Fokas Method applied on Boundary Value Problems for axisymmetric potentials.

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- Solving Riemann-Hilbert problems, we can reconstruct W everywhere and then get u.

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- $L_\alpha(u) = 0 \Leftrightarrow L_{2-\alpha}(x^{\alpha-1}u) = 0$.

$\alpha = -2m,\ m \in \mathbb{N}$.

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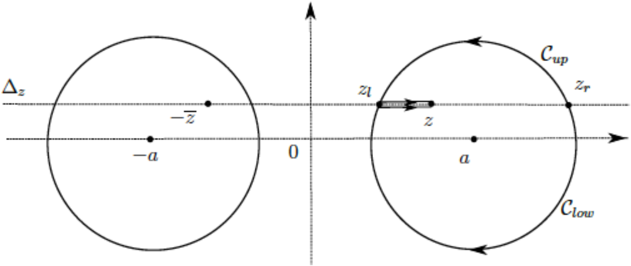

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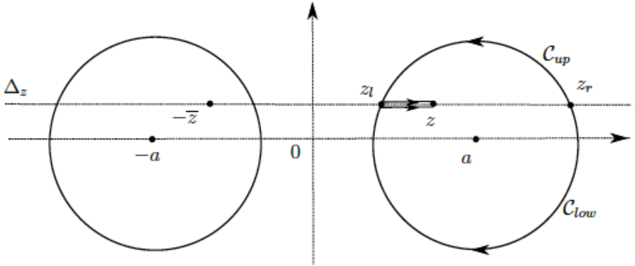

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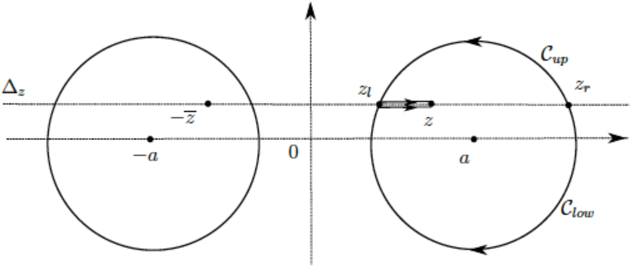

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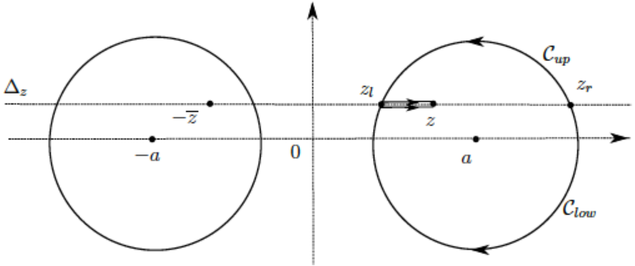

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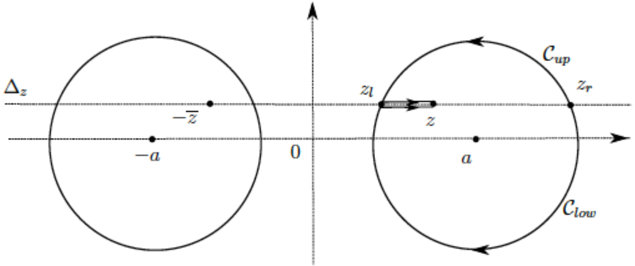
- 

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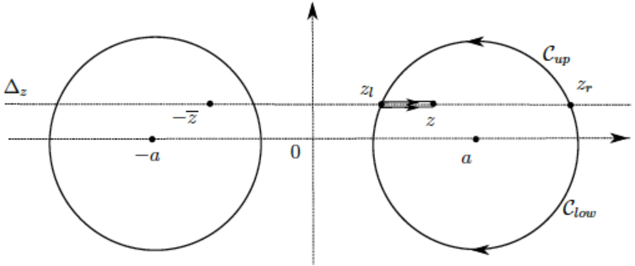

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- $\phi \sim_{k\to+\infty} \frac{u(z)-u(z_r)}{k^{2m-1}}$

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- $$u(z) - u(z_r) = 2\,\mathrm{Re}\, a_r - \frac{1}{\pi}\mathrm{Im}\int_{(z,z_r)} \widetilde{J}(z,k')dk'$$

Computation of the residue a_r of $\widetilde{\phi}$ in z_r

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$$w(z',k) = (k + \overline{z}')^{-m-1}((k - iy')u_t + ix'u_n)\tau^{-1}(z')dz'$$

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- m integrations by parts give $\widetilde{\phi}_{z_r,-\overline{z}_r}$, hence a_r.

Let u be a solution of the equation $\Delta u + \alpha x^{-1}\partial_x u = 0$, $\alpha = -2m$, $m \in \mathbb{N}$, in the domain $\mathcal{D}$ with smooth tangential and (outer) normal derivatives u_t and u_n on the boundary $\mathcal{C}$.

$$u(z) = -\frac{1}{\pi}\mathrm{Im} \int_{(z,z_r)} \left((k-z)(k+\bar{z})\right)^m J(z,k)dk + 2\mathrm{Re}\, a_r + u(z_r), \tag{1}$$

where the quantity a_r can be explicitly computed in terms of the tangential derivatives along $\mathcal{C}$ of u_t and u_n, up to order $m-1$, at z_r. The function $J(z,k)$ is given by

$$J(z,k) = -\int_{\mathcal{C}} W(z',k),$$

where $W(z,k)$ is the differential form

$$W(z,k) = \left((k-z)(k+\bar{z})\right)^{-m-1} \left((k+\bar{z})u_z(z)dz + (k-z)u_{\bar{z}}(z)d\bar{z}\right) \tag{2}$$

$$= \left((k-z)(k+\bar{z})\right)^{-m-1} \left((k-iy)u_t(z) + ixu_n(z)\right) ds, \tag{3}$$

with $z = x + iy$ and ds the length element on $\mathcal{C}$.

The case $\alpha = -1$.

- $W(z,k) = \dfrac{(k+\overline{z})u_z dz + (k-z)u_{\overline{z}} d\overline{z}}{\sqrt{(k-z)(k+\overline{z})}}$

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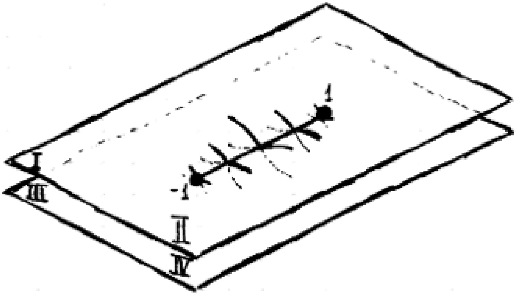

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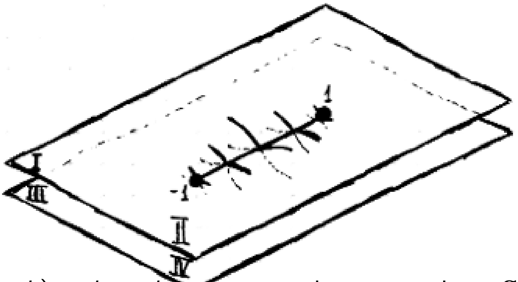

- $\lambda_1(z,k) \sim k$ as $k \to \infty_1$ on the upper sheet $\mathbb{S}_{z,1}$
- $\lambda_2(z,k) \sim -k$ as $k \to \infty_2$ on the upper sheet $\mathbb{S}_{z,2}$

Integration of the form.

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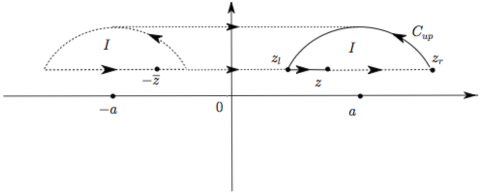

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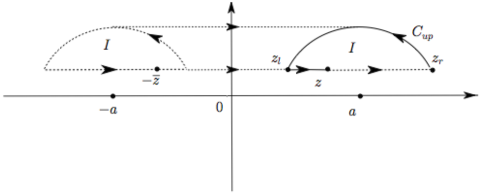

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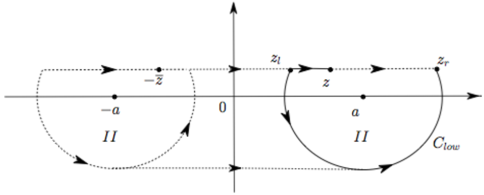

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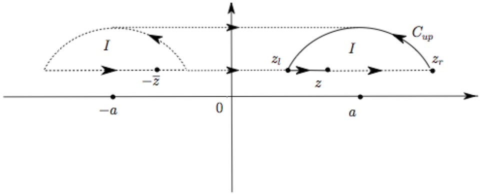

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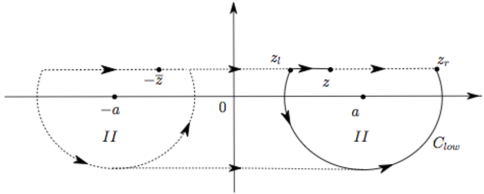
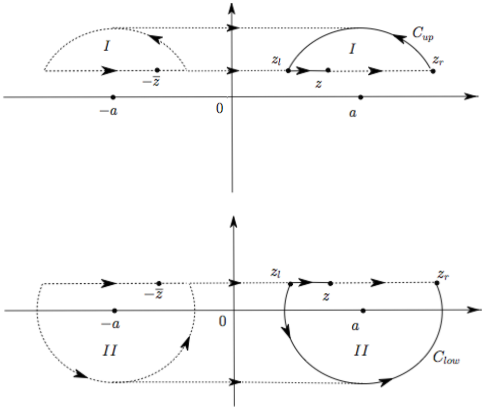
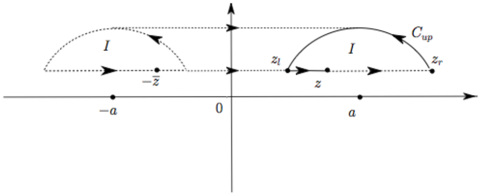
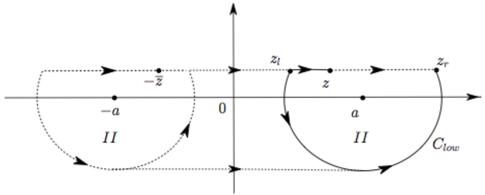


- $\phi(z,\infty_1) = -\phi(z,\infty_2)$.

▶


C_{up}
I
I
z_l
z_r
z
$-\overline{z}$
$-a$
0
a
z_l
z_r
z
$-\overline{z}$
$-a$
0
a
II
II
C_{low}

- 

- 

$$\phi(z,k) = \frac{1}{4i\pi}\int_{\mathcal{C}_{up}\cup -\bar{\mathcal{C}}_{up}} J(z,k')\left(\frac{\lambda(z,k)}{\lambda_1(z,k')}+1\right)\frac{dk'}{k'-k}$$

$$+\frac{1}{4i\pi}\int_{\mathcal{C}_{low}\cup -\bar{\mathcal{C}}_{low}} J(z,k')\left(\frac{\lambda(z,k)}{\lambda_2(z,k')}+1\right)\frac{dk'}{k'-k}, \quad (4)$$

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- f is real valued , $f(z) = g(z) + \overline{g}(1/z)$, $g \in \mathbb{H}(\mathbb{D})$, $\overline{g}(1/z) \in \mathbb{H}(\mathbb{C} \setminus \mathbb{D})$.

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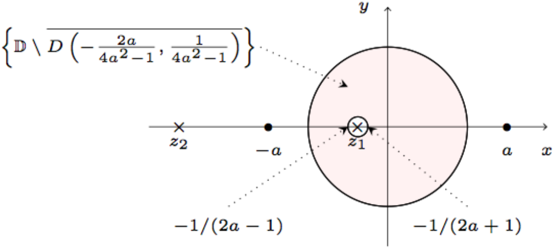

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- Let $\xi = -(k+a)^{-1}$, we get

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$$\forall \xi \in \mathbb{D} \setminus \overline{D\left(-\frac{2a}{4a^2-1}, \frac{1}{4a^2-1}\right)}.$$

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- Thank you for your attention !