Q|} \int_Q |\varphi - \varphi_Q|^2 \, d\mu \right)^{\frac{1}{2}} \right\|_{\mathcal{M}}, \right. \\ &\left. \sup_{|Q|>1} \left\| \left(\frac{1}{|Q|} \int_Q |\varphi|^2 \, d\mu \right)^{\frac{1}{2}} \right\|_{\mathcal{M}} \right\}. \end{aligned}$$

Set $bmo^c\left(\mathbb{R}^d; \mathcal{M}\right)$ be the space of all $\varphi \in L^\infty\left(\mathcal{M}; L_c^2\left(\mathbb{R}^d, \frac{dt}{1+|t|^{d+1}}\right)\right)$ such that $\|\varphi\|_{bmo^c} < \infty$. $bmo^r\left(\mathbb{R}^d; \mathcal{M}\right)$ be the space of all $\varphi \in L^\infty\left(\mathcal{M}; L_c^2\left(\mathbb{R}^d, \frac{dt}{1+|t|^{d+1}}\right)\right)$ such that $\|\varphi^*\|_{bmo^c} < \infty$.

$$bmo\left(\mathbb{R}^d; \mathcal{M}\right) = bmo^c\left(\mathbb{R}^d; \mathcal{M}\right) \cap bmo^r\left(\mathbb{R}^d; \mathcal{M}\right).$$

The duality between h_1 and bmo

It is easy to see from the definitions of $h_1^c\left(\mathbb{R}^d;\mathcal{M}\right)$ and $bmo^c\left(\mathbb{R}^d;\mathcal{M}\right)$, that

$$\begin{aligned} H_1^c\left(\mathbb{R}^d;\mathcal{M}\right) &\subset h_1^c\left(\mathbb{R}^d;\mathcal{M}\right), \\ BMO^c\left(\mathbb{R}^d;\mathcal{M}\right) &\supset bmo^c\left(\mathbb{R}^d;\mathcal{M}\right). \end{aligned}$$

And by following the steps in Mei's paper, similarly, we can also prove the duality in the local case that

$$\left(h_1^c\left(\mathbb{R}^d;\mathcal{M}\right)\right)^* = bmo^c\left(\mathbb{R}^d;\mathcal{M}\right).$$

Replacing the poisson kernel

Let $\Phi \in \mathcal{S}$ with $\int \Phi(s)\, ds = 0$. Assume that Φ is nondegenerate in the following sense:

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \exists 0 < \varepsilon < 1 \text{ such that } \hat{\Phi}(\varepsilon\xi) \neq 0.$$

For any $f \in L_1\left(\mathcal{M}; L_c^2\left(\mathbb{R}^d, \frac{dt}{1+|t|^{d+1}}\right)\right) + L_\infty\left(\mathcal{M}; L_c^2\left(\mathbb{R}^d, \frac{dt}{1+|t|^{d+1}}\right)\right)$, we can define the radial and conic square function of f associated to Φ as follows

$$\begin{aligned} g_\Phi^c(f)(s) &= \left(\int_0^1 |\Phi_\varepsilon * f(s)|^2 \frac{d\varepsilon}{\varepsilon}\right)^{\frac{1}{2}}, \\ s_\Phi^c(f)(s) &= \left(\int_\Gamma |\Phi_\varepsilon * f(s+t)|^2 \frac{dt d\varepsilon}{\varepsilon^{d+1}}\right)^{\frac{1}{2}}, s \in \mathbb{R}^d, \end{aligned}$$

where $\Gamma = \left\{(t,\varepsilon) \in \mathbb{R}_+^{d+1} : |t| < \varepsilon < 1\right\}$

If we take $\Phi_\varepsilon = 2\pi I\left(P_\varepsilon^1 - P_\varepsilon^0\right)$, where I is Riesz potential, the square functions defined here coincide with the previous definitions with respect to the poisson kernels on the strip.

Theorem (Xu, Xiao, X. 2015)

Let $1 \leq p < \infty$ *, we have*

$$\left\|G_{\Phi}^{c}(f)(s)\right\|_{L_p} \approx \left\|S_{\Phi}^{c}(f)(s)\right\|_{L_p} \approx \|f\|_{H_p^c}.$$

The key is to view $\Phi_{\cdot}(s)(\Phi_{\cdot}(s)$ being the function $\varepsilon \mapsto \Phi_{\varepsilon}(s))$ as a H-valued kernel of Calderón-Zygmund operator, with $H = L_2((0,\infty), \frac{d\varepsilon}{\varepsilon})$ and $H = L_2(\Gamma, \frac{dtd\varepsilon}{\varepsilon^{d+1}})$.

Lemma

Let $k:\mathbb{R}^d\to H$ *be a Hilbert-valued kernel, assume that*

a) $\sup_{\xi\in\mathbb{R}^d}\left\|\widehat{k}(\xi)\right\|_H<\infty$;

b) $\|k(s-t)-k(s)\|_H\lesssim\frac{|t|}{|s-t|^{d+1}},\forall\,|s|>2\,|t|>0.$

Then the operator $k^c: f\mapsto(k\otimes 1_{\mathcal{M}})*f$ *is bounded*

i) from $bmo_0^{\alpha}(\mathbb{R}^d,\mathcal{M})$ *to* $bmo^{\alpha}(\mathbb{R}^d,B(H)\overline{\bigotimes}\mathcal{M})$ *, where* $\alpha=c$ *,* $\alpha=r$ *or* α *is void;*

ii) and from $h_1^c(\mathbb{R}^d,\mathcal{M})$ *to* $h_1^c\left(\mathbb{R}^d,B(H)\overline{\bigotimes}\mathcal{M}\right)$.

Corollary (X.)

Let $1\leq p<\infty$ *, we have*

$$\|g_\Phi^c(f)(s)\|_{L_p}+\|\varphi*f\|_{L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})}\approx\|s_\Phi^c(f)(s)\|_{L_p}+\|\varphi*f\|_{L_p}\approx\|f\|_{h_p^c}.$$

Relation between h_1 and H_1

Theorem

Let $\varphi \in \mathcal{S}$, $\int \varphi = 1$, *then* $\|f - \varphi * f\|_{h_1^c} \leq C \|f\|_{h_1^c}$, *so* $f \in h_1^c \Longrightarrow f - \varphi * f \in H_1^c$.

Application: Pseudo-differential operator

Definition

Pseudo-differential operator is a mapping $f \mapsto T(f)$ given by

$$T(f)(s) = \int \widehat{f}(\xi)\, \sigma(s, \xi)\, e^{2\pi i s\xi} d\xi,$$

σ is called the **symbol** of T, and if σ satisfies

$$|D_s^{\alpha} D_{\xi}^{\beta} \sigma(s, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{-|\beta|},$$

then we say the symbol σ is of **order zero**, this class of symbols is denoted by S^0.

The general pseudo-differential operators have a parallel description in terms of their kernels,

$$T(f)(s) = \int_{\mathbb{R}^d} K(s, s-t) f(t) dt,$$

where $K(s,t) = (\sigma(s,\xi))^{\vee}$.

Lemma

Let T be a pseudo-differential operator with symbol in S^0 and let $\Phi \in \mathcal{S}$ with vanishing mean and nondegenerate. Then $T_\varepsilon(f) = \Phi_\varepsilon * T(f)$ admits a kernel $K_\varepsilon(s,t)$ and a symbol $\sigma_\varepsilon(s,\xi)$ which satisfy

$$
\begin{aligned}
\Big(\int_0^1 |D_s^\alpha D_\xi^\beta K_\varepsilon(s,t)|^2 \frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} &\leq C_{\alpha,\beta}|t|^{-d-|\beta|},\\
\Big(\int_0^1 |D_s^\alpha D_\xi^\beta \sigma_\varepsilon(s,\xi)|^2 \frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} &\leq c_{\alpha,\beta}(1+|\xi|)^{-|\beta|}.
\end{aligned}
$$

The boundedness of pseudo-differential operators on h_1^c

Lemma

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\Big(\int_0^1 |D_s^\alpha D_\xi^\beta K_\varepsilon(s,t)|^2 \frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} &\leq C_{\alpha,\beta}|t|^{-d-|\beta|}, \\
\Big(\int_0^1 |D_s^\alpha D_\xi^\beta \sigma_\varepsilon(s,\xi)|^2 \frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} &\leq c_{\alpha,\beta}(1+|\xi|)^{-|\beta|}.
\end{aligned}$$

Theorem (X.)

Let $T(f)(s) = \int \widehat{f}(\xi)\,\sigma(s,\xi)\, e^{2\pi i s\xi} d\xi$ and $T \in S^0$, then for any $f \in h_1^c(\mathbb{R}^d, \mathcal{M})$, we have $\|T(f)\|_{h_1^c} \leq C\,\|f\|_{h_1^c}$.

Thank you !