Lacunary Fourier series for compact quantum groups

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Outline

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Woronowicz's compact quantum groups

- G a compact group $\rightsquigarrow$ comultiplication on $C(G)$:
  $\Delta_G : C(G) \to C(G \times G), \quad \Delta_G(f)(s,t) = f(st), \quad s,t \in G.$
  $(C(G), \Delta_G)$ determines G.
- G compact abelian $\Rightarrow$ dual $\hat{G}$ discrete. $C(G) = C_r^*(\hat{G})$; $G = \hat{\hat{G}}$.
- Γ a discrete group $\rightsquigarrow$ comultiplication on $C_r^*(\Gamma)$
  $\Delta_{C_r^*(\Gamma)} : C_r^*(\Gamma) \to C_r^*(\Gamma) \otimes C_r^*(\Gamma), \quad \lambda(\gamma) \mapsto \lambda(\gamma) \otimes \lambda(\gamma), \quad \gamma \in \Gamma.$
  $(C_r^*(\Gamma), \Delta_{C_r^*(\Gamma)})$ determines Γ. (View $\hat{\Gamma} = (C_r^*(\Gamma), \Delta_{C_r^*(\Gamma)})$.)

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- A compact quantum group is a pair $\mathbb{G} = (A, \Delta)$, where:

  A : a unital C*-algebra, $\quad \Delta : A \to A \otimes A \quad$ a *-homomorphism s.t.

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta,$$

$$\overline{\mathrm{span}}((1 \otimes A)\Delta(A)) = \overline{\mathrm{span}}((A \otimes 1)\Delta(A)) = A \otimes A.$$

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  $A := C(\mathbb{G})$ is called the algebra of "continuous functions" on $\mathbb{G}$.
- There exists a Haar state h on $C(\mathbb{G})$ which is "translate invariant".

Towards Fourier analysis: the "dual group"

Recall the Fourier transform on a compact abelian group G,

$$\mathcal{F} : \text{functions on } G \to \text{functions on Pontryagin dual } \hat{G}.$$

- G cpt non-abelian: replace $\hat{G}$ by $\mathrm{Irr}(G)$ (irreducible representations)
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- Unitary representation of $\mathbb{G}$: $u = [u_{ij}]_{i,j=1}^{n} \in \mathbb{M}_n(C(\mathbb{G}))$ unitary s.t.

$$\forall 1 \le j, k \le n,\ \Delta(u_{jk}) = \sum_{p=1}^{n} u_{jp} \otimes u_{pk}.$$

  $\mathrm{Irr}(\mathbb{G})$: equivalent class of all such irreducible representations u. For each $\pi \in \mathrm{Irr}(\mathbb{G})$ choose a representative $u^{(\pi)}$.

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- All such matrix coefficients $u_{ij}^{(\pi)}$ $(\pi \in \mathrm{Irr}(\mathbb{G}))$ spans a **dense** algebra of "polynomials" $\mathrm{Pol}(\mathbb{G}) \subset C(\mathbb{G})$.
  Completions of $\mathrm{Pol}(\mathbb{G})$ (wrt different topologies) $\Rightarrow$:

  $$L^2(\mathbb{G}), \quad C_r(\mathbb{G}) (\subset B(L^2(\mathbb{G}))), \quad L^\infty(\mathbb{G}) (\subset B(L^2(\mathbb{G}))).$$

  h extends to a normal faithful state on $L^\infty(\mathbb{G})$ (maybe NON-tracial).

Towards Fourier analysis: the "dual group"

In the framework of locally compact quantum groups, there is a "dual" discrete quantum group, denoted $\hat{\mathbb{G}}$, subject to the following $*$-algebra

$$c_c(\hat{\mathbb{G}}) := \oplus_{\pi\in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}(\mathbb{C}).$$

We may define the Fourier transform

$$\mathcal{F} : \mathrm{Pol}(\mathbb{G}) \to c_c(\hat{\mathbb{G}}), \quad x \mapsto \hat{x},$$

where

$$\hat{x}(\pi) = (h \otimes \mathrm{id})((u^{(\pi)})^*(x\otimes 1)) = [h(u_{ji}^{(\pi)^*} x)]_{i,j=1}^{n_\pi}, \quad \pi \in \mathrm{Irr}(\mathbb{G}).$$

(Recall: for a compact group G, $f \in C(G)$,

$$\hat{f}(\pi) = \int_G u^{(\pi)}(g)^* f(g) dm(g) = \left[\int_G u_{ji}^{(\pi)^*} f dm\right]_{i,j=1}^{n_\pi} .)$$

Fourier series

Briefly, we obtain the Fourier transform

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The map can be extended to L^p-spaces. (Note h on $L^\infty(\mathbb{G})$ may be not tracial!)

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$$L^p(\mathbb{G}) = (L^\infty(\mathbb{G}), L^1(\mathbb{G}))_{1/p}, \quad 1 \leq p \leq \infty.$$

Define $\ell^p(\hat{\mathbb{G}})$ on $c_c(\hat{\mathbb{G}})$ similarly.

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$$\mathcal{F} : L^p(\mathbb{G}) \to \ell^q(\hat{\mathbb{G}}) \text{ contraction}, \quad 1/p + 1/q = 1.$$

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- (Plancherel theorem) $\mathcal{F} : L^2(\mathbb{G}) \to \ell^2(\hat{\mathbb{G}})$ unitary.

More remark on $\hat{\mathbb{G}}$: non-unimodularity

Recall: a discrete group is always unimodular.
But the discrete quantum group $\hat{\mathbb{G}}$ can be non-unimodular.

There are different "left/right invariant" Haar weights on $\ell^\infty(\hat{\mathbb{G}})$, and a modular element F linking them,

$$F = (F_\pi)_{\pi \in \mathrm{Irr}(\mathbb{G})}, \quad F_\pi \in \mathbb{M}_{n_\pi}(\mathbb{C}).$$

It is possible $\|F\| = \sup_\pi \|F_\pi\| = +\infty$.

In fact, F is trivial iff h on $L^\infty(\mathbb{G})$ is tracial. Woronowicz used this F to implement the modular property of non-tracial Haar state h on $\mathbb{G}$.

Fourier multipliers

- **Multipliers**: Each $a = (a_\pi)_{\pi \in \mathrm{Irr}(\mathbb{G})} \in \prod_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}$ induces a map

$$m_a : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G}), \quad m_a x = \mathcal{F}^{-1}(a\hat{x}).$$

We say a is a left bounded multiplier on $L^p(\mathbb{G})$ if m_a extends to a bdd map on $L^p(\mathbb{G})$. (similar def. for right multipliers)
$M(L^p(\mathbb{G})) =$\{left & right bdd multipliers on $L^p(\mathbb{G})$\} $\subset B(L^p(\mathbb{G}))$.

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- Daws, Neufang, Junge, Ruan (09'-12'): study completely bounded multiplier on $L^\infty(\mathbb{G})$. Easy to establish

$$\|a\|_{\ell^\infty(\hat{\mathbb{G}})} \leq \|a\|_{M_{cb}(L^\infty(\mathbb{G}))}.$$

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Proposition (Partially communicated by M. Junge)

$$\|F^{\frac{1}{4}-\frac{1}{2p}} a F^{-\frac{1}{4}+\frac{1}{2p}}\|_{\ell^\infty(\hat{\mathbb{G}})} \leq \|a\|_{M(L^p(\mathbb{G}))}.$$

Outline

Sidon sets: classical settings

Definition Let G be a compact abelian group and $\Gamma = \hat{G}$ be the dual group. A subset $E \subset \Gamma$ is called a Sidon set if

$$\forall (\alpha_\gamma) \subset \mathbb{C}, \quad \sum_{\gamma \subset E} |\alpha_\gamma| \sim \| \sum_{\gamma \in E} \alpha_\gamma \gamma \|_{C(G)}.$$

- Various characterizations: interpolation of bounded measures, multipliers, $\Lambda(p)$-estimations, unconditional bases...
- Typical examples: Rademacher functions; lacunarity in $\mathbb{Z}$...

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Noncommutative generalizations:

- $G \rightsquigarrow$ compact non-abelian group, $\Gamma \rightsquigarrow \mathrm{Irr}(G)$;
- $\Gamma \rightsquigarrow$ discrete non-abelian group, $C(G) \rightsquigarrow C(\hat{\Gamma}) := C_r^*(\Gamma)$ group C*-algebra. (Recall the quantum group $\hat{\Gamma}$)

$$\mathrm{Irr}(\hat{\Gamma}) = \Gamma, \mathrm{Pol}(\hat{\Gamma}) = \mathbb{C}\Gamma, C_r(\hat{\Gamma}) = C_r^*(\Gamma), L^\infty(\hat{\Gamma}) = VN(\Gamma).$$

Sidon sets: classical settings

Let G be a compact group.
For $f \in L^\infty(G)$ and $\pi \in \mathrm{Irr}(G)$, recall $\hat{f}(\pi) = \int_G f(g) u^{(\pi)}(g)^* dm(g)$.
The ℓ^1-norm on $\hat{f}$ is explicitly given by

$$\|\hat{f}\|_1 = \sum_{\pi \in \mathrm{Irr}(G)} d_\pi \mathrm{Tr}(|\hat{f}(\pi)|).$$

Theorem (Figà-Talamanca) Consider $E \subset \mathrm{Irr}(G)$. TFAE:

1. E is a Sidon set, i.e.,

$$\mathrm{supp}(\hat{f}) \subset E \Rightarrow \|\hat{f}\|_1 \sim \|f\|_\infty;$$

2. $\oplus_{\pi \in E} \mathbb{M}_{n_\pi} = \{\hat{\mu}|_E : \mu \in M(G)\}$;
3. $\oplus^{c_0}_{\pi \in E} \mathbb{M}_{n_\pi} = \{\hat{f}|_E : f \in L^1(G)\}$.

Sidon sets: quantum group setting

Let $\mathbb{G}$ be a compact quantum group. F the modular element for $\hat{\mathbb{G}}$.
For $x \in L^\infty(\mathbb{G})$ and $\pi \in \mathrm{Irr}(\mathbb{G})$, recall $\hat{x}(\pi) = (h \otimes \mathrm{id})((u^{(\pi)})^*(x \otimes 1))$.
The norm on $\ell^1(\hat{\mathbb{G}})$ is explicitly given by

$$\|\hat{x}\|_1 = \sum_{\pi \in \mathrm{Irr}(\mathbb{G})} d_\pi \mathrm{Tr}(|\hat{x}(\pi) F_\pi|). \quad (d_\pi = \mathrm{Tr}(F_\pi))$$

Theorem (W.) Consider $E \subset \mathrm{Irr}(\mathbb{G})$. TFAE:

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3. $\oplus^{c_0}_{\pi \in E} \mathbb{M}_{n_\pi} = \{\hat{x}|_E : x \in L^1(\mathbb{G})\}$.

Sidon sets: classical settings revisited

If $\mathbb{G} = \hat{\Gamma}$ for a discrete group Γ, recall

$$\mathrm{Irr}(\hat{\Gamma}) = \Gamma \quad \text{and} \quad x = \sum_{\gamma} \hat{x}(\gamma)\lambda(\gamma) \in VN(\Gamma).$$

The previous theorem improves a result of Picardello: (Picardello proved in the special case that Γ is amenable)

Corollary Let Γ be a discrete group (not necessarily amenable). TFAE:

1. $E \subset \Gamma$ is a Sidon set, i.e.,

$$\forall (\alpha_\gamma) \subset \mathbb{C}, \quad \sum_{\gamma \in E} |\alpha_\gamma| \sim \| \sum_{\gamma \in E} \alpha_\gamma \lambda(\gamma) \|_{VN(\Gamma)};$$

2. $E \subset \Gamma$ is a strong Sidon set, i.e.,

$$c_0(E) = \{ f|_E : f \in A(\Gamma)(\cong L^1(\hat{\Gamma})) \}.$$

Remark: various generalizations vs amenability

- $\mathbb{G}$ is called coamenable if $\epsilon : u_{ij}^{(\pi)} \mapsto \delta_{ij}$ is bdd on $C_r(\mathbb{G})$.
  Recall our convention: $\hat{\Gamma} = (C_r^*(\Gamma), \Delta_{C_r^*(\Gamma)})$ being a compact quantum group. $\hat{\Gamma}$ is coamenable iff Γ is amenable.
- If $\mathbb{G}$ is not coamenable, there are various (non-equivalent) analogue of Sidon sets: weak Sidon sets, interpolation sets of multipliers, unconditional Sidon sets, Leinert sets, etc.
  (For $\hat{\Gamma}$: Pisier 95')

Sidon sets $\Rightarrow \Lambda(p)$-sets

Definition $E \subset \mathrm{Irr}(\mathbb{G})$ is a $\Lambda(p)$-set if

$$\|x\|_p \sim \|x\|_1, \quad \mathrm{supp}(\hat{x}) \subset E.$$

- Case for compact groups: Hewitt-Ross, Marcus-Pisier;
- Case for (dual of) discrete groups: Picardello, Harcharras...
- Case for compact quantum groups: more delicate –

  Theorem (Blendek-Michaliček 13') Let $\mathbb{G}$ be a CQG s.t. the Haar state h is tracial on $C(\mathbb{G})$. Denote

  $$\chi_\pi = \sum_{i=1}^{n_\pi} u_{ii}^{(\pi)}, \quad \pi \in \mathrm{Irr}(\mathbb{G}).$$

  If $E \subset \mathrm{Irr}(\mathbb{G})$ is a Helgason-Sidon set ($\subsetneq$ Sidon set), then for all $(c_\pi)_{\pi \in E} \subset \mathbb{C}$ and $x = \sum_{\pi \in E} c_\pi \chi_\pi$,

  $$\|x\|_2 \sim \|x\|_1.$$

Sidon sets $\Rightarrow \Lambda(p)$-sets, Fourier multipliers

Theorem (W.) Let $2 < p < \infty$ and $E \subset \mathrm{Irr}(\mathbb{G})$. Then:

$$\|x\|_p \sim \|x\|_1, \quad \mathrm{supp}(\hat{x}) \subset E,$$

if and only if

$$\forall\, a \in \oplus_{\pi \in E} \mathbb{M}_{n_\pi}, \quad \exists\, b \in M(L^p(\mathbb{G})), \quad b|_E = a.$$

Remark: the completely bdd version of above thm is not established yet.
(For $\mathbb{G} = \hat{\Gamma}$ dual of discrete group: Harcharras 99';
For $\mathbb{G} = G$ compact group: Hare-Mohanty 15'.)

Corollary Any Sidon set for $\mathbb{G}$ is a $\Lambda(p)$-set for $1 < p < \infty$.

Observations on non-modularity I

Recall that we have a modular element F for $\hat{\mathbb{G}}$.

Proposition Let $E \subset \mathrm{Irr}(\mathbb{G})$ a $\Lambda(p)$-set for $2 < p < \infty$. Then

$$\sup_{\pi \in E} \|F_\pi\| < +\infty.$$

Drinfeld-Jimbo deformation: a compact semi-simple Lie group $G \rightsquigarrow$ a compact quantum group G_q $(0 < q < 1)$, with $\hat{G}_q$ non-unimodular.

Corollary $\mathrm{SU}(2)_q$ does not admit $\Lambda(p)$-set for any $2 < p < \infty$.

Observations on non-modularity II

More words on noncommutative L^p:
Let h be non-tracial on $\mathbb{G}$ (equivalently $\hat{\mathbb{G}}$ non-unimodular)

- According to Kosaki 84': For each $0 \leq \theta \leq 1$, there is a complex interpolation scale $(L^p_{(\theta)}(\mathbb{G}))_{1 \leq p \leq \infty}$ between $L^\infty(\mathbb{G})$ and $L^\infty(\mathbb{G})_*$, which are isometrically isomorphic but not equal.

  In the previous slides we have indeed taken $L^p(\mathbb{G}) = L^p_{(0)}(\mathbb{G})$.
- Casper 13': The boundedness of Fourier transform depends on θ.

Proposition Our definition of $\Lambda(p)$-sets is independent of θ.
That is, let $2 < p < \infty$, $0 \leq \theta, \theta' \leq 1$ and $E \subset \mathrm{Irr}(\mathbb{G})$. Then:

$$\|x\|_{L^p_{(\theta)}(\mathbb{G})} \sim \|x\|_{L^1_{(\theta)}(\mathbb{G})}, \quad \mathrm{supp}(\hat{x}) \subset E,$$

if and only if

$$\|x\|_{L^p_{(\theta')}(\mathbb{G})} \sim \|x\|_{L^1_{(\theta')}(\mathbb{G})}, \quad \mathrm{supp}(\hat{x}) \subset E.$$

Existence of $\Lambda(p)$-sets

Theorem (Bożejko; W.) Let (M, φ) be a vNa equipped with a normal faithful state φ. Let $B = \{x_i \in M : i \geq 1\}$ be an orthogonal system wrt φ s.t. $\sup_i \|x_i\|_\infty < \infty$. Then for each $1 \leq p < \infty$, there exists an infinite subset $\{x_{i_k} : k \geq 1\} \subset B$

$$\forall (c_k) \subset \mathbb{C}, \quad \| \sum_{k \geq 1} c_k x_{i_k} \|_{L^p(M)} \sim \Big(\sum_{k \geq 1} |c_k|^2 \Big)^{\frac{1}{2}}.$$

Corollary Let $\mathbb{G}$ be a CQG. Let $E \subset \mathrm{Irr}(\mathbb{G})$ be an infinite subset with $\sup_{\pi \in E} d_\pi < \infty$. Then for each $1 \leq p < \infty$, there exists an infinite subset $E' \subset E$ s.t.

$$\|x\|_p \sim \|x\|_1, \quad \mathrm{supp}(\hat{x}) \subset E'.$$

Thank you very much!