Inner Functions and Inverse Spectral theory

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A classical problem

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$$R(\Lambda) = \sup\{a : Ker T_{\bar{S}^a J_\Lambda} = 0\},$$

$S(z) = e^{iz}$ and J_Λ is an MIF with $\sigma(J) = \Lambda$.

Model Spaces

Paley Wiener Space

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What about uniqueness sets of K_Θ?

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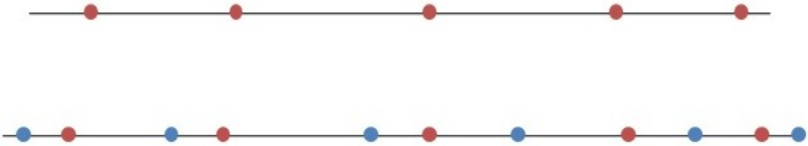
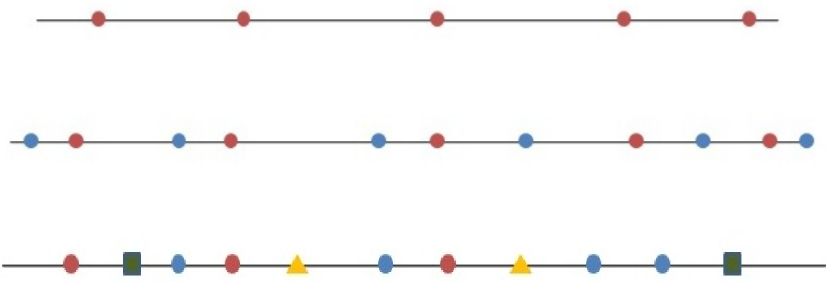
Question

Can we reconstruct potential **uniquely** from spectral data?





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An MIF Θ is a bounded analytic function on $\mathbb{C}_+$, with a meromorphic continuation on $\mathbb{C}$ such that $|\Theta| = 1$ on $\mathbb{R}$.

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$\{x \in \mathbb{R} | \Theta(x) = e^{i\alpha}\}$

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Marchenko

The Weyl -m function (and hence the Weyl-inner function) determines the potential uniquely.

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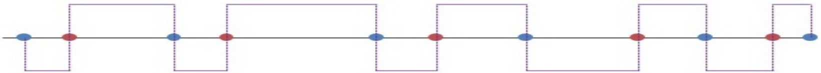

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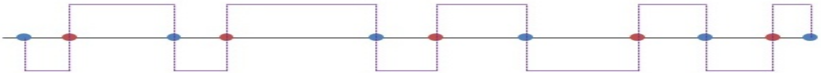

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Lemma (Mitkovski, R.)

The sequence Λ uniquely defines Φ if and only if the set $\Lambda \cup \{a\}$ is a uniqueness set for $K^2_{\Phi^2}$, for all $a \in \mathbb{R} \setminus \Lambda$.

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Lemma (Mitkovski, R.)

If Φ is a Weyl inner function, then Λ does not uniquely defines Φ iff for some MIF Θ with $\sigma(\Theta) = \Lambda$, there is an non-zero $f \in \ker^2_{\Phi^2\Theta}$ and $f(a) = 0$ for some $a \in \mathbb{R} \setminus \Lambda$.

What it means: Schrödinger case

Let L be a Schrödinger operator, with potential q. Let $\Lambda = \{\lambda_n\}_n$ be a sequence such that $\lambda_n \in \sigma(L, \alpha_n, D)$. Then, this data uniquely identifies q iff $\Lambda \cup \{a\}$ is a uniqueness set for $K^2_{\Phi^2}$, for any point $a \in \mathbb{R} \setminus \Lambda$.
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Question: What can we say in the case of MIFs corresponding to a different class of operators?

Some more questions

Makarov and Poltoratski

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In fact the other MIF Φ_2 can be written as

$$\Phi_2 = e^{ic}\Phi\frac{\bar{G}}{G},$$

where c is some real constant.

Question: In which cases can we 'close the gap'?

Thank you!