Dispersive inequalities via heat semigroup

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   - Strichartz estimates in various settings
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$$\begin{cases} i\partial_t u + \Delta u = F(u) \\ u(0,x) = u_0(x) \end{cases}, \quad t \in \mathbb{R},\ x \in \mathbb{R}^d. \qquad \text{(NLS)}$$

- Duhamel's formula:
  $u(t,x) = e^{it\Delta}u_0(x) - i\int_0^t e^{i(t-s)\Delta}F(u(s,x))ds.$
- Existence, uniqueness: Contraction principle.
  Relies on Strichartz estimates: $\forall\ 2 \leq p, q \leq +\infty$

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2} \Rightarrow \|e^{it\Delta}u_0\|_{L^p_t L^q_x} \lesssim \|u_0\|_{L^2}. \qquad (1)$$

- Via a TT^* argument, interpolation with $\|e^{it\Delta}\|_{L^2\to L^2} \lesssim 1$, and Hardy-Littlewood-Sobolev inequality (Keel-Tao), (1) reduces to $L^1 - L^\infty$ dispersion inequality:

$$\|e^{it\Delta}\|_{L^1\to L^{+\infty}} \lesssim |t|^{-\frac{d}{2}}. \tag{2}$$

- (2) can be obtained by a complexification of the heat semigroup $(e^{t\Delta})_{t\geq 0}$.
- In $\mathbb{R}^d$ we have an explicit formulation of the heat semigroup kernel:

$$p_t(x,y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}.$$

Question: What do we know in other settings ?

Examples

- Outside of a smooth convex domain of $\mathbb{R}^d$ with Laplace-Beltrami operator: global-in-time estimates with loss of $\frac{1}{p}$ derivatives [Burq-Gérard-Tzvetkov].
- Compact riemannian manifold: local-in-time estimates with loss of $\frac{1}{p}$ derivatives [Burq-Gérard-Tzvetkov].
- Asymptotically hyperbolic manifolds: local-in-time estimates without loss [Bouclet].
- Laplacian with a smooth potential, infinite manifolds with boundary with one trapped orbit: local-in-time estimates with $\frac{1}{p} + \varepsilon$ loss of derivatives [Christianson].

Remark: One cannot expect global-in-time estimates in a compact setting.

Theorem [Burq-Gérard-Tzvetkov, '04]

Let $\mathcal{M}$ be a compact riemannian manifold of dimension d. If $\varphi \in C_0^\infty(\mathbb{R}_+)$ then for all $h \in]0,1]$:

$$\|e^{it\Delta}\varphi(h^2\Delta)\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{d}{2}}, \quad |t| \lesssim h.$$

C_0^∞ are not adapted to our problem. We substitute them by a familly of C^∞ functions well suited to the semigroup setting:

$$\psi_m(x) = x^m e^{-x}, \; m \in \mathbb{N}.$$

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The space:
(X, d, μ) is a metric measured space with μ satisfying a doubling property:

$$\forall x \in X,\ \forall r > 0,\ \mu(B(x, 2r)) \leq C\mu(B(x, r)). \tag{3}$$

Then there exists a homogeneous dimension d such that:

$$\forall x \in X, \forall r > 0, \forall \lambda \geq 1, \mu(B(x, \lambda r)) \lesssim \lambda^d \mu(B(x, r)).$$

Examples

Euclidean space $\mathbb{R}^d$, open sets of $\mathbb{R}^d$, smooth manifolds of dimension d, some fractals sets, Lie groups, Heisenberg group,...

The operator:

- H is a self-adjoint nonnegative operator, densely defined on $L^2(X)$.
- H generates a L^2-holomorphic semigroup $(e^{-tH})_{t\geq 0}$ (Davies).
- Davies-Gaffney estimates:

$$\forall t > 0,\ \forall E, F \subset X,\ \|e^{-tH}\|_{L^2(E)\to L^2(F)} \lesssim e^{-\frac{d(E,F)^2}{4t}} \qquad \text{(DG)}$$

- Typical on-diagonal upper estimates:

$$\forall t > 0,\ \forall x \in X,\ 0 \le p_t(x,x) \lesssim \frac{1}{\mu(B(x,\sqrt{t}))} \qquad \text{(DUE)}$$

- Self-improve (Coulhon-Sikora) into full gaussian estimates:

$$\forall t > 0,\ \forall x,y \in X,\ 0 \le p_t(x,y) \lesssim \frac{1}{\mu(B(x,\sqrt{t}))} e^{-\frac{d(x,y)^2}{4t}}. \qquad \text{(UE)}$$

Remark:

$$\text{(DUE)} \Rightarrow \text{(UE)} \Rightarrow \text{(DG)}.$$

Some cases where the previous estimates hold:

Examples

- (DUE): Δ on a domain with boundary conditions, semigroup generated by a self-adjoint operator of divergence form $H = -\mathrm{div}(A\nabla)$ with A a real bounded elliptic matrix on $\mathbb{R}^d$;
- (UE): $H = -\sum_{i=1}^{d} X_i^2$ where X_i are vector fields satisfying Hörmander condition on a Lie group or a riemannian manifold with bounded geometry;
- (DG): most second order self-adjoint differential operators, Laplace-Beltrami on a riemannian manifold, Schrödinger operator with potential...

The function spaces:

- $L^1 - L^\infty$ estimate seems out of reach.
- We prove instead $H^1 - \mathrm{BMO}$ dispersion.
- H^1 and BMO adapted to the semigroup (equivalent to the classical H^1 of Coifman-Weiss, and BMO of John-Nirenberg).
- Atomic decomposition (Bernicot-Zhao).
- Interpolate with Lebesgue spaces, and the intermediate spaces are corresponding Lebesgue spaces.

The question we investigate is how to prove an $H^1 - \mathrm{BMO}$ dispersive estimate:

$$\|e^{itH}\psi_m(h^2H)\|_{H^1\to\mathrm{BMO}} \lesssim |t|^{-\frac{d}{2}}.$$

Remark: Write $e^{itH}\psi_m(h^2H) = (h^2H)^m e^{-zH}$ with $z = h^2 - it$.

- $|t| \leq 1$ (i.e. t independant of h) is difficult.
- $|t| \leq h^2$ is straightforward by analytic continuation of (UE) (since $\mathrm{Re}(z) \simeq |z| \geq |t|$).
- $h^2 \leq |t| \leq h$ is dealt by Burq-Gérard-Tzvetkov ('04) in the compact riemannian manifold setting (using pseudo-differential tools).
- We will treat the case $h^2 \leq |t| \leq h^{1+\varepsilon}$ (for all $\varepsilon > 0$).

Hypothesis $(H_m(A))$

An operator T satisfies Hypothesis $(H_m(A))$ if:

$$\forall r > 0, \ \|T\psi_m(r^2H)\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \lesssim A\mu(B_r)^{\frac{1}{2}}\mu(\widetilde{B_r})^{\frac{1}{2}}, \quad (H_m(A))$$

for any two balls B_r, $\widetilde{B_r}$ of radius r.

Remarks:

- We intend to use hypothesis $(H_m(A))$ for $T = e^{itH}\psi_{m'}(h^2H)$ and $A = |t|^{-\frac{d}{2}}$.
- Hypothesis $(H_m(A))$ is weaker than the $L^1 - L^\infty$ estimate by Cauchy-Schwarz inequality.

Theorem 1 [Bernicot, S., '14]

Let T be a self-adjoint operator commuting with H and satisfying $\|T\|_{L^2\to L^2} \lesssim 1$. If T satisfies $(H_m(A))$ for $m \geq \frac{d}{2}$, then for all $p \in (1,2)$:

$$\|T\|_{H^1\to \mathrm{BMO}} \lesssim A \quad \text{thus} \quad \|T\|_{L^p\to L^{p'}} \lesssim A^{\frac{1}{p}-\frac{1}{p'}}.$$

That theorem reduces the $H^1 - \mathrm{BMO}$ and $L^p - L^{p'}$ estimates to a microlocalized $L^2(B_r) - L^2(\widetilde{B_r})$ one.

Ideas of the proof:

- Use the atomic structure of H^1.
- Use an approximation of the identity well suited to our setting $(e^{-sH})_{s>0}$
- Interpolate between $H^1 - \mathrm{BMO}$ and $L^2 - L^2$.

Summary of theorem 1

$H_m(A) \Rightarrow H^1 \to \mathrm{BMO}$ and $L^p \to L^{p'}$ dispersive estimates.

Wave propagator

For $f \in L^2$, we note $\cos(t\sqrt{H})f$ the unique solution at time t of the wave problem:

$$\begin{cases} \partial_t^2 u + Hu = 0 \\ u_{|t=0} = f \\ \partial_t u_{|t=0} = 0 \end{cases} .$$

The wave propagator is the map $f \mapsto \cos(t\sqrt{H})f$.

Finite speed propagation

For any disjoint open sets $U_1, U_2 \subset X$, and any $f_1 \in L^2(U_1)$, $f_2 \in L^2(U_2)$, we have:

$$\forall 0 < t < d(U_1, U_2), \ < \cos(t\sqrt{H}) f_1, f_2 >= 0. \tag{4}$$

We have the equivalence (Coulhon-Sikora '06):

$$\text{(DG)} \Leftrightarrow \text{(4)}.$$

Remark: If $\cos(t\sqrt{H})$ has a kernel K_t, (4) means that K_t is supported in the "light cone":

$$\operatorname{supp} K_t \subset \{(x,y) \in X^2, \ d(x,y) \leq t\}.$$

Assumption on the wave propagator

There exists $\kappa \in (0, \infty]$ and an integer ℓ such that for all $s \in (0, \kappa)$, for all $r > 0$ and any two balls B_r, $\widetilde{B_r}$ of radius r, we have:

$$\| \cos(s\sqrt{H})\psi_\ell(r^2 H)\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}} \left(\frac{r}{r+|L-s|}\right)^{\frac{d+1}{2}}$$

where $L = d(B_r, \widetilde{B_r})$.

Remark: κ is linked to the geometry of the space X (its injectivity radius for example).

Theorem 2 [Bernicot, S. '14]

Under the previous assumption on the wave propagator, for all $m \geq \max\{\frac{d}{2}, \ell + \lceil \frac{d-1}{2} \rceil\}$:

1. If $\kappa = +\infty$: e^{itH} satisfies $(H_m(|t|^{-\frac{d}{2}}))$ for all $t \in \mathbb{R}$.
2. If $\kappa < +\infty$: for all $\varepsilon > 0$ and $h > 0$ with $|t| \leq h^{1+\varepsilon}$ and all integer $m' \geq 0$, $e^{itH}\psi_{m'}(h^2 H)$ satisfies $(H_m(|t|^{-\frac{d}{2}}))$.

- In the first case we obtain global-in-time Strichartz estimates without loss of derivatives.
- In the second case we recover local-in-time Strichartz estimates with $\frac{1}{p} + \varepsilon$ loss of derivatives.

Ideas of the proof:

- Cauchy formula $\Rightarrow e^{-zH} = \int_0^{+\infty} \cos(s\sqrt{H}) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{\pi z}}$ with $z = h^2 - it$;
- Integrate by parts when s is small;
- Use assumption on $\cos(s\sqrt{H})$ when $s < \kappa$;
- Use the exponential decay of $e^{-\frac{s^2}{4z}}$ when s is large.

Summary of theorem 2

$L^2(B_r) \to L^2(\widetilde{B_r})$ dispersion for the wave propagator $\Rightarrow H_m(|t|^{-\frac{d}{2}})$.

Some cases where we can check $L^2(B_r) \to L^2(\widetilde{B_r})$ dispersion for the wave propagator to apply Theorem 1 and 2 and recover Strichartz estimates:

Examples

- $X = \mathbb{R}^d$ with $H = -\Delta$ $(\kappa = +\infty)$;
- $X = \mathbb{R}^d$ with $H = -\mathrm{div}(A\nabla)$ where $A \in C^{1,1}$ $(\kappa < +\infty)$;
- Compact riemannian manifolds with Laplace-Beltrami operator (κ depends on the injectivity radius);
- Non-compact riemannian manifolds with bounded geometry (κ given by the geometry);
- Non-trapping asymptotically conic manifolds with $H = -\Delta + V$ ([Hassel-Yang '15]).

One thing to remember:

$$L^2(B_r) - L^2(\widetilde{B_r}) \text{ dispersive estimates for the wave propagator}$$

$$\Downarrow$$

$$H^1 - \text{BMO dispersive estimates for the Schrödinger operator}$$

$$\Downarrow$$

$$L^pL^q \text{ Strichartz inequalities for the Schrödinger operator}$$

- A good understanding of the wave propagator in various settings will help to detect whereas the method can apply:
  - The proof of (DG) $\Leftrightarrow$ (4) may allow us to show that gaussian upper bounds (UE) imply a dispersion for $\cos(s\sqrt{H})$;
  - Weaken the assumption on $\cos(s\sqrt{H})$, in particular near the boundary of the light cone;
  - Klainerman's commuting vector fields method may give a suitable $L^1 - L^\infty$ dispersive estimates for $\cos(s\sqrt{H})$ in various settings (mild assumption on the geometry of X, or $H = -\mathrm{div}(A\nabla)$ with no/minimal regularity on A);
- Find new examples where we can apply our method to derive Strichartz estimates in general settings;
- Perturbation of H with a potential V with no regularity;

Thank you for your attention !