Matrix Weights: On the Way to the Linear Bound

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Calderòn-Zygmund Operators

Let $\Delta = \{(x, x) : x \in \mathbb{R}\}$ be the diagonal of $\mathbb{R} \times \mathbb{R}$.
$K : \mathbb{R} \times \mathbb{R} \setminus \Delta \to \mathbb{C}$ is a standard Calderón-Zygmund kernel if there exists $\delta > 0$ such that

$$|K(x, y)| \leq \frac{C}{|x - y|},$$

$$|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq C_\delta \frac{|y - z|}{|x - y|^{1+\delta}},$$

for $x, y, z \in \mathbb{R}$ with $|x - y| > 2|y - z|$.

CZOs asociated to a kernel

An operator T is called a Calderòn-Zygmund operator on $\mathbb{R}$ associated to K, if

$$Tf(x) = \int_{\mathbb{R}} K(x,y)f(y)\,dy, \qquad x \notin \operatorname{supp} f.$$

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Example: The Hilbert transform

$$Hf(x) = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy.$$

Theorem 1 (David and Journé, 1984) *Let T be a CZO associated to a standard kernel. Then the following are equivalent:*

A T *defines a bounded linear operator* $L^2(\mathbb{R}) \to L^2(\mathbb{R})$

Theorem 2 (David and Journé, 1984) *Let T be a CZO associated to a standard kernel. Then the following are equivalent:*

A *T defines a bounded linear operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$*

B *The following three conditions hold:*

- *(i)* $|\langle T\chi_I, \chi_I \rangle| \lesssim |I|$ *for all intervals $I \subset \mathbb{R}$ (weak boundedness)*
- *(ii)* $T1 \in BMO(\mathbb{R})$
- *(iii)* $T^*1 \in BMO(\mathbb{R})$

Idea: A CZ operator consists of parts which are controlled by the BMO norm of $T1$, T^*1 (so-called paraproducts) and a part which is "close to the diagonal".

Let $f \in L^2(\mathbb{R})$,

$$f(x) = \sum_{I \in \mathcal{D}} h_I(x) \hat{f}(I).$$

For a sequence (α_I), define the Haar multiplier

$$T_\alpha f = \sum_{I \in \mathcal{D}} h_I \alpha_I \hat{f}(I).$$

If $(\alpha_I) = (\varepsilon_I) \in \{-1, 1\}^{\mathcal{D}}$, T_ε is a dyadic martingale transform.

Haar multipliers, martingale transforms, and dyadic shifts

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If $(\alpha_I) = (\varepsilon_I) \in \{-1, 1\}^{\mathcal{D}}$, T_ε is a dyadic martingale transform.
A dyadic shift S with parameters (m,n) is given by

$$Sf = \sum_{L \in \mathcal{D}} \sum_{J \in \mathcal{D}^m(L)} h_J(x) \sum_{I \in \mathcal{D}^n(L)} c_{I,J} \hat{f}(I).$$

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Then $k = \max\{n, m\} + 1$ is called the complexity of S.

Theorem 3 (Hytönen 2010) *Let T be a CZO on $\mathbb{R}$ with the standard kernel estimates, the weak boundedness property $|\langle T\chi_I, \chi_I\rangle| \leq C|I|$ for all intervals I, and the conditions $T(1) = b_1, T^*(1) = b_2 \in BMO(\mathbb{R})$. Then for $f, g \in C_c^1(\mathbb{R})$,*

$$\langle Tf, g\rangle_{L^2(\mathbb{R}),L^2(\mathbb{R})} =$$

$$C \cdot \mathbb{E}_\Omega \langle \left(\Pi_{b_1}^\omega + (\Pi_{b_2}^\omega)^* + \sum_{m,n=0}^{\infty} \tau(m,n) S_\omega^{mn} \right) f, g\rangle_{L^2(\mathbb{R}),L^2(\mathbb{R})},$$

where C depends only on the constants in the standard estimates of the kernel K and the weak boundedness property, S_ω^{mn} is a dyadic Haar shift of parameters (m,n) on the dyadic system $\mathcal{D}^\omega$ and $\tau(m,n) \lesssim P(\max\{m,n\})2^{-\delta \max\{m,n\}}$, with P a polynomial.

Weighted norm inequalities

Let w be a weight function on $\mathbb{R}$, $1 < p < \infty$. We say that w is an A_p-weight, if

$$[w]_{A_p} = \sup_{I \text{ interval}} \langle w \rangle_I \langle w^{-\frac{1}{p-1}} \rangle_I^{p-1} < \infty.$$

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Theorem 5 (Hunt, Muckenhoupt, Wheeden 1974) *Let $1 < p < \infty$ and w be a weight on $\mathbb{R}$. Then the Hilbert transform $H : L^p_w(\mathbb{R}) \to L^p_w(\mathbb{R})$ is bounded, if and only if w is an A_p-weight.*

What is the sharp bound of the norm of $\|T\|_{L^p_w \to L^p_w}$ in terms of $[w]_{A_p}$?

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A_2-Conjecture: $\|T\|_{L^2_w \to L^2_w} \leq C_T [w]_{A_2}$.

Some History of Sharp A_2 Bounds

- Maximal function (Buckley 1993)
- Dyadic square function (Hukovic, Nazarov, Treil, Volberg 1999)
- Dyadic martingale transforms (Wittwer 2000)
- $3/2$ bound for the Hilbert transform (Petermichl, Pott 2001)
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- $T(1)$ Theorem for two-weighted dyadic shifts (Nazarov, Treil, Volberg 2008)
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The Bellmann function proof of Treil I

We assume the linear bound for martingale transforms:

$$\sup_{\varepsilon} \langle T_\varepsilon f, g \rangle = \sum_I |\langle f \rangle_{I^+} - \langle f \rangle_{I^-}| |\langle g \rangle_{I^+} - \langle g \rangle_{I^-}| \cdot |I|$$

$$\leq C[w]_{A_2} \|f\|_{L^2_w} \|g\|_{L^2_{w^{-1}}}.$$

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Define $\mathcal{B}_X(\mathbf{f},\mathbf{F},\mathbf{U},\mathbf{g},\mathbf{G},\mathbf{V}) :=$

$$= \quad |I_0|^{-1}\sup\sum_{I\subseteq I_0} |\langle f\rangle_{I^+} - \langle f\rangle_{I^-}||\langle g\rangle_{I^+} - \langle g\rangle_{I^-}| \cdot |I|,$$

with the sup over all functions f, g and A_2 weights w such that

$$\langle f\rangle_{I_0} = \mathbf{f}, \quad \left\langle wf^2\right\rangle_{I_0} = \mathbf{F}, \quad \langle g\rangle_{I_0} = \mathbf{g}, \quad \left\langle w^{-1}g^2\right\rangle_{I_0} = \mathbf{G},$$

$$\sup_{I\subset I_0}\langle w\rangle_I\langle w^{-1}\rangle_I \leq X, \quad \langle w\rangle_{I_0} = \mathbf{U}, \quad \langle w^{-1}\rangle_{I_0} = \mathbf{V}.$$

By the assumption of the linear bound,

$$\mathcal{B}_X(\mathbf{f}, \mathbf{F}, \mathbf{U}, \mathbf{g}, \mathbf{G}, \mathbf{V}) \leq CX\mathbf{F}^{1/2}\mathbf{G}^{1/2}. \tag{1}$$

Proof II

By the assumption of the linear bound,

$$\mathcal{B}_X(\mathbf{f}, \mathbf{F}, \mathbf{U}, \mathbf{g}, \mathbf{G}, \mathbf{V}) \leq C X \mathbf{F}^{1/2} \mathbf{G}^{1/2}. \tag{1}$$

Write $(\mathbf{f}, \mathbf{F}, \mathbf{U}, \mathbf{g}, \mathbf{G}, \mathbf{V}) = \mathbf{A}$. For $\mathbf{A} = \frac{1}{2}(\mathbf{A}^+ + \mathbf{A}^-)$,

$$\frac{1}{2}\left(\mathcal{B}_X(\mathbf{A}^+) + \mathcal{B}_X(\mathbf{A}^-)\right) + |\mathbf{f}^+ - \mathbf{f}^-||\mathbf{g}^+ - \mathbf{g}^-| \leq \mathcal{B}_X(\mathbf{A}) \tag{2}$$

By the assumption of the linear bound,

$$\mathcal{B}_X(\mathbf{f}, \mathbf{F}, \mathbf{U}, \mathbf{g}, \mathbf{G}, \mathbf{V}) \leq C X \mathbf{F}^{1/2} \mathbf{G}^{1/2}. \tag{1}$$

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$$\frac{1}{2}\left(\mathcal{B}_X(\mathbf{A}^+) + \mathcal{B}_X(\mathbf{A}^-)\right) + |\mathbf{f}^+ - \mathbf{f}^-||\mathbf{g}^+ - \mathbf{g}^-| \leq \mathcal{B}_X(\mathbf{A}) \tag{2}$$

Moreover, each function $\mathcal{B}$ with the boundedness property (1) und the convexity property (2) proves the linear bound for martingale transforms.

For a dyadic shift of complexity n, control instead

$$|I_0|^{-1} \sup \sum_{L \subseteq I_0} \sum_{I,J \in \mathcal{D}^n(L)} \frac{|\langle f \rangle_I - \langle f \rangle_L|}{2^n} \frac{|\langle g \rangle_J - \langle g \rangle_L|}{2^n} \cdot |L|.$$

For some bounded real sequence $(\alpha_I^L)_{I \in \mathcal{D}^n(L)}$ of average zero,

$$\sum_{I,J \in \mathcal{D}^n(L)} \frac{|\langle f \rangle_I - \langle f \rangle_L|}{2^n} \frac{|\langle g \rangle_J - \langle g \rangle_L|}{2^n} \cdot |L|$$

$$\lesssim \sum_{I,J \in \mathcal{D}^n(L)} \alpha_I^L \alpha_J^L \frac{\langle f \rangle_I - \langle f \rangle_L}{2^n} \frac{\langle g \rangle_J - \langle g \rangle_L}{2^n} \cdot |L|.$$

But for each such sequence,

$$\sum_{I,J\in\mathcal{D}^n(L)} \alpha_I^L \alpha_J^L \frac{\langle f\rangle_I - \langle f\rangle_L}{2^n} \frac{\langle g\rangle_J - \langle g\rangle_L}{2^n} \cdot |L| =$$

$$\left(\sum_{I\in\mathcal{D}^n(L)} \frac{1+\alpha_I^L}{2^n}(\langle f\rangle_I - \langle f\rangle_L)\right)\left(\sum_{J\in\mathcal{D}^n(L)} \frac{1+\alpha_J^L}{2^n}(\langle g\rangle_J - \langle g\rangle_L)\right)|L|$$

$$\lesssim \mathcal{B}_X(A_L) - \frac{1}{2^n}\sum_{I\in\mathcal{D}^n(L)} \mathcal{B}_X(A_I).$$

Proof IV

But for each such sequence,

$$\sum_{I,J\in\mathcal{D}^n(L)} \alpha_I^L \alpha_J^L \frac{\langle f\rangle_I - \langle f\rangle_L}{2^n} \frac{\langle g\rangle_J - \langle g\rangle_L}{2^n} \cdot |L| =$$

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$$\lesssim \mathcal{B}_X(A_L) - \frac{1}{2^n}\sum_{I\in\mathcal{D}^n(L)} \mathcal{B}_X(A_I).$$

Adding $L \in \mathcal{D}(I_0)$ gives a telescopic sum and the desired bound.

Matrix weights

A matrix weight is a measurable function $W : \mathbb{R} \to \mathrm{Mat}(d \times d, \mathbb{C})$ which is positive and invertible almost everywhere.

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$$[W]_{A_2} := \sup_I \left\| \left\langle W(t) \right\rangle^{1/2} \left\langle W^{-1}(t) \right\rangle^{1/2} \right\|^2 < \infty.$$

Theorem 9 (Treil, Volberg, Christ, Goldberg) *Let W be a matrix A_p weight, $1 < p < \infty$. Then for each CZO T associated to a standard kernel,*

$$T : L^p_W(\mathbb{R}, \mathbb{C}^d) \to L^p_W(\mathbb{R}, \mathbb{C}^d) \textit{ is bounded.}$$

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Theorem 10 (Treil, Volberg, Christ, Goldberg) *Let W be a matrix A_p weight, $1<p<\infty$. Then for each CZO T associated to a standard kernel,*

$$T: L^p_W(\mathbb{R},\mathbb{C}^d)\to L^p_W(\mathbb{R},\mathbb{C}^d) \text{ is bounded.}$$

What is the sharp bound for $\|T\|_{L^2_W\to L^2_W}$ in terms of $[W]_{A_2}$?

Now let $f \in L^2(\mathbb{R}, \mathbb{C}^d)$,

$$f(x) = \sum_{I \in \mathcal{D}} h_I(x) \hat{f}(I).$$

For a sequence (σ_I) of $d \times d$ matrices, define the Haar multiplier

$$T_\sigma f = \sum_{I \in \mathcal{D}} h_I \sigma_I \hat{f}(I).$$

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It is easy to see that in order for T_σ to be bounded on $L^2_W(\mathbb{R}, \mathbb{C}^d)$, one needs

$$\|\sigma\|_{\infty,W} = \sup_I \|\langle W \rangle_I^{1/2} \sigma_I \langle W \rangle_I^{-1/2}\| < \infty.$$

Let $N : [1, \infty) \to [1, \infty)$ be given by

$$N(X) = \sup \|T_\sigma\|_{L^2_W(\mathbb{R},\mathbb{C}^d) \to L^2_W(\mathbb{R},\mathbb{C}^d)},$$

where the supremum is taken over all $d \times d$ matrix A_2 weights W with $[W]_{A_2} \le X$ and T_σ with $\|\sigma\|_{\infty,W} \le 1$.

Theorem 11 (Pott, Stoica 2015) *Let W be a $d \times d$ matrix A_2 weight. Let T be a Calderón-Zygmund operator on $\mathbb{R}$ associated to a standard kernel. Suppose that $|\langle T\chi_I, \chi_I\rangle| \le C|I|$ for all intervals I, and $T(1) = T^*(1) = 0$. Then*

$$\|T\|_{L^2_W(\mathbb{R},\mathbb{C}^d) \to L^2_W(\mathbb{R},\mathbb{C}^d)} \le C \cdot dN([W]_{A_2}),$$

where C depends only on the constants in the standard estimates and the weak boundedness property.

Remaining obstacles

To prove the linear bound for matrix weights for all CZOs, it therefore remains to prove it for Haar multipliers and for paraproducts. [Bickel, Petermichl, Wick 2014] prove a $[W]_{A_2}^{3/2} \log [W]_{A_2}$ bound for Haar multipliers, listing also a strategy to prove the linear bound:

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1. prove a matrix version of the Weighted Carleson Embedding Theorem for matrix weights -just done [Culiuc, Treil 2015]
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[Isralowitz, Kwon, Pott 2015] prove the $[W]_{A_2}^{3/2}\log[W]_{A_2}$ bound for paraproducts.

An Application

Suppose the A_2 Conjecture holds for matrix weights. Let T be a CZO which is bounded on $L^2(\mathbb{R})$. Then

$$\|[T,b]\|_{L^2_w\to L^2_w} \lesssim C_T\left(\frac{1}{R}[w]_{A_2}+R[w]_{A_2}[w]^2_{A_\infty}\right)\|b\|_{BMO} \quad (R>0).$$

In particular, choosing $R=[w]^{-1}_{A_\infty}$,

$$\|[T,b]\|_{L^2_w\to L^2_w} \lesssim C_T[w]_{A_2}[w]_{A_\infty}\|b\|_{BMO}$$

Moreover,

$$\|\,[\ldots[T,b],\ldots,b]\,\|_{L^2_w\to L^2_w} \lesssim C_T[w]_{A_2}\left([w]_{A_\infty}+[w^{-1}]_{A_\infty}\right)^n\|b\|^n_{BMO},$$

(compare with the results in Chung, Pereyra, Perez 2012 and Hytönen, Perez 2013).

Proof I

Let w be an A_2-weight, b a BMO function of norm 1. Define the matrix weight

$$W = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix},$$

$$W^{-1} = \begin{pmatrix} 1 & 0 \\ -\bar{b} & 1 \end{pmatrix} \begin{pmatrix} w^{-1} & 0 \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$$

Then

$$\begin{aligned} \|T\|_{L^2_W(\mathbb{C}^d) \to L^2_W(\mathbb{C}^2)} &= \|W^{1/2} T W^{-1/2}\|_{L^2(\mathbb{C}^2) \to L^2(\mathbb{C}^2)} \\ &= \left\| \begin{pmatrix} w^{1/2} T w^{-1/2} & w^{1/2}[T, b] w^{-1/2} \\ 0 & w^{1/2} T w^{-1/2} & 0 \end{pmatrix} \right\|_{L^2(\mathbb{C}^2) \to L^2(\mathbb{C}^2)} \end{aligned}$$

and thus

Proof II

$$
\begin{aligned}
& \|[T,b]\|_{L^2_w \to L^2_w} = \|w^{1/2}[T,b]w^{-1/2}\|_{L^2 \to L^2} \\
\lesssim \quad & \sup_I \mathbf{tr}\left(\langle W\rangle_I \langle W^{-1}\rangle_I\right) \\
= \quad & \sup_I ⨍_I ⨍_I w(x)w^{-1}(y)(2+|b(x)-b(y)|^2)dxdy \\
\lesssim \quad & [w]_{A_2} \\
& \quad + \sup_I \left(⨍_I ⨍_I (w(x)w^{-1}(y))^r\right)^{1/r} \left(⨍_I ⨍_I |b(x)-b(y)|^{2r'}\right)^{1/r'} \\
\leq \quad & [w]_{A_2}(1+\|b\|^2_{BMO,2r'}) \\
\lesssim \quad & [w]_{A_2}(1+([w]_{A_\infty}+[w^{-1}]_{A_\infty})^2\|b\|^2_{BMO})
\end{aligned}
$$

for $r = \frac{4[w]_{A_\infty}}{4[w]_{A_\infty}-1}$, $r' = 4[w]_{A_\infty}$.

For iterated commutators, consider the matrix weight

$$\begin{pmatrix} 1 & b & \dots & \frac{b^n}{n!} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} w & 0 & \dots & 0 \\ 0 & w & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ -\bar{b} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{(-\bar{b})^n}{n!} & \ddots & -\bar{b} & 1 \end{pmatrix}.$$

This yields the weighted commutator matrix

$$\begin{pmatrix} T & [T,b] & \dots & \frac{1}{n!}\,[[[T,b],\dots],b] \\ 0 & T & [T,b] & \vdots \\ \vdots & \ddots & \ddots & [T,b] \\ 0 & \dots & \dots & T \end{pmatrix},$$

Checking the matrix A_2 constant gives

$$
\begin{aligned}
& \|[[[T,b],\ldots],b]\|_{L^2_w \to L^2_w} = \\
\lesssim\ & \sup_I \mathbf{tr}\left(\langle W\rangle_I \langle W^{-1}\rangle_I\right) \\
\lesssim\ & \sup_I \fint_I \fint_I w(x) w^{-1}(y)(2+|b(x)-b(y)|^{2n})\,dx\,dy \\
\lesssim\ & [w]_{A_2} \\
& + \sup_I \left(\fint_I \fint_I (w(x) w^{-1}(y))^r\right)^{1/r} \left(\fint_I \fint_I |b(x)-b(y)|^{2nr'}\right)^{1/r'} \\
\leq\ & [w]_{A_2}(1+\|b\|^{2n}_{BMO,2nr'}) \\
\lesssim\ & [w]_{A_2}(1+([w]_{A_\infty}+[w^{-1}]_{A_\infty})^{2n}\|b\|^{2n}_{BMO}),
\end{aligned}
$$

which yields the second result.

Calderon-Zygmund Operators on UMD spaces

A Banach space X is a UMD (unconditional for martingale differences) space if there exists a constant C_p for one (or equivalently, for all) $p \in (1, \infty)$ such that

$$\Big\| \sum_{k=1}^{n} \varepsilon_k d_k \Big\|_{L^p(X)} \leq C_p \Big\| \sum_{k=1}^{n} d_k \Big\|_{L^p(X)},$$

for all $n \geq 1$, all X-valued martingale difference sequences $\{d_k\}_{k=1}^{n}$, and all choices of signs $\{\varepsilon_k\}_{k=1}^{n}$ of signs from $\{\pm 1\}$.

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for all $n \geq 1$, all X-valued martingale difference sequences $\{d_k\}_{k=1}^n$, and all choices of signs $\{\varepsilon_k\}_{k=1}^n$ of signs from $\{\pm 1\}$. For each $p \in (1, \infty)$, the smallest constant in the previous inequality is denoted by $\beta_{p,X}$ and is called the UMD$_p$ constant of X.

Theorem 12 (Burkholder, Bourgain) *Let X be a Banach space. Then X is a UMD space, if and only if the Hilbert transform extends to a bounded linear operator $L^p(\mathbb{R}, X) \to L^p(\mathbb{R}, X)$ for some (and equivalently, to all) $1 < p < \infty$.*

Theorem 13 (Figiel's T1 Theorem) *Let X be a UMD space, $1 < p < \infty$, and $\beta_{p,X}$ be the UMD_p constant of X. Let T be a Calderón-Zygmund operator on $\mathbb{R}$ which satisfies the standard kernel estimates, the weak boundedness property $|\langle T\chi_I, \chi_I\rangle| \leq C|I|$ for all intervals I, and the conditions $T(1), T^*(1) \in BMO$. Then*

$$\|T\|_{L^p(X)\to L^p(X)} \leq C\beta_{p,X}^2,$$

where C depends only on the constants in the standard estimates and the weak boundedness property, and the BMO norm of $T(1), T^*(1)$.

Theorem 14 (Pott, Stoica 2013) *Let X be a UMD space, $1 < p < \infty$, and $\beta_{p,X}$ be the UMD_p constant of X. Let K be an even standard kernel with smoothness $\delta > 1/2$ and T be a Calderón-Zygmund operator on $\mathbb{R}$ associated to K. Suppose that T satisfies the weak boundedness property $|\langle T\chi_I, \chi_I\rangle| \leq C|I|$ for all intervals I, and the vanishing paraproduct conditions $T(1) = T^*(1) = 0$. Then*

$$\|T\|_{L^p(X)\to L^p(X)} \leq C_T \beta_{p,X},$$

where C_T depends only on the constants in the standard estimates and the weak boundedness property.

By definition of $\beta_{p,X}$

$$\sum_I |\langle\langle f\rangle_{I^+} - \langle f\rangle_{I^-}, \langle g\rangle_{I^+} - \langle g\rangle_{I^-}\rangle_{X,X^*}| \cdot |I|$$

$$\leq \beta_{p,X} \|f\|_{L^p(X)} \|g\|_{L^{p'}(X^*)}.$$

By definition of $\beta_{p,X}$

$$\sum_I |\langle\langle f\rangle_{I^+} - \langle f\rangle_{I^-}, \langle g\rangle_{I^+} - \langle g\rangle_{I^-}\rangle_{X,X^*}| \cdot |I| \leq \beta_{p,X}\|f\|_{L^p(X)}\|g\|_{L^{p'}(X^*)}.$$

Define $\mathcal{B}(\mathbf{f}, \mathbf{F}, \mathbf{g}, \mathbf{G}) :=$

$$= |I_0|^{-1} \sup \sum_{I\subseteq I_0} |\langle\langle f\rangle_{I^+} - \langle f\rangle_{I^-}, \langle g\rangle_{I^+} - \langle g\rangle_{I^-}\rangle_{X,X^*}| \cdot |I|,$$

with the sup over all functions f, g such that

$$\langle f\rangle_{I_0} = \mathbf{f}, \quad \left\langle \|f\|_X^p \right\rangle_{I_0} = \mathbf{F}, \quad \langle g\rangle_{I_0} = \mathbf{g}, \quad \left\langle \|g\|_{X^*}^{p'} \right\rangle_{I_0} = \mathbf{G},$$

Thus $\mathcal{B}_X(\mathbf{f}, \mathbf{F}, \mathbf{g}, \mathbf{G}) \leq \beta_{p,X}\mathbf{F}^{1/p}\mathbf{G}^{1/p'}$.

Write $(\mathbf{f}, \mathbf{F}, \mathbf{g}, \mathbf{G}) = \mathbf{A}$. For $\mathbf{A} = \frac{1}{2}(\mathbf{A}^{+} + \mathbf{A}^{-})$,

$$\frac{1}{2}\big(\mathcal{B}_X(\mathbf{A}^{+}) + \mathcal{B}_X(\mathbf{A}^{-})\big) + \big|\langle \mathbf{f}^{+} - \mathbf{f}^{-}, \mathbf{g}^{+} - \mathbf{g}^{-}\rangle_{X,X^*}\big| \leq \mathcal{B}_X(\mathbf{A}) \quad (3)$$

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$$\frac{1}{2}\Big(\mathcal{B}_X(\mathbf{A}^+) + \mathcal{B}_X(\mathbf{A}^-)\Big) + \left|\langle \mathbf{f}^+ - \mathbf{f}^-, \mathbf{g}^+ - \mathbf{g}^- \rangle_{X,X^*}\right| \leq \mathcal{B}_X(\mathbf{A}) \quad (3)$$

Moreover, each function $\mathcal{B}$ with the boundedness property (1) und the convexity property (2) proves the linear bound for martingale transforms.

Proof III

For a dyadic shift of complexity n, control instead

$$|I_0|^{-1} \sup \sum_{L \subseteq I_0} \sum_{I,J \in \mathcal{D}^n(L)} \frac{1}{2^{2n}} |\langle \langle f \rangle_I - \langle f \rangle_L, \langle g \rangle_J - \langle g \rangle_L \rangle_{X,X^*}| \cdot |L|.$$

For any bounded real sequence $(\alpha_I^L)_{I \in \mathcal{D}^n(L)}$ of average zero,

$$\sum_{I,J \in \mathcal{D}^n(L)} \frac{\alpha_I^L}{2^n} \frac{\alpha_J^L}{2^n} \langle \langle f \rangle_I - \langle f \rangle_L, \langle g \rangle_J - \langle g \rangle_L \rangle \cdot |L|$$

$$\lesssim \mathcal{B}(A_L) - \frac{1}{2^n} \sum_{I \in \mathcal{D}^n(L)} \mathcal{B}(A_I).$$

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$$\lesssim \mathcal{B}(A_L) - \frac{1}{2^n} \sum_{I \in \mathcal{D}^n(L)} \mathcal{B}(A_I).$$

by the convexity properties of the Bellman function.

Lemma 15 *Let Λ^L be the real symmetric $2^n \times 2^n$ matrix with entries*

$$\Lambda^L_{I,J} = \frac{1}{2^{2n}} \left(\langle \langle f \rangle_I - \langle f \rangle_L, \langle g \rangle_J - \langle g \rangle_L \rangle_{X,X^*} + \langle \langle f \rangle_J - \langle f \rangle_L, \langle g \rangle_I - \langle g \rangle_L \rangle_{X,X^*} \right)$$

Then for a suitably chosen bounded real sequence $\alpha^L = (\alpha^L_I)_{I \in \mathcal{D}^n(L)}$ of average zero and with sup norm 1, we have that

$$2^{-n/2} \sum_{I,J \in \mathcal{D}^n(L)} \left| \Lambda^L_{I,J} \right| \cdot |L|$$

$$\lesssim \mathbf{tr}(\alpha^L \otimes \alpha^L \cdot \Lambda^L)|L| \lesssim \mathcal{B}(A_L) - \frac{1}{2^n} \sum_{I \in \mathcal{D}^n(L)} \mathcal{B}(A_I).$$

Let $\mathcal{S}$ denote the Schur multipliers on the $2^n \times 2^n$ matrices. By Grothendieck's inequality,

$$\mathcal{S} \simeq l^\infty_{2^n} \hat{\otimes} l^\infty_{2^n}.$$

Thus the matrix Λ^L can be paired with all symmetric real Schur multipliers of norm 1. But for any $2^n \times 2^n$ matrix M,

$$\|M\|_{\mathcal{S}} \leq 2^{n/2} \|M\|_{l^\infty(n\times n)},$$

and this is sharp (Davidson, Donsig 2007). This means that

$$\begin{aligned}\sup_{\alpha^L} \mathbf{tr}(\alpha^L \otimes \alpha^L \cdot \Lambda^L)|L| &\gtrsim 2^{-n/2} \sup_{\|M\|_{l^\infty} \leq 1} \mathbf{tr}(M \cdot \Lambda^L)|L| \\ &= \sum_{I,J\in\mathcal{D}^n(L)} \left|\Lambda^L_{I,J}\right| \cdot |L|. \qquad \square\end{aligned}$$