Mixed Commutators vs Mixed BMO

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Toeplitz and Hankel operators on subspaces of Hilbert spaces of functions are compositions of multiplication operators and orthogonal projection on these spaces - so it is natural to investigate how their norms are related to the infinity norm of their symbols.

Norms of Toeplitz operators - classical case

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Toeplitz operators from H^2 to H^2 are defined by:

$$T_\phi = P^+ \circ M_\phi$$

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The operator $T_\phi = P^+ \circ M_\phi$ is bounded if and only if its symbol ϕ is a bounded function (in $L^\infty(T)$) ; and $\|T_\phi\| = \|\phi\|_\infty$. (Brown-Halmos(1963))

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Because, if W is the operator M_z (unitary on L^2 then M_ϕ is the strong limit of the sequence $W^{*n} T_\phi P^+ W^n$; so $\|M_\phi\| \leq \|T_\phi\|$.

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With a little work, the same reasoning works in several variables (with $W = M_{z_1 z_2 \cdots z_n}$).

A more varied selection of Toeplitz's

Recently a lot of work has been done on what are called *Truncated Toeplitz operators* - that is, operators of the form $P^E \circ M_\phi$ where E is a 'model space' or W^* invariant subspace of H^2.

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The classical example of such a space is $\mathcal{P}_n$ the polynomials of degree n where the operators are Toeplitz matrices

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdot & a_{-n} \\ a_1 & a_0 & a_{-1} & \cdot & a_{-(n-1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_n & \cdot & \cdot & \cdot & a_0 \end{pmatrix}$$

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And the symbol is any function ϕ such that $\hat{\phi}(k) = a_k$ for $k = -n, \cdots n$ so it is not at all unique. But, one rarely finds a symbol with infinity norm equal to the norm of the operator (in the analytic case you can, using the commutant lifting theorem).

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Another case where it is very difficult to characterize the symbols of bounded Toeplitz is for operators on the Bergman spaces (again the easily treated case is for an 'analytic' symbol)....but it's not the time to discuss this!

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So, if ϕ is analytic $H_\phi = 0$ and only the antianalytic part of ϕ matters.

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This time, only the analytic part of the symbol matters. The characterization of symbols of bounded Hankels comes from the Nehari theorem.

Nehari Theorem

Theorem

$$H_b : H^2(\mathbb{T}) \to H^2(\mathbb{T})$$

is bounded iff there is a bounded function β *with* $P^+(\beta) = P^+(b)$*.*
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The proof here uses the factorization of H^1 functions into two H^2 functions; then Hahn-Banach and the $L^1 - L^\infty$ duality to get the L^∞ function.

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The proof here uses the factorization of H^1 functions into two H^2 functions; then Hahn-Banach and the $L^1 - L^\infty$ duality to get the L^∞ function. And the several variable case is *very* different - and best

discussed in the framework of commutators with Hilbert transforms.

Hilbert Transform

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This map is trivially bounded on $L^2(\mathbb{T})$ thanks to Plancherel:

$$\sum_{k\geq 0} |\hat{u}(k)|^2 \leq \sum_{k} |\hat{u}(k)|^2$$

It is also bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$ but not on $L^1(\mathbb{T})$.

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We set $H^1_{Re} = \{f \in L^1 : Hf \in L^1\}$, (these are the real parts of the boundary values of functions in the analytic space $H^1(\mathbb{D})$) and set:

$$\|f\|_{H^1_{Re}} = \|f\|_1 + \|Hf\|_1.$$

Thus, the Hilbert transform **is** bounded on H^1_{Re}.

BMO

BMO (bounded mean oscillation) is the Banach space of all functions $f \in L^1_{loc}(\mathbb{T})$ for which

$$\|f\|_{BMO} = \sup_I \frac{1}{|I|} \int_I |f - c_I| < \infty$$

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The amazing discovery by Charles Fefferman in 1970 was (the multivariable version) that *BMO* is actually the dual space of H^1_{Re}.

BMO

In fact, functions g in BMO can be written in the form $g = g_0 + Hg_1$; where $g_0, g_1 \in L^\infty$. and given the (equivalent) norm

$$\|g\|_{BMO} = \sup\{\|g_0\|_\infty, \|g_1\|_\infty\}$$

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This shows that $L^\infty \subset BMO$ and also that:

$P^+(L^\infty) \subset BMO$

so functions of the form $P^+ f$ for $f \in L^\infty$ are in BMO, producing 'logarithmic infinities'.

n-variable Hilbert transforms

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We define these transforms for the two variable case, the n-variable case is done in the same way. The space $H^2 \otimes L^2$ is the closed subspace of functions in $L^2(\mathbb{T}^2)$ whose biharmonic extension to the bidisk is analytic in the first variable.
We write P_1 for the orthogonal projection onto this subspace and P_2 for the orthogonal projection onto $L^2 \otimes H^2$ (defined in the same way).

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Then we define the '*jth*' variable Hilbert transform $H_j = -iP_j + iP_j^{\perp}$;

2-variable product BMO

The ' product BMO' is defined, for $n = 2$ by

$$\phi \in BMO(\mathbb{T}^2) \iff \phi = g_1 + H_1(g_2) + H_2(g_3) + H_1(H_2(g_4)) \qquad (*)$$

with all the $g_i \in L^\infty$.

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with all the $g_i \in L^\infty$. The norm is defined by

$$\|\phi\|_{BMO} = \inf\{\max_j \|g_j\|_\infty\}$$

where the inf is taken over all decompositions of the form $(*)$.

Duality of product BMO

Now we can define $H^1_{Re}(\mathbb{T}^2)$ to be

$$\{f \in L^1 : H_1(f), H_2(f), H_1(H_2(f)) \in L^1(\mathbb{T}^2)\}$$

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The n-variable product bmo and H^1_{Re} are defined in exactly the same way.

Commutators with Hilbert transform vs Hankels

The commutator of the Hilbert transform and the multiplication operator M_b gives us a couple of 'Hankel operators'.

$$[M_b, H]f = M_b \cdot Hf - H(M_b f) =$$
$$-i\big[(P^+ + P^-)M_b(P^+ - P^-)f - (P^+ - P^-)M_b(P^+ + P^-)f\big]$$

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So Nehari's theorem tells us that $[M_b, H]$ is bounded on L^2 iff $P^+ b \in BMO$ and $P^- b \in BMO$ iff $b \in BMO$. and that

$$\|[M_b, H]\|_{2\to 2} \lesssim \|b\|_{BMO}$$

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In fact it is also true that

$$\|b\|_{BMO} \lesssim \|[M_b, H]\|_{2\to 2}$$

Various Commutators in two dimensions

For a symbol $b(x, y)$ that depends on two variables, there are several natural choices:

1) $[M_b, R_i]$ where $i = 1, 2$ and R_i is the Riesz transform in the ith direction. $\rightarrow$ 1 parameter BMO

2) $[M_b, H_1 H_2]$ where H_i are Hilbert transforms in the $i - th$ variable, $i = 1, 2$. Simple case of a product CZO. $\rightarrow$ little BMO

3) $[[M_b, H_1], H_2]$, simplest case of an iterated commutator. $\rightarrow$ product BMO

Commutators and multi-variable Hankels

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The Sadosky-Ferguson paper shows that little BMO are the symbols of BIG Hankel operators. These operators are the several variable form of 1-variable Hankels, using projection on the orthogonal complement:

If P is orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{T}^n)$
Then the 'big Hankel operator with symbol ϕ is defined by:

$$H_\phi : H^2(\mathbb{T}^n) \to (H^2(\mathbb{T}^n))^\perp$$

$$H_\phi(f) = P^\perp(\phi f)$$

More general BMOs - work with Petermichl, Pipher, Ou

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For example, we call $BMO_{(12)3}$ the Banach space of functions $b \in L^1(\mathbb{T}^3)$ such that the families $(b(\cdot, x_2, \cdot))_{x_2 \in \mathbb{T}}$ and $(b(x_1, \cdot, \cdot))_{x_1 \in \mathbb{T}}$ are uniformly bounded in product BMO. This is the space

$$L^\infty(\mathbb{T}^3) + H_1(\mathbb{T}^3) + H_2(\mathbb{T}^3) + H_1 H_2(\mathbb{T}^3)$$

equipped with the norm $\max\{\|g_i\|_\infty$.

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equipped with the norm $\max\{\|g_i\|_\infty$.

We are looking at the characterizations of these types of spaces in terms of their preduals, commutators and Hankel-type operators.

(This involves certain types of weak factorization of their pre-duals.)

Mixed BMO and its predual

We have the following results: To characterize mixed BMO as a dual space:

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Theorem

A function $f \in L^1(\mathbb{T}^3)$ satisfies

$$\sup_{\|\varphi\|_{BMO_{(13)2}}} \left| \int_{\mathbb{T}^3} f \varphi dm \right| < \infty$$

if and only if there exist functions $f' \in H^1_{Re}(\mathbb{T} \times \mathbb{T}) \otimes L^1(\mathbb{T})$ and $f'' \in L^1(\mathbb{T}) \otimes H^1_{Re}(\mathbb{T} \times \mathbb{T})$ such that $f = f' + f''$.

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Theorem

Let $b \in L^1(\mathbb{T}^3)$. The the following are equivalent.

1. $b \in BMO_{(12)3}$
2. *The commutators* $[H_3, [H_1, b]]$ *and* $[H_3, [H_2, b]]$ *are bounded on* $L^2(\mathbb{T}^3)$
3. *The commutator* $[H_3, [H_2 H_1, b]]$ *is bounded on* $L^2(\mathbb{T}^3)$.

Mixed BMO and it's Hankels

Now, using the commutator theorem, one can characterize the mixed BMO functions in terms of 'hankel types' that is, operators of type $P^{\perp} M_b P$ as follows:

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Theorem

- *The commutators* $[H_3, [H_1, b]]$ *and* $[H_3, [H_2, b]]$ *are bounded on* $L^2(\mathbb{T}^3)$ *if and only if all eight operators* $P_i P_3 b P_i^{\perp} P_3^{\perp}, P_i^{\perp} P_3 b P_i P_3^{\perp}, P_i P_3 b P_i^{\perp} P_3^{\perp}$ *with* $i \in \{1, 2\}$ *are bounded on* $L^2(\mathbb{T}^3)$.
- *The commutator* $[H_3, [H_2 H_1, b]]$ *is bounded on* $L^2(\mathbb{T}^3)$ *if and only if all four Hankels* $P_3 Q_{12} b Q_{12}^{\perp} P_3^{\perp}, P_3^{\perp} Q_{12}^{\perp} b Q_{12} P_3, P_3 Q_{12}^{\perp} b Q_{12} P_3^{\perp}, P_3^{\perp} Q_{12} b Q_{12}^{\perp} P_3$ *with* $Q_{12} = P_1 P_2 + P_1^{\perp} P_2^{\perp}$ *and* $Q_{12}^{\perp} = P_1^{\perp} P_2 + P_1 P_2^{\perp}$, *are bounded on* $L^2(\mathbb{T}^3)$.

Mixed BMO and it's Hankels

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Higher order Journe commutators and characterizations of multi-parameter BMO; Stefanie Petermichl; Yumeng Ou; Elizabeth Strouse
To appear: Advances in Mathematics

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