Matrix Weights: On the Way to the Linear Bound

Sandra Pott

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Calderòn-Zygmund Operators Standard kernels CZOs asociated to a kernel T1 Theorem

Haar multipliers, martingale transforms, and

dyadic shifts

The Representation Theorem of Hytönen

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces

Calderòn-Zygmund Operators

Standard kernels CZOs asociated to a kernel

T1 Theorem

Haar multipliers,

martingale

 $transforms, \ and$

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The Representation Theorem of Hytönen

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces Let $\Delta = \{(x, x) : x \in \mathbb{R}\}$ be the diagonal of $\mathbb{R} \times \mathbb{R}$. $K : \mathbb{R} \times \mathbb{R} \setminus \Delta \to \mathbb{C}$ is a standard Calderón-Zygmund kernel if there exists $\delta > 0$ such that

$$|K(x,y)| \le \frac{C}{|x-y|},$$

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \le C_{\delta} \frac{|y-z|}{|x-y|^{1+\delta}},$$

for $x, y, z \in \mathbb{R}$ with |x - y| > 2|y - z|.

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Calderòn-Zygmund Operators Standard kernels

CZOs asociated ▷ to a kernel

T1 Theorem

Haar multipliers,

martingale

transforms, and

dyadic shifts

The Representation Theorem of Hytönen

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces An operator T is called a Calderòn-Zygmund operator on $\mathbb R$ associated to K, if

$$Tf(x) = \int_{\mathbb{R}} K(x, y) f(y) \, dy, \qquad x \notin \operatorname{supp} f.$$

Standard kernels CZOs asociated ▷ to a kernel

T1 Theorem

Haar multipliers,

martingale

transforms, and

dyadic shifts

The Representation Theorem of Hytönen

Weighted norm inequalities

Matrix weights

An Application

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Example: The Hilbert transform

$$Hf(x) = p.v.\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x - y} dy.$$

T1 Theorem

Calderòn-Zygmund Operators

Standard kernels CZOs asociated to a kernel

▷ T1 Theorem Haar multipliers, martingale transforms, and dyadic shifts

The Representation Theorem of Hytönen

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces **Theorem 1 (David and Journé, 1984)** Let *T* be a CZO associated to a standard kernel. Then the following are equivalent:

A T defines a bounded linear operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$

T1 Theorem

Calderòn-Zygmund Operators

Standard kernels CZOs asociated to a kernel

▷ T1 Theorem Haar multipliers.

martingale

transforms, and

dyadic shifts

The Representation Theorem of Hytönen

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces **Theorem 2 (David and Journé, 1984)** Let *T* be a CZO associated to a standard kernel. Then the following are equivalent:

A T defines a bounded linear operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ B The following three conditions hold:

(i) $|\langle T\chi_I, \chi_I \rangle| \lesssim |I|$ for all intervals $I \subset \mathbb{R}$ (weak boundedness) (ii) $T1 \in BMO(\mathbb{R})$ (iii) $T^*1 \in BMO(\mathbb{R})$

Idea: A CZ operator consists of parts which are controlled by the BMO norm of T1, T^*1 (so-called paraproducts) and a part which is "close to the diagonal".

Standard kernels CZOs asociated to a kernel

T1 Theorem Haar multipliers,

martingale transforms. and

 \triangleright dyadic shifts

The Representation Theorem of Hytönen

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces Let $f \in L^2(\mathbb{R})$,

$$f(x) = \sum_{I \in \mathcal{D}} h_I(x)\hat{f}(I).$$

For a sequence (α_I) , define the Haar multiplier

$$T_{\alpha}f = \sum_{I \in \mathcal{D}} h_I \alpha_I \hat{f}(I).$$

If $(\alpha_I) = (\varepsilon_I) \in \{-1, 1\}^{\mathcal{D}}$, T_{ε} is a dyadic martingale transform.

Standard kernels CZOs asociated to a kernel

T1 Theorem Haar multipliers, martingale

transforms, and

dyadic shifts

The Representation Theorem of Hytönen

Weighted norm inequalities

Matrix weights

An Application

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If $(\alpha_I) = (\varepsilon_I) \in \{-1, 1\}^{\mathcal{D}}$, T_{ε} is a dyadic martingale transform. A dyadic shift S with parameters (m,n) is given by

$$Sf = \sum_{L \in \mathcal{D}} \sum_{J \in \mathcal{D}^m(L)} h_J(x) \sum_{I \in \mathcal{D}^n(L)} c_{I,J} \hat{f}(I).$$

Standard kernels CZOs asociated to a kernel

T1 Theorem Haar multipliers, martingale

transforms, and

dyadic shifts

The Representation Theorem of Hytönen

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces

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Then $k = \max\{n, m\} + 1$ is called the complexity of S.

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T1 Theorem Haar multipliers, martingale transforms, and dyadic shifts

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Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces **Theorem 3 (Hytönen 2010)** Let T be a CZO on \mathbb{R} with the standard kernel estimates, the weak boundedness property $|\langle T\chi_I, \chi_I \rangle| \leq C|I|$ for all intervals I, and the conditions $T(1) = b_1, T^*(1) = b_2 \in BMO(\mathbb{R})$. Then for $f, g \in C_c^1(\mathbb{R})$,

$$\langle Tf,g\rangle_{L^{2}(\mathbb{R}),L^{2}(\mathbb{R})} = C \cdot \mathbb{E}_{\Omega} \langle \left(\Pi_{b_{1}}^{\omega} + (\Pi_{b_{2}}^{\omega})^{*} + \sum_{m,n=0}^{\infty} \tau(m,n) S_{\omega}^{mn} \right) f,g\rangle_{L^{2}(\mathbb{R}),L^{2}(\mathbb{R})},$$

where C depends only on the constants in the standard estimates of the kernel K and the weak boundedness property, S^{mn}_{ω} is a dyadic Haar shift of parameters (m, n) on the dyadic system \mathcal{D}^{ω} and $\tau(m, n) \lesssim P(\max\{m, n\})2^{-\delta \max\{m, n\}}$, with P a polynomial.

Calderòn-Zygmund
Operators

Weighted norm \triangleright inequalities A_p -weights and the Hunt-Muckenhoupt-Wheeden Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen The Bellmann function proof of Treil I Proof II Proof III Proof IV Matrix weights An Application

Calderon-Zygmund Operators on UMD spaces

Weighted norm inequalities

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 A_p -weights and the Hunt-Muckenhoupt-Wheeden ▷ Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen The Bellmann function proof of Treil I Proof II Proof III Proof IV Matrix weights An Application Calderon-Zygmund Operators on UMD

spaces

Let w be a weight function on $\mathbb{R}, \ 1 We say that <math display="inline">w$ is an $A_p\text{-weight},$ if

$$[w]_{A_p} = \sup_{I \text{ interval}} \langle w \rangle_I \langle w^{-\frac{1}{p-1}} \rangle_I^{p-1} < \infty.$$

9 / 36

Weighted norm inequalities

 A_p -weights and the Hunt-Muckenhoupt-Wheeden > Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen The Bellmann function proof of Treil I Proof II Proof III

Proof IV

Matrix weights

An Application

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Theorem 5 (Hunt, Muckenhoupt, Wheeden 1974) Let 1 and <math>w be a weight on \mathbb{R} . Then the Hilbert transform $H: L^p_w(\mathbb{R}) \to L^p_w(\mathbb{R})$ is bounded, if and only if w is an A_p -weight.

What is the sharp bound of the norm of $||T||_{L^p_w \to L^p_w}$ in terms of $[w]_{A_p}$?

Weighted norm inequalities

 A_p -weights and the Hunt-Muckenhoupt-Wheeden > Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen The Bellmann function proof of Treil I Proof II Proof III

Proof IV

Matrix weights

An Application

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What is the sharp bound of the norm of $||T||_{L^p_w \to L^p_w}$ in terms of $[w]_{A_p}$?

$$A_2$$
-Conjecture: $||T||_{L^2_w \to L^2_w} \le C_T[w]_{A_2}$.

Calderòn-Zygmund	\Box Maximal function (Dualday 1002)
Operators	\Box Maximal function (Buckley 1993)
Weighted norm inequalities	 Dyadic square function (Hukovic, Nazarov, Treil, Volberg 1999)
Hunt-Muckenhoupt- Wheeden	Dyadic martingale transforms (Wittwer 2000)
Theorem Some History of	\Box 3/2 bound for the Hilbert transform (Petermichl, Pott 2001)
Sharp A₂ ▷ Bounds	Hilbert transform (Petermichl 2007)
The Representation Theorem of Hytönen	Paraproducts (Beznosova 2008)
The Bellmann function proof of Treil I	$\Box T(1) \text{ Theorem for two-weighted dyadic shifts (Nazarov, Treil Volberg 2008)}$
Proof II	Uilbort transform again (Lacov, Deguara, Determicht 2000)
Proof III	\Box midert transform again (Lacey, Reguera, Petermichi 2009)
Proof IV	all Calderon-Zygmund operators (Hytönen 2010)
Matrix weights	□ all CZOs (Treil 2011)
An Application	$\Box A_2 - A_{\infty}$ bound for all CZOs (Hytönen, Perez 2013)
Calderon-Zygmund Operators on UMD spaces	

Calderòn-Zygmund Operators Weighted norm	Maximal function (Buckley 1993) Dyadic square function (Hukovic, Nazarov, Treil, Volbe	rg
inequalities	1999)	0
A _p -weights and the Hunt-Muckenhoupt- Wheeden	Dyadic martingale transforms (Wittwer 2000)	
Theorem Some History of	3/2 bound for the Hilbert transform (Petermichl, Pott 2	2001)
Sharp A_2 \triangleright Bounds	Hilbert transform (Petermichl 2007)	
The Representation	Paraproducts (Beznosova 2008)	
The Bellmann function proof of Treil I	T(1) Theorem for two-weighted dyadic shifts (Nazarov, Volberg 2008)	Treil,
Proof II	Volberg 2000)	
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Matrix weights	all CZOs (Treil 2011)
An Application	$A_2 - A_{\infty}$ bound for all CZOs (Hytönen Perez 2013)
Calderon-Zygmund Operators on UMD	112 11∞ bound for an 0200 (Figuren, Forez 2010)
spaces	

Calderòn-Zygmund Operators Weighted norm inequalities Ap-weights and the Hunt-Muckenhoupt- Wheeden Theorem Some History of Sharp A2 ▷ Bounds The Representation The Representation The Bellmann function proof of Treil I Proof II Proof II Proof II Proof IV Matrix weights An Application	 Maximal fund Dyadic square 1999) Dyadic marti $3/2$ bound for Hilbert trans Paraproducts T(1) Theore Volberg 2008 Hilbert trans all Calderon- all CZOs (Tr A_2-A_{∞} bour 	ction (Buckley 1993) e function (Hukovic, Nazarov, Treil, Volberg ngale transforms (Wittwer 2000) or the Hilbert transform (Petermichl, Pott 2001) form (Petermichl 2007) 6 (Beznosova 2008) m for two-weighted dyadic shifts (Nazarov, Treil, 3) form again (Lacey, Reguera, Petermichl 2009) Zygmund operators (Hytönen 2010) eil 2011) of for all CZOs (Hytönen, Perez 2013)
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Calderon-Zygmund Operators on UMD spaces		

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An Application	A_2 - A_∞ bound for all CZOs (Hytönen, Perez 2013)
Calderon-Zygmund	
Operators on UMD	
spaces	

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An Application	A_2 - A_∞ bound for all CZOs (Hytönen, Perez 2013)
Calderon-Zygmund Operators on UMD spaces	

Calderon-Zygmund Maximal function (Buckley 1993) Operators Weighted norm inequalities 1999) A_p -weights and the Hunt-Muckenhoupt- \square Wheeden Theorem Some History of Sharp A_2 \triangleright Bounds The Representation \square Theorem of Hytönen The Bellmann function proof of Volberg 2008) Treil I Proof II Proof III Proof IV \square Matrix weights all CZOs (Treil 2011) \square An Application Calderon-Zygmund Operators on UMD spaces

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Weighted norm inequalities A_n -weights and the Hunt-Muckenhoupt-Wheeden Theorem Some History of Sharp A_2 Bounds The Representation Theorem of ▷ Hytönen The Bellmann function proof of Treil I Proof II Proof III Proof IV

Matrix weights

An Application

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$$\langle Tf,g\rangle_{L^{2}(\mathbb{R}),L^{2}(\mathbb{R})} = C \cdot \mathbb{E}_{\Omega} \langle \left(\Pi_{b_{1}}^{\omega} + (\Pi_{b_{2}}^{\omega})^{*} + \sum_{m,n=0}^{\infty} \tau(m,n) S_{\omega}^{mn} \right) f,g\rangle_{L^{2}(\mathbb{R}),L^{2}(\mathbb{R})},$$

where C depends only on the constants in the standard estimates of the kernel K and the weak boundedness property, S^{mn}_{ω} is a dyadic Haar shift of parameters (m, n) on the dyadic system \mathcal{D}^{ω} and $\tau(m, n) \lesssim P(\max\{m, n\})2^{-\delta \max\{m, n\}}$, with P a polynomial.

Calderòn-Zygmund Operators Weighted norm inequalities A_p -weights and the Hunt-Muckenhoupt-Wheeden Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen The Bellmann function proof of ▷ Treil I Proof II Proof III Proof IV Matrix weights An Application Calderon-Zygmund Operators on UMD spaces

We assume the linear bound for martingale transforms:

$$\sup_{\varepsilon} \langle T_{\varepsilon} f, g \rangle = \sum_{I} |\langle f \rangle_{I^{+}} - \langle f \rangle_{I^{-}} || \langle g \rangle_{I^{+}} - \langle g \rangle_{I^{-}} |\cdot|I|$$

 $\leq C[w]_{A_2} \|f\|_{L^2_w} \|g\|_{L^2_{w^{-1}}}.$

Weighted norm inequalities

 A_p -weights and the Hunt-Muckenhoupt-Wheeden Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen

The Bellmann function proof of ▷ Treil I

Proof II

Proof II

Proof III

Proof IV

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces We assume the linear bound for martingale transforms:

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$$\leq C[w]_{A_{2}} ||f||_{L^{2}_{w}} ||g||_{L^{2}_{w^{-1}}}.$$

Define
$$\mathcal{B}_X(\mathbf{f},\mathbf{F},\mathbf{U},\mathbf{g},\mathbf{G},\mathbf{V}):=$$

$$= |I_0|^{-1} \sup \sum_{I \subseteq I_0} |\langle f \rangle_{I^+} - \langle f \rangle_{I^-} ||\langle g \rangle_{I^+} - \langle g \rangle_{I^-}| \cdot |I|,$$

with the sup over all functions f, g and A_2 weights w such that

$$\langle f \rangle_{I_0} = \mathbf{f}, \quad \left\langle w f^2 \right\rangle_{I_0} = \mathbf{F}, \quad \langle g \rangle_{I_0} = \mathbf{g}, \quad \left\langle w^{-1} g^2 \right\rangle_{I_0} = \mathbf{G},$$

$$\sup_{I \subset I_0} \langle w \rangle_I \langle w^{-1} \rangle_I \le X, \quad \langle w \rangle_{I_0} = \mathbf{U}, \quad \langle w^{-1} \rangle_{I_0} = \mathbf{V}.$$

Calderòn-Zygmund Operators

Weighted norm inequalities A_p -weights and the Hunt-Muckenhoupt-Wheeden Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen The Bellmann function proof of Treil I ▷ Proof II Proof III Proof IV Matrix weights An Application Calderon-Zygmund

Operators on UMD spaces

By the assumption of the linear bound,

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\mathcal{B}_X(\mathbf{f}, \mathbf{F}, \mathbf{U}, \mathbf{g}, \mathbf{G}, \mathbf{V}) \le C X \mathbf{F}^{1/2} \mathbf{G}^{1/2}. (1)
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Calderòn-Zygmund Operators

Weighted norm inequalities A_p -weights and the Hunt-Muckenhoupt-Wheeden Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen The Bellmann function proof of Treil I

▷ Proof II

Proof III

Proof IV

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces By the assumption of the linear bound,

$$\mathcal{B}_{X}(\mathbf{f}, \mathbf{F}, \mathbf{U}, \mathbf{g}, \mathbf{G}, \mathbf{V}) \leq CX\mathbf{F}^{1/2}\mathbf{G}^{1/2}.$$
(1)
Write $(\mathbf{f}, \mathbf{F}, \mathbf{U}, \mathbf{g}, \mathbf{G}, \mathbf{V}) = \mathbf{A}.$ For $\mathbf{A} = \frac{1}{2}(\mathbf{A}^{+} + \mathbf{A}^{-}),$
 $\frac{1}{2}(\mathcal{B}_{X}(\mathbf{A}^{+}) + \mathcal{B}_{X}(\mathbf{A}^{-})) + |\mathbf{f}^{+} - \mathbf{f}^{-}||\mathbf{g}^{+} - \mathbf{g}^{-}|$
 $\leq \mathcal{B}_{X}(\mathbf{A})$ (2)

Calderòn-Zygmund Operators

Weighted norm inequalities A_p -weights and the Hunt-Muckenhoupt-Wheeden Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen The Bellmann function proof of Treil I \triangleright Proof II Proof III

Proof IV

Matrix weights

An Application

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$$\mathcal{B}_{X}(\mathbf{f}, \mathbf{F}, \mathbf{U}, \mathbf{g}, \mathbf{G}, \mathbf{V}) \leq CX\mathbf{F}^{1/2}\mathbf{G}^{1/2}.$$
(1)
Write $(\mathbf{f}, \mathbf{F}, \mathbf{U}, \mathbf{g}, \mathbf{G}, \mathbf{V}) = \mathbf{A}.$ For $\mathbf{A} = \frac{1}{2}(\mathbf{A}^{+} + \mathbf{A}^{-}),$
 $\frac{1}{2}(\mathcal{B}_{X}(\mathbf{A}^{+}) + \mathcal{B}_{X}(\mathbf{A}^{-})) + |\mathbf{f}^{+} - \mathbf{f}^{-}||\mathbf{g}^{+} - \mathbf{g}^{-}|$
 $\leq \mathcal{B}_{X}(\mathbf{A})$ (2)

Moreover, each function \mathcal{B} with the boundedness property (1) und the convexity property (2) proves the linear bound for martingale transforms.

Calderòn-Zygmund Operators

Weighted norm inequalities

 A_p -weights and the Hunt-Muckenhoupt-Wheeden Theorem Some History of Sharp A_2 Bounds The Representation

Theorem of Hytönen

The Bellmann

function proof of

Treil I

Proof II

▷ Proof III

Proof IV

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces For a dyadic shift of complexity n, control instead

$$|I_0|^{-1} \sup \sum_{L \subseteq I_0} \sum_{I,J \in \mathcal{D}^n(L)} \frac{|\langle f \rangle_I - \langle f \rangle_L|}{2^n} \frac{|\langle g \rangle_J - \langle g \rangle_L|}{2^n} \cdot |L|.$$

For some bounded real sequence $(\alpha_I^L)_{I \in \mathcal{D}^n(L)}$ of average zero,

$$\sum_{I,J\in\mathcal{D}^{n}(L)} \frac{|\langle f\rangle_{I} - \langle f\rangle_{L}|}{2^{n}} \frac{|\langle g\rangle_{J} - \langle g\rangle_{L}|}{2^{n}} \cdot |L|$$
$$\lesssim \sum_{I,J\in\mathcal{D}^{n}(L)} \alpha_{I}^{L} \alpha_{J}^{L} \frac{\langle f\rangle_{I} - \langle f\rangle_{L}}{2^{n}} \frac{\langle g\rangle_{J} - \langle g\rangle_{L}}{2^{n}} \cdot |L|.$$

Proof IV

Calderòn-Zygmund Operators

Weighted norm inequalities

 A_p -weights and the Hunt-Muckenhoupt-Wheeden Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen The Bellmann function proof of

Treil I

Proof II

Proof III

▷ Proof IV

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces But for each such sequence,

$$\sum_{I,J\in\mathcal{D}^{n}(L)} \alpha_{I}^{L} \alpha_{J}^{L} \frac{\langle f \rangle_{I} - \langle f \rangle_{L}}{2^{n}} \frac{\langle g \rangle_{J} - \langle g \rangle_{L}}{2^{n}} \cdot |L| =$$

$$\sum_{I\in\mathcal{D}^{n}(L)} \frac{1 + \alpha_{I}^{L}}{2^{n}} (\langle f \rangle_{I} - \langle f \rangle_{L}) \left(\sum_{J\in\mathcal{D}^{n}(L)} \frac{1 + \alpha_{J}^{L}}{2^{n}} (\langle g \rangle_{J} - \langle g \rangle_{L}) \right) |L|$$

$$\lesssim \mathcal{B}_{X}(A_{L}) - \frac{1}{2^{n}} \sum_{I\in\mathcal{D}^{n}(L)} \mathcal{B}_{X}(A_{I}).$$

Proof IV

Calderòn-Zygmund Operators

Weighted norm inequalities A_p -weights and the

Hunt-Muckenhoupt-Wheeden Theorem Some History of Sharp A_2 Bounds The Representation Theorem of Hytönen The Bellmann

function proof of

Treil I

Proof II

Proof III

▷ Proof IV

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces But for each such sequence,

$$\sum_{I,J\in\mathcal{D}^{n}(L)} \alpha_{I}^{L} \alpha_{J}^{L} \frac{\langle f \rangle_{I} - \langle f \rangle_{L}}{2^{n}} \frac{\langle g \rangle_{J} - \langle g \rangle_{L}}{2^{n}} \cdot |L| =$$

$$\left(\sum_{I\in\mathcal{D}^{n}(L)} \frac{1 + \alpha_{I}^{L}}{2^{n}} (\langle f \rangle_{I} - \langle f \rangle_{L}) \right) \left(\sum_{J\in\mathcal{D}^{n}(L)} \frac{1 + \alpha_{J}^{L}}{2^{n}} (\langle g \rangle_{J} - \langle g \rangle_{L}) \right) |L|$$

$$\lesssim \mathcal{B}_{X}(A_{L}) - \frac{1}{2^{n}} \sum_{I\in\mathcal{D}^{n}(L)} \mathcal{B}_{X}(A_{I}).$$

Adding $L \in \mathcal{D}(I_0)$ gives a telescopic sum and the desired bound.

Weighted norm inequalities

▷ Matrix weights

Matrix weights Matrix Haar

multipliers

The reduction to Haar multipliers in

the matrix case

Remaining obstacles

An Application

Calderon-Zygmund Operators on UMD spaces

Matrix weights

Matrix weights

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

Matrix weights Matrix Haar

multipliers

The reduction to Haar multipliers in

the matrix case

Remaining obstacles

An Application

Calderon-Zygmund Operators on UMD spaces A matrix weight is a measurable function $W : \mathbb{R} \to \operatorname{Mat}(d \times d, \mathbb{C})$ which is positive and invertible almost everywhere.

Matrix weights

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

Matrix weights
 Matrix Haar
 multipliers
 The reduction to
 Haar multipliers in

the matrix case Remaining obstacles

An Application

Calderon-Zygmund Operators on UMD spaces A matrix weight is a measurable function

 $W : \mathbb{R} \to \operatorname{Mat}(d \times d, \mathbb{C})$ which is positive and invertible almost everywhere. W satisfies the matrix A_2 Muckenhoupt condition, if

$$[W]_{A_2} := \sup_{I} \left\| \left\langle W(t) \right\rangle^{1/2} \left\langle W^{-1}(t) \right\rangle^{1/2} \right\|^2 < \infty.$$

Theorem 9 (Treil, Volberg, Christ, Goldberg) Let W be a matrix A_p weight, 1 . Then for each CZO <math>T associated to a standard kernel,

$$T: L^p_W(\mathbb{R}, \mathbb{C}^d) \to L^p_W(\mathbb{R}, \mathbb{C}^d)$$
 is bounded.

Matrix weights

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

Matrix weights
 Matrix Haar
 multipliers
 The reduction to
 Haar multipliers in
 the matrix case

Remaining obstacles

An Application

Calderon-Zygmund Operators on UMD spaces A matrix weight is a measurable function

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Theorem 10 (Treil, Volberg, Christ, Goldberg) Let W be a matrix A_p weight, 1 . Then for each CZO <math>T associated to a standard kernel,

$$T: L^p_W(\mathbb{R}, \mathbb{C}^d) \to L^p_W(\mathbb{R}, \mathbb{C}^d)$$
 is bounded.

What is the sharp bound for $||T||_{L^2_W \to L^2_W}$ in terms of $[W]_{A_2}$?

Weighted norm inequalities

Matrix weights

Matrix weights

Matrix Haar ▷ multipliers

The reduction to Haar multipliers in the matrix case

Remaining obstacles

An Application

Calderon-Zygmund Operators on UMD spaces Now let $f\in L^2(\mathbb{R},\mathbb{C}^d)$,

$$f(x) = \sum_{I \in \mathcal{D}} h_I(x)\hat{f}(I).$$

For a sequence (σ_I) of $d \times d$ matrices, define the Haar multiplier

$$T_{\sigma}f = \sum_{I \in \mathcal{D}} h_I \sigma_I \hat{f}(I).$$

Weighted norm inequalities

Matrix weights

Matrix weights

Matrix Haar ▷ multipliers

The reduction to Haar multipliers in the matrix case

Remaining obstacles

An Application

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For a sequence (σ_I) of $d \times d$ matrices, define the Haar multiplier

$$T_{\sigma}f = \sum_{I \in \mathcal{D}} h_I \sigma_I \hat{f}(I).$$

It is easy to see that in order for T_{σ} to be bounded on $L^2_W(\mathbb{R}, \mathbb{C}^d)$, one needs

$$\|\sigma\|_{\infty,W} = \sup_{I} \|\langle W \rangle_{I}^{1/2} \sigma_{I} \langle W \rangle_{I}^{-1/2} \| < \infty.$$

Weighted norm inequalities

Matrix weights Matrix weights Matrix Haar multipliers

The reduction to Haar multipliers ▷ in the matrix case Remaining obstacles

An Application

Calderon-Zygmund Operators on UMD spaces Let $N:[1,\infty) \to [1,\infty)$ be given by

$$N(X) = \sup \|T_{\sigma}\|_{L^2_W(\mathbb{R},\mathbb{C}^d) \to L^2_W(\mathbb{R},\mathbb{C}^d)},$$

where the supremum is taken over all $d \times d$ matrix A_2 weights W with $[W]_{A_2} \leq X$ and T_{σ} with $\|\sigma\|_{\infty,W} \leq 1$.

Theorem 11 (Pott, Stoica 2015) Let W be a $d \times d$ matrix A_2 weight. Let T be a Calderón-Zygmund operator on \mathbb{R} associated to a standard kernel. Suppose that $|\langle T\chi_I, \chi_I \rangle| \leq C|I|$ for all intervals I, and $T(1) = T^*(1) = 0$. Then

$$||T||_{L^2_W(\mathbb{R},\mathbb{C}^d)\to L^2_W(\mathbb{R},\mathbb{C}^d)} \le C \cdot dN([W]_{A_2}),$$

where C depends only on the constants in the standard estimates and the weak boundedness property.

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

Matrix weights Matrix Haar multipliers The reduction to Haar multipliers in the matrix case Remaining ▷ obstacles

An Application

Calderon-Zygmund Operators on UMD spaces To prove the linear bound for matrix weights for all CZOs, it therefore remains to prove it for Haar multipliers and for paraproducts. [Bickel, Petermichl, Wick 2014] prove a $[W]_{A_2}^{3/2} \log[W]_{A_2}$ bound for Haar multipliers, listing also a strategy to prove the linear bound:

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

Matrix weights Matrix Haar multipliers The reduction to Haar multipliers in the matrix case Remaining ▷ obstacles

An Application

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1. prove a matrix version of the Weighted Carleson Embedding Theorem for matrix weights

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

Matrix weights Matrix Haar multipliers The reduction to Haar multipliers in the matrix case Remaining ▷ obstacles

An Application

Calderon-Zygmund Operators on UMD spaces To prove the linear bound for matrix weights for all CZOs, it therefore remains to prove it for Haar multipliers and for paraproducts. [Bickel, Petermichl, Wick 2014] prove a $[W]_{A_2}^{3/2} \log[W]_{A_2}$ bound for Haar multipliers, listing also a strategy to prove the linear bound:

1. prove a matrix version of the Weighted Carleson Embedding Theorem for matrix weights -just done [Culiuc, Treil 2015]

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

Matrix weights Matrix Haar multipliers The reduction to Haar multipliers in the matrix case Remaining ▷ obstacles

An Application

Calderon-Zygmund Operators on UMD spaces To prove the linear bound for matrix weights for all CZOs, it therefore remains to prove it for Haar multipliers and for paraproducts. [Bickel, Petermichl, Wick 2014] prove a $[W]_{A_2}^{3/2} \log[W]_{A_2}$ bound for Haar multipliers, listing also a strategy to prove the linear bound:

 prove a matrix version of the Weighted Carleson Embedding Theorem for matrix weights -just done [Culiuc, Treil 2015]
 verify the test condition

 $\sum_{I\subseteq J} \hat{W}(I) \langle W\rangle_I^{-1} \hat{W}(I) \lesssim [W]_{A_2} W(J) \text{ - this gives the linear }$

bound for the weighted square function

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

Matrix weights Matrix Haar multipliers The reduction to Haar multipliers in the matrix case Remaining ▷ obstacles

An Application

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bound for the weighted square function

3. prove the linear bound for Haar multipliers.

Weighted norm inequalities

Matrix weights

Matrix weights Matrix Haar multipliers The reduction to Haar multipliers in the matrix case Remaining ▷ obstacles

An Application

Calderon-Zygmund Operators on UMD spaces To prove the linear bound for matrix weights for all CZOs, it therefore remains to prove it for Haar multipliers and for paraproducts. [Bickel, Petermichl, Wick 2014] prove a $[W]_{A_2}^{3/2} \log[W]_{A_2}$ bound for Haar multipliers, listing also a strategy to prove the linear bound:

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3. prove the linear bound for Haar multipliers.

[Isralowitz, Kwon, Pott 2015] prove the $[W]_{A_2}^{3/2} \log[W]_{A_2}$ bound for paraproducts.

Weighted norm inequalities

Matrix weights

An Application An application of the matrix linear bound Proof I

Proof II

Proof III

Proof IV

Calderon-Zygmund Operators on UMD spaces

An Application

Weighted norm inequalities

Matrix weights

An Application

An application of the matrix linear ▷ bound Proof I Proof II Proof III Proof IV Calderon-Zygmund Operators on UMD

spaces

Suppose the A_2 Conjecture holds for matrix weights. Let T be a CZO which is bounded on $L^2(\mathbb{R})$. Then

$$\|[T,b]\|_{L^2_w \to L^2_w} \lesssim C_T \left(\frac{1}{R} [w]_{A_2} + R[w]_{A_2} [w]^2_{A_\infty}\right) \|b\|_{BMO} \quad (R>0).$$

In particular, choosing $R = [w]_{A_{\infty}}^{-1}$,

$$||[T,b]||_{L^2_w \to L^2_w} \lesssim C_T[w]_{A_2}[w]_{A_\infty} ||b||_{BMO}$$

Moreover,

 $\| [\dots [T, b], \dots, b] \|_{L^2_w \to L^2_w} \lesssim C_T[w]_{A_2} \left([w]_{A_\infty} + [w^{-1}]_{A_\infty} \right)^n \| b \|_{BMO}^n,$

(compare with the results in Chung, Pereyra, Perez 2012 and Hytönen, Perez 2013).

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

An Application An application of the matrix linear bound

▷ Proof I

Proof II

Proof III

Proof IV

Calderon-Zygmund Operators on UMD spaces Let w be an A_2 -weight, b a BMO function of norm 1. Define the matrix weight

$$W = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \overline{b} & 1 \end{pmatrix},$$
$$W^{-1} = \begin{pmatrix} 1 & 0 \\ -\overline{b} & 1 \end{pmatrix} \begin{pmatrix} w^{-1} & 0 \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$$

Then

$$\begin{split} \|T\|_{L^2_W(\mathbb{C}^d) \to L^2_W(\mathbb{C}^2)} &= \|W^{1/2} T W^{-1/2}\|_{L^2(\mathbb{C}^2) \to L^2(\mathbb{C}^2)} \\ &= \left\| \begin{pmatrix} w^{1/2} T w^{-1/2} & w^{1/2} [T, b] w^{-1/2} \\ 0 & w^{1/2} T w^{-1/2} & 0 \end{pmatrix} \right\|_{L^2(\mathbb{C}^2) \to L^2(\mathbb{C}^2)} \end{split}$$

and thus

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

An Application An application of the

matrix linear bound

Proof I

▷ Proof II

Proof III

Proof IV

Calderon-Zygmund Operators on UMD spaces

$$\begin{split} \|[T,b]\|_{L^2_w \to L^2_w} &= \|w^{1/2}[T,b]w^{-1/2}\|_{L^2 \to L^2} \\ \lesssim \sup_{I} \operatorname{tr} \left(\langle W \rangle_I \langle W^{-1} \rangle_I \right) \\ &= \sup_{I} \oint_{I} \oint_{I} w(x)w^{-1}(y)(2+|b(x)-b(y)|^2) dx dy \\ \lesssim [w]_{A_2} \\ &\quad + \sup_{I} \left(\oint_{I} \int_{I} (w(x)w^{-1}(y))^r \right)^{1/r} \left(\oint_{I} \int_{I} |b(x)-b(y)|^{2r'} \right)^{1/r'} \\ &\leq [w]_{A_2}(1+\|b\|_{BMO,2r'}^2) \\ \lesssim [w]_{A_2}(1+([w]_{A_\infty}+[w^{-1}]_{A_\infty})^2 \|b\|_{BMO}^2) \\ \end{split}$$
for $r = \frac{4[w]_{A_\infty}}{4[w]_{A_\infty}-1}, r' = 4[w]_{A_\infty}$.

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

An Application An application of the matrix linear bound

Proof I

Proof II

▷ Proof III

Proof IV Calderon-7

Calderon-Zygmund Operators on UMD spaces For iterated commutators, consider the matrix weight

$$\begin{pmatrix} 1 & b & \dots & \frac{b^n}{n!} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} w & 0 & \dots & 0 \\ 0 & w & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ -\overline{b} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{(-\overline{b})^n}{n!} & \ddots & -\overline{b} & 1 \end{pmatrix}$$

This yields the weighted commutator matrix

$$\begin{pmatrix} T & [T,b] & \dots & \frac{1}{n!} \left[[[T,b],\dots],b \right] \\ 0 & T & [T,b] & \vdots \\ \vdots & \ddots & \ddots & [T,b] \\ 0 & \dots & \dots & T \end{pmatrix},$$

Proof IV

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

An Application An application of the matrix linear bound

Proof I

Proof II

Proof III

▷ Proof IV

Calderon-Zygmund Operators on UMD spaces Checking the matrix A_2 constant gives

$$\begin{split} \| [[[T, b], \dots], b] \|_{L^2_w \to L^2_w} &= \\ \lesssim \sup_{I} \operatorname{tr} \left(\langle W \rangle_I \langle W^{-1} \rangle_I \right) \\ \lesssim \sup_{I} \int_{I} \int_{I} w(x) w^{-1}(y) (2 + |b(x) - b(y)|^{2n}) dx dy \\ \lesssim [w]_{A_2} \\ &+ \sup_{I} \left(\int_{I} \int_{I} (w(x) w^{-1}(y))^r \right)^{1/r} \left(\int_{I} \int_{I} |b(x) - b(y)|^{2nr'} \right)^{1/r'} \\ \leq [w]_{A_2} (1 + ||b||^{2n}_{BMO, 2nr'}) \\ \lesssim [w]_{A_2} (1 + ([w]_{A_\infty} + [w^{-1}]_{A_\infty})^{2n} ||b||^{2n}_{BMO}), \end{split}$$

which yields the second result.

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on ▷ UMD spaces CZ operators on UMD spaces The theorems of Burkholder, Bourgain, and Figiel

The linear bound for even CZ operators on UMD spaces The Bellmann function proof of Treil adapted Proof II Proof III The Lemma Proof of the Lemma

Calderon-Zygmund Operators on UMD spaces

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces

CZ operators on UMD spaces The theorems of Burkholder, Bourgain, and Figiel

The linear bound for even CZ operators on UMD spaces The Bellmann function proof of Treil adapted Proof II Proof III The Lemma Proof of the Lemma A Banach space X is a UMD (unconditional for martingale differences) space if there exists a constant C_p for one (or equivalently, for all) $p \in (1, \infty)$ such that

$$\left\|\sum_{k=1}^{n}\varepsilon_{k}d_{k}\right\|_{L^{p}(X)} \leq C_{p}\left\|\sum_{k=1}^{n}d_{k}\right\|_{L^{p}(X)},$$

for all $n \ge 1$, all X-valued martingale difference sequences $\{d_k\}_{k=1}^n$, and all choices of signs $\{\varepsilon_k\}_{k=1}^n$ of signs from $\{\pm 1\}$.

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces

CZ operators on UMD spaces The theorems of Burkholder, Bourgain, and Figiel

The linear bound for even CZ operators on UMD spaces The Bellmann function proof of Treil adapted Proof II Proof III The Lemma Proof of the Lemma A Banach space X is a UMD (unconditional for martingale differences) space if there exists a constant C_p for one (or equivalently, for all) $p \in (1, \infty)$ such that

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for all $n \ge 1$, all X-valued martingale difference sequences $\{d_k\}_{k=1}^n$, and all choices of signs $\{\varepsilon_k\}_{k=1}^n$ of signs from $\{\pm 1\}$. For each $p \in (1, \infty)$, the smallest constant in the previous inequality is denoted by $\beta_{p,X}$ and is called the UMD_p constant of X.

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces

CZ operators on UMD spaces

The theorems of Burkholder, Bourgain, and ▷ Figiel

The linear bound for even CZ operators on UMD spaces The Bellmann function proof of Treil adapted Proof II Proof III The Lemma Proof of the Lemma **Theorem 12 (Burkholder, Bourgain)** Let X be a Banach space. Then X is a UMD space, if and only if the Hilbert transform extends to a bounded linear operator $L^p(\mathbb{R}, X) \to L^p(\mathbb{R}, X)$ for some (and equivalently, to all) 1 .

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces

CZ operators on UMD spaces The theorems of Burkholder, Bourgain, and Figiel > The linear bound for even CZ operators

on UMD spaces The Bellmann function proof of Treil adapted

Proof II

Proof III

The Lemma

Proof of the Lemma

Theorem 13 (Figiel's T1 Theorem) Let X be a UMD space, $1 , and <math>\beta_{p,X}$ be the UMD_p constant of X. Let T be a Calderón-Zygmund operator on \mathbb{R} which satisfies the standard kernel estimates, the weak boundedness property $|\langle T\chi_I, \chi_I \rangle| \leq C|I|$ for all intervals I, and the conditions $T(1), T^*(1) \in BMO$. Then

 $||T||_{L^p(X)\to L^p(X)} \le C\beta_{p,X}^2,$

where C depends only on the constants in the standard estimates and the weak boundedness property, and the BMO norm of $T(1), T^*(1)$.

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces CZ operators on

UMD spaces The theorems of Burkholder, Bourgain, and Figiel

The linear bound for even CZ operators on ▷ UMD spaces The Bellmann function proof of Treil adapted Proof II Proof III The Lemma Proof of the Lemma **Theorem 14 (Pott, Stoica 2013)** Let X be a UMD space, $1 , and <math>\beta_{p,X}$ be the UMD_p constant of X. Let K be an even standard kernel with smoothness $\delta > 1/2$ and T be a Calderón-Zygmund operator on \mathbb{R} associated to K. Suppose that T satisfies the weak boundedness property $|\langle T\chi_I, \chi_I \rangle| \leq C|I|$ for all intervals I, and the vanishing paraproduct conditions $T(1) = T^*(1) = 0$. Then

$$|T||_{L^p(X)\to L^p(X)} \le C_T \beta_{p,X},$$

where C_T depends only on the constants in the standard estimates and the weak boundedness property.

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces CZ operators on

UMD spaces The theorems of Burkholder,

Bourgain, and Figiel

The linear bound for even CZ operators on UMD spaces The Bellmann function proof of ▷ Treil adapted Proof II Proof III The Lemma Proof of the Lemma

By definition of
$$\beta_{p,X}$$

$$\sum_{I} |\langle \langle f \rangle_{I^+} - \langle f \rangle_{I^-}, \langle g \rangle_{I^+} - \langle g \rangle_{I^-} \rangle_{X,X^*} | \cdot |I|$$

 $\leq \beta_{p,X} \|f\|_{L^p(X)} \|g\|_{L^{p'}(X^*)}.$

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces

CZ operators on UMD spaces The theorems of Burkholder, Bourgain, and Figiel

The linear bound for even CZ operators on UMD spaces

The Bellmann function proof of ▷ Treil adapted Proof II Proof III

The Lemma

Proof of the Lemma

By definition of $\beta_{p,X}$

$$\sum_{I} |\langle \langle f \rangle_{I^+} - \langle f \rangle_{I^-}, \langle g \rangle_{I^+} - \langle g \rangle_{I^-} \rangle_{X,X^*} | \cdot |I|$$

 $\leq \beta_{p,X} \|f\|_{L^p(X)} \|g\|_{L^{p'}(X^*)}.$

Define $\mathcal{B}(\mathbf{f}, \mathbf{F}, \mathbf{g}, \mathbf{G}) :=$

$$= |I_0|^{-1} \sup \sum_{I \subseteq I_0} |\langle \langle f \rangle_{I^+} - \langle f \rangle_{I^-}, \langle g \rangle_{I^+} - \langle g \rangle_{I^-} \rangle_{X,X^*} |\cdot|I|,$$

with the sup over all functions f, g such that

$$\langle f \rangle_{I_0} = \mathbf{f}, \quad \left\langle \|f\|_X^p \right\rangle_{I_0} = \mathbf{F}, \quad \langle g \rangle_{I_0} = \mathbf{g}, \quad \left\langle \|g\|_{X^*}^{p'} \right\rangle_{I_0} = \mathbf{G},$$

Thus $\mathcal{B}_X(\mathbf{f}, \mathbf{F}, \mathbf{g}, \mathbf{G}) \leq \beta_{p,X} \mathbf{F}^{1/p} \mathbf{G}^{1/p'}.$

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces CZ operators on

UMD spaces The theorems of Burkholder, Bourgain, and Figiel

The linear bound for even CZ operators on UMD spaces The Bellmann function proof of Treil adapted ▷ Proof II Proof III The Lemma Proof of the Lemma

Write
$$(\mathbf{f}, \mathbf{F}, \mathbf{g}, \mathbf{G}) = \mathbf{A}$$
. For $\mathbf{A} = \frac{1}{2} (\mathbf{A}^+ + \mathbf{A}^-)$,
 $\frac{1}{2} (\mathcal{B}_X(\mathbf{A}^+) + \mathcal{B}_X(\mathbf{A}^-)) + |\langle \mathbf{f}^+ - \mathbf{f}^-, \mathbf{g}^+ - \mathbf{g}^- \rangle_{X,X^*}|$
 $\leq \mathcal{B}_X(\mathbf{A})$ (3)

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces CZ operators on

UMD spaces The theorems of Burkholder, Bourgain, and Figiel

The linear bound for even CZ operators on UMD spaces The Bellmann function proof of Treil adapted ▷ Proof II Proof III The Lemma Proof of the Lemma

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 $\leq \mathcal{B}_X(\mathbf{A})$ (3)

Moreover, each function \mathcal{B} with the boundedness property (1) und the convexity property (2) proves the linear bound for martingale transforms.

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces CZ operators on

UMD spaces The theorems of Burkholder,

Bourgain, and Figiel The linear bound for even CZ operators on UMD spaces

The Bellmann function proof of

Treil adapted

Proof II

▷ Proof III

The Lemma

Proof of the Lemma

For a dyadic shift of complexity n, control instead

$$|I_0|^{-1} \sup \sum_{L \subseteq I_0} \sum_{I,J \in \mathcal{D}^n(L)} \frac{1}{2^{2n}} |\langle \langle f \rangle_I - \langle f \rangle_L, \langle g \rangle_J - \langle g \rangle_L \rangle_{X,X^*} |\cdot|L|.$$

For any bounded real sequence $(\alpha_I^L)_{I \in \mathcal{D}^n(L)}$ of average zero,

$$\sum_{I,J\in\mathcal{D}^n(L)} \frac{\alpha_I^L}{2^n} \frac{\alpha_J^L}{2^n} \langle \langle f \rangle_I - \langle f \rangle_L, \langle g \rangle_J - \langle g \rangle_L \rangle \cdot |L|$$
$$\lesssim \mathcal{B}(A_L) - \frac{1}{2^n} \sum_{I\in\mathcal{D}^n(L)} \mathcal{B}(A_I).$$

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces CZ operators on

UMD spaces The theorems of Burkholder,

Bourgain, and Figiel

The linear bound for even CZ operators on UMD spaces The Bellmann function proof of Treil adapted Proof II ▷ Proof III The Lemma

Proof of the Lemma

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For any bounded real sequence $(\alpha_I^L)_{I \in \mathcal{D}^n(L)}$ of average zero,

$$\sum_{I,J\in\mathcal{D}^n(L)} \frac{\alpha_I^L}{2^n} \frac{\alpha_J^L}{2^n} \langle \langle f \rangle_I - \langle f \rangle_L, \langle g \rangle_J - \langle g \rangle_L \rangle \cdot |L|$$
$$\lesssim \mathcal{B}(A_L) - \frac{1}{2^n} \sum_{I\in\mathcal{D}^n(L)} \mathcal{B}(A_I).$$

by the convexity properties of the Bellman function.

The Lemma

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces CZ operators on

UMD spaces The theorems of Burkholder, Bourgain, and Figiel

The linear bound for even CZ operators on UMD spaces The Bellmann function proof of Treil adapted Proof II Proof III ▷ The Lemma Proof of the Lemma **Lemma 15** Let Λ^L be the real symmetric $2^n \times 2^n$ matrix with entries

$$\Lambda_{I,J}^{L} = \frac{1}{2^{2n}} \left(\langle\!\langle f \rangle_{I} - \langle f \rangle_{L}, \langle g \rangle_{J} - \langle g \rangle_{L} \rangle_{X,X^{*}} + \langle\!\langle f \rangle_{J} - \langle f \rangle_{L}, \langle g \rangle_{I} - \langle g \rangle_{L} \rangle_{X,X^{*}} \right)$$

Then for a suitably chosen bounded real sequence $\alpha^L = (\alpha_I^L)_{I \in \mathcal{D}^n(L)}$ of average zero and with sup norm 1, we have that

$$2^{-n/2} \sum_{I,J \in \mathcal{D}^n(L)} |\Lambda_{I,J}^L| \cdot |L| \\ \lesssim \operatorname{tr}(\alpha^L \otimes \alpha^L \cdot \Lambda^L) |L| \lesssim \mathcal{B}(A_L) - \frac{1}{2^n} \sum_{I \in \mathcal{D}^n(L)} \mathcal{B}(A_I).$$

Proof of the Lemma

Calderòn-Zygmund Operators

Weighted norm inequalities

Matrix weights

An Application

Calderon-Zygmund Operators on UMD spaces CZ operators on UMD spaces The theorems of

Burkholder, Bourgain, and Figiel

The linear bound for even CZ operators on UMD spaces The Bellmann function proof of Treil adapted Proof II Proof III The Lemma Proof of the ▷ Lemma Let S denote the Schur multipliers on the $2^n \times 2^n$ matrices. By Grothendieck's inequality,

$$\mathcal{S} \simeq l_{2^n}^\infty \hat{\otimes} l_{2^n}^\infty.$$

Thus the matrix Λ^L can be paired with all symmetric real Schur multipliers of norm 1. But for any $2^n \times 2^n$ matrix M,

$$\|M\|_{\mathcal{S}} \le 2^{n/2} \|M\|_{l^{\infty}(n \times n)},$$

and this is sharp (Davidson, Donsig 2007). This means that

$$\begin{split} \sup_{\alpha^L} \mathbf{tr}(\alpha^L \otimes \alpha^L \cdot \Lambda^L) |L| \gtrsim 2^{-n/2} \sup_{\|M\|_{l^{\infty}} \leq 1} \mathbf{tr}(M \cdot \Lambda^L) |L| \\ &= \sum_{I,J \in \mathcal{D}^n(L)} |\Lambda^L_{I,J}| \cdot |L|. \quad \Box \end{split}$$