The non-commutative Khintchine inequalities in L_p, $0 < p \leq 2$

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Almost unconditional series

B Banach space
$x_n \in B$
Question When does the series

$$\sum \pm x_n$$

converge for almost all choices of signs ?
More formally: (ε_n) i.i.d. random variables on $(\Omega, \mathbb{P})$ with

$$\mathbb{P}(\varepsilon_n = \pm 1) = 1/2$$

When does the random series

$$\sum \varepsilon_n x_n$$

converge a.s. ?

Khintchine inequalities

Scalar case $B = \mathbb{R}$ or $\mathbb{C}$

Answer to the question: $\sum \varepsilon_n x_n$ converge a.s.

IFF $\sum |x_n|^2 < \infty$

IFF $\sum \varepsilon_n x_n$ converges in L_2

IFF $\sum \varepsilon_n x_n$ converges in L_p for ALL $0 < p < \infty$

Khintchine inequalities

For any $0 < p < \infty$ there are constants $A_p > 0$ and $B_p > 0$ such that for any sequence $x = (x_n)$ in ℓ_2 we have

$$A_p \left(\sum |x_n|^2 \right)^{1/2} \leq \left(\int \left| \sum x_n \varepsilon_n \right|^p d\mathbb{P} \right)^{1/p} \leq B_p \left(\sum |x_n|^2 \right)^{1/2}.$$

Note that $A_p = 1$ when $p \geq 2$ and $B_p = 1$ when $p \leq 2$ are the obvious cases since the $L_p(\mathbb{P})$-norm is monotone increasing in p and $\| \sum x_n \varepsilon_n \|_{L_2(\mathbb{P})} = (\sum |x_n|^2)^{1/2}$.

Kahane inequalities

For any $0 < p < q < \infty$ there is a constant $K(p,q)$ such that for any Banach space B and any finite subset $x_1, \ldots, x_n$ in an arbitrary Banach space B we have

$$\left\|\sum \varepsilon_k x_k\right\|_{L_p(B)} \leq \left\|\sum \varepsilon_k x_k\right\|_{L_q(B)} \leq K(p,q) \left\|\sum \varepsilon_k x_k\right\|_{L_p(B)}.$$

In particular

$$\forall 0 < p < \ \infty \quad \|\sum \varepsilon_k x_k\|_{L_p(B)} \simeq \|\sum \varepsilon_k x_k\|_{L_2(B)}$$

Moreover

$$\sum \varepsilon_n x_n \text{ converge a.s. in B \ IFF \ it converges in } L_p(B)$$

Our initial Question When does the series

$$\sum \varepsilon_n x_n$$

converge for almost all choices of signs ?
is now reduced to: Find a "computable" equivalent for

$$\|\sum \varepsilon_n x_n\|_{L_p(B)}$$

and we can choose the p that we like.

.....cf Talagrand, Latała and Bednorz for, say, $B = \ell_\infty$

Example: $B = L_p(m)$

Consider $B = L_p(m)$
Then the Khintchine inequality in L_p and Fubini imply:

$$\|\sum \varepsilon_n x_n\|_{L_p(B)} \simeq \|(\sum |x_n|^2)^{1/2}\|_B$$

Moreover

$$\sum \varepsilon_n x_n \text{ converge a.s. in B} \;\; \text{IFF} \;\; \|(\sum |x_n|^2)^{1/2}\|_B < \infty$$

Non-commutative Khintchine inequalities

$B = L_p(M, \tau)$
M von Neumann alg.
τ standard trace (normal, faithful, semi-finite): $\tau(x^*x) = \tau(xx^*)$
Basic example: $M = B(H)$ equipped with usual trace $x \mapsto \mathrm{tr}(x)$
Then

$$L_p(M, \tau) = \text{Schatten } p - \text{class } S_p$$

Note

$$\|(\sum |x_n|^2)^{1/2}\|_B \text{ still makes sense with } |x| = (x^*x)^{1/2}$$

However

$$\|(\sum |x_n|^2)^{1/2}\|_B \not\approx \|(\sum |x_n^*|^2)^{1/2}\|_B$$

while

$$\|\sum \varepsilon_n x_n\|_{L_p(B)} = \|\sum \varepsilon_n x_n^*\|_{L_p(B)}$$

Non-commutative Khintchine inequalities

The case $1 < p < \infty$ is due to **F. Lust-Piquard 1986**
There are positive constants α_p, β_p such that for any finite sequence $x = (x_1, \ldots, x_n)$ in $B = S_p$ (or $B = L_p(\tau)$) we have

$$\frac{1}{\beta_p} |||(x_k)|||_p \leq \| \sum \varepsilon_n x_n \|_{L_p(B)} \leq \alpha_p |||(x_k)|||_p \tag{1}$$

where $|||(x_k)|||_p$ is defined as follows:
If $2 \leq p < \infty$

$$|||(x_k)|||_p = \max \left\{ \left\| \left(\sum x_k^* x_k \right)^{\frac{1}{2}} \right\|_p, \left\| \left(\sum x_k x_k^* \right)^{\frac{1}{2}} \right\|_p \right\}$$

and if $0 < p \leq 2$:

$$|||(x_k)|||_p \stackrel{\text{def}}{=} \inf_{x_k = a_k + b_k} \left\{ \left\| \left(\sum a_k^* a_k \right)^{\frac{1}{2}} \right\|_p + \left\| \left(\sum b_k b_k^* \right)^{\frac{1}{2}} \right\|_p \right\}.$$

Note that $\beta_p = 1$ if $p \geq 2$, while $\alpha_p = 1$ if $p \leq 2$

The case $p = 1$ is in 1991 joint work with F. Lust-Piquard
There we give 2 proofs.
One of the proofs shows that the non-commutative Khintchine inequality for $p = 1$
is essentially equivalent to the
"little non-commutative Grothendieck inequality"
that I had proved in 1978

Example of Application

Consider $x_{ij} \in \mathbb{C}$
Question When does

$$[\pm x_{i,j}] \in S_p$$

for almost all choices of independent signs $\pm$?
i.e. When does $[\varepsilon_{i,j} x_{i,j}] \in S_p$ a.s. ?
Answer
•IFF $\sum_i (\sum_j |x_{ij}|^2)^{p/2} + \sum_j (\sum_i |x_{ij}|^2)^{p/2} < \infty$ when $2 \le p < \infty$
• IFF $\exists$ a decomposition $x_{ij} = a_{ij} + b_{ij}$ such that
$\sum_i (\sum_j |a_{ij}|^2)^{p/2} + \sum_j (\sum_i |b_{ij}|^2)^{p/2} < \infty$ when $1 \le p \le 2$
This follows from

$$\| \sum \varepsilon_{i,j} x_{ij} e_{i,j} \|_{L_P(S_p)} \simeq |||(x_{ij} e_{i,j})|||_p$$

Digression: Cotype 2

When $0 < p \leq 2$ the L_p and $L_p(\tau)$ spaces are **of cotype 2**

Definition

B is of cotype 2 if there is C such that

$$(\sum \|x_n\|^2)^{1/2} \leq C \|\sum \varepsilon_n x_n\|_{L_2(B)}$$

$$\sum \varepsilon_n x_n \text{ converges a.s. in B } \Rightarrow (\sum \|x_n\|^2)^{1/2} < \infty$$

The case $0 < p < 1$ remained open since our 1991 paper.
In JFA 2009 I made an attempt to solve the problem.
I proposed to use an extrapolation method (idea originated in Maurey's work, already used to prove my non-com Grothendieck theorem)
The idea was to introduce a priori for $0 < p \le 2$ the property

$$(K_p) \quad \begin{cases} \exists \beta_p \text{ such that for any finite sequence} \\ x = (x_k) \text{ in } B = L_p(\tau) \text{ we have} \\ |||x|||_p \le \beta_p \| \sum \varepsilon_n x_n \|_{L_p(B)} \text{ where} \end{cases}$$

$$|||(x_k)|||_p \stackrel{\text{def}}{=} \inf_{x_k = a_k + b_k} \left\{ \left\| \left(\sum a_k^* a_k \right)^{\frac{1}{2}} \right\|_p + \left\| \left(\sum b_k b_k^* \right)^{\frac{1}{2}} \right\|_p \right\}.$$

Extrapolation Idea: to prove $(K_q) \Rightarrow (K_p) \ \ \forall 0 < p < q < 2$.
Note: For the extrapolation argument (ε_n) can be replaced by any general orthonormal sequence ($\exists$ analogy with Rudin's $\Lambda(p)$-sets)

The key ingredient for this is a new form of (non-commutative) Hölder inequality involving the Jordan product

Notation:

$$x \cdot y = (xy + yx)/2$$

$$J(x)(y) = x \cdot y$$

$$\mathcal{D} = \{f > 0 \mid \tau(f) = 1 \text{ densities}\}$$

Fix $0 < p < 2$

For $x = (x_k)$ (finite sequence) and $q \in [p, 2]$

$$C_q(x) = \inf_{f \in \mathcal{D}} \left\{ \left\| J(f^{\frac{1}{p} - \frac{1}{q}})^{-1} (\sum \varepsilon_n x_n) \right\|_{L_q(\mathbb{P} \times \tau)} \right\}$$

Let

$$J(f^{\frac{1}{p} - \frac{1}{q}})^{-1} (\sum \varepsilon_n x_n) = \sum \varepsilon_n y_n.$$

Then

$$x_n = (f^{\frac{1}{p} - \frac{1}{q}} y_n + y_n f^{\frac{1}{p} - \frac{1}{q}})/2.$$

$0 < p < 2$ fixed, but $q \in [p, 2]$ varies

$$C_q(x) = \inf_{f \in \mathcal{D}} \left\{ \left\| J(f^{\frac{1}{p} - \frac{1}{q}})^{-1} (\sum \varepsilon_n x_n) \right\|_{L_q(\mathbb{P} \times \tau)} \right\}$$

$$C_p(x) = \| \sum \varepsilon_n x_n \|_{L_p(\mathbb{P} \times \tau)} \qquad\qquad C_2(x) \simeq |||x|||_p$$

Extrapolation idea to prove $(K_q) \Rightarrow (K_p)$ requires:
for $p < q < 2$ and $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$

$$C_q(x) \lesssim C_p(x)^{1-\theta} C_2(x)^{\theta}$$

Indeed, assuming (K_q) we have for $y_n = J(f^{\frac{1}{p} - \frac{1}{q}})^{-1}(x_n)$

$$(K_q) \Rightarrow |||y|||_q \lesssim \| \sum \varepsilon_n y_n \|_{L_q(B)}$$

and choosing f realizing (up to a factor) the $\inf_{f \in \mathcal{D}}$

$$|||y|||_q \lesssim C_q(x) \lesssim C_p(x)^{1-\theta} C_2(x)^{\theta}$$

Sublemma: $|||x|||_p \lesssim |||y|||_q$ for any $f \in \mathcal{D}$
Conclusion: $|||x|||_p \lesssim C_p(x)^{1-\theta} |||x|||_p^{\theta} \Rightarrow |||x|||_p \lesssim C_p(x)$

Proof of Sublemma is not difficult but uses $\forall f, g \in \mathcal{D}$

$$|||(f^{\frac{1}{p}-\frac{1}{q}} z_n g^{\frac{1}{q}-\frac{1}{2}})|||_p \lesssim |||(z_n)|||_2 = (\sum \|z_n\|_2)^{1/2}$$

which follows from the 3 line lemma
and hence

$$|||J(f^{\frac{1}{p}-\frac{1}{q}})J(g^{\frac{1}{q}-\frac{1}{2}})z_n|||_p \lesssim |||(z_n)|||_2 = (\sum \|z_n\|_2)^{1/2}$$

Main point is Hölder type inequality

$$(*) \quad C_q(x) \lesssim C_p(x)^{1-\theta} C_2(x)^{\theta}$$

which actually reduces indeed to a form of Hölder inequality:
Define r by $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$ then (*) follows from:
$\forall F \in B^+_{L_r}$ (e.g. $F = f^{\frac{1}{p}-\frac{1}{2}} = f^{\frac{1}{r}}$)

$$(**) \quad \|J(F^{1-\theta})x\|_q \lesssim \|J(F)x\|_p^{1-\theta} \|x\|_2^{\theta}$$

Review of commutative case

In the commutative case, with m instead of τ, it is easy to check that for any $q \in [p, 2]$

$$C_q(x) = \inf_{f \in \mathcal{D}} \|f^{-(\frac{1}{p}-\frac{1}{q})}(\mathbb{E}|\sum \varepsilon_n x_n|^q)^{1/q}\|_{L_q(m)}$$

and hence since $\|g\|_p = \inf_{f \in \mathcal{D}} \|f^{-(\frac{1}{p}-\frac{1}{q})} g\|_q$

$$C_q(x) = \|(\mathbb{E}|\sum \varepsilon_n x_n|^q)^{1/q}\|_{L_p(m)}$$

Fix $0 < p < 2$**.** Then for all $q \in (p, 2)$ with $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$

$$C_q(x) \leq C_p(x)^{1-\theta} C_2(x)^{\theta}$$

of course by classical Khintchine ineq.

$$\forall q \in [2, p] \quad C_q(x) \simeq \|(\sum |x_n|^2)^{1/2}\|_{L_p(m)}$$

Return to non-commutative case Problem is to show $\forall F \in B^+_{L_r}$ (e.g. $F = f^{\frac{1}{p}-\frac{1}{2}} = f^{\frac{1}{r}}$)

$$\|J(F^{1-\theta})x\|_q \lesssim \|J(F)x\|_p^{1-\theta}\|x\|_2^{\theta}$$

where

$$J(x) = \frac{xy + yx}{2}$$

When $1 \leq p < q < 2$, this holds cf. [P, JFA 2009] (using interpolation results notably by Junge-Parcet)
This uses the boundedness, $L_p \to L_p$ if $p > 1$ or $L_1 \to \mathrm{weak}\ L_1$ of the triangular projection

$$P: \ [x_{ij}] \mapsto [x_{ij} 1_{i \leq j}]$$

Unbounded for $p < 1$!

For $p > 1$ the proof of (*) is very simple because the boundedness $L_p \to L_p$ of the triangular projection implies

$$\|J(F^{1-\theta})x\|_p \simeq \|F^{1-\theta}x\|_p + \|xF^{1-\theta}\|_p$$

so we are reduced to work with one sided multiplication for which Hölder inequalities are well known (by the 3 line lemma)

Case $0 < p < 1$

In [P. JFA2009], I could not prove

$$(**) \quad \|J(F^{1-\theta})x\|_q \lesssim \|J(F)x\|_p^{1-\theta}\|x\|_2^{\theta}$$

However I observed that the value of θ is irrelevant and that it suffices to have a very weak estimate (valid $\forall x, F$)

$$(***) \quad \forall\delta > 0 \quad \|J(F^{1-\theta})x\|_q \leq w(\delta)\|J(F)x\|_p + \delta\|x\|_2$$

with

$$\lim_{\delta\to 0} w(\delta) = 0$$

this is enough for the extrapolation proof to work
But I was stuck, I could not prove (***)...
In Fall 2014, Éric Ricard wrote to me that :

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We worked together on [P and Ricard, JIMJ to appear in 2016] and proved the following version of Hölder inequality
With θ as before such that

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2} \text{ and } \frac{1}{r} = \frac{1}{p} - \frac{1}{2}$$

For any $0 < \theta' < 1$ be such that

$$1 - \theta' < (p/2)(1-\theta)$$

$$\forall F \in B_{L_r}^{+} \quad \|J(F^{1-\theta})x\|_q \lesssim \|J(F)x\|_p^{1-\theta'} \|x\|_2^{\theta'}$$

A key ingredient: Complex Uniform Convexity of $L_p(\tau)$ (Q. Xu) for $0 < p < 1$

Application:

Non-commutative Khintchine (K_p) holds for $0 < p < 1$

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More generally
Let x be in $L_s(\tau)$, and let $f \in L_1^+$ with $\|f\|_1 = 1$. Note that $\|f^\alpha\|_r = 1$ ($\alpha = 1/r$). Let $0 < \theta < 1$. Let q be determined by

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{s} \text{ and let } \alpha = \frac{1}{r} = \frac{1}{p} - \frac{1}{s}.$$

Theorem (P.-Ricard)

Let $0 < p < q < s \leq \infty$. Let α, θ be as above. Then for any θ' such that $1 - \theta' < (p/2)(1-\theta)$ there is a constant C such that for any $x \in L_s(\tau)$ and $f \in L_1(\tau)^+$ with $\|f\|_1 = 1$, and for any unitaries $V, W \in M$ commuting with f we have

$$\left\| xWf^{\alpha(1-\theta)} + Vf^{\alpha(1-\theta)}x \right\|_q \leq C \left\| xWf^\alpha + Vf^\alpha x \right\|_p^{1-\theta'} \|x\|_s^{\theta'}. \quad (2)$$

In particular for any choice of sign ± 1 we have

$$\left\| xf^{\alpha(1-\theta)} \pm f^{\alpha(1-\theta)}x \right\|_q \leq C \left\| xf^\alpha \pm f^\alpha x \right\|_p^{1-\theta'} \|x\|_s^{\theta'}. \quad (3)$$

Application to the Mazur maps

The Mazur map $M_{p,q} : L_p(\tau) \to L_q(\tau)$ $(0 < p, q < \infty)$ given by

$$M_{p,q}(f) = f|f|^{\frac{p-q}{q}}$$

uniform homeomorphism between unit spheres (due to Raynaud)
Question: For which $0 < \gamma \leq 1$ is it γ-Hölder,i.e. $\exists C$ such that $\forall g, h \in L_p(\tau)$ with $\|g\|_p = \|h\|_p = 1$ we have

$$\big\|M_{p,q}(g) - M_{p,q}(h)\big\|_q \leq C\big\|g - h\big\|_p^\gamma.$$

If $1 \leq p, q < \infty$, $M_{p,q}$ is Hölder with exponent $\min\{1, \frac{p}{q}\}$ as for commutative integration (due to É. Ricard) Actually, Ricard proved that for $0 < p, q < \infty$, $M_{p,q}$ is γ-Hölder IFF
$\forall x = x^* \in L_\infty(\tau)$, $\|x\|_\infty = 1$, $\forall \phi \in L_p(\tau)^+$, $\|\phi\|_p = 1$,

$$\big\|x\phi^{\frac{p}{q}} \pm \phi^{\frac{p}{q}}x\big\|_q \lesssim \big\|x\phi \pm \phi x\big\|_p^\gamma. \tag{4}$$

But this is the same as (3) with $s = \infty$ $\phi = f^\alpha$ and $\gamma = 1 - \theta'$

and hence we obtain

Theorem

For any $0 < p, q < \infty$ and any semifinite von Neumann algebra, the Mazur map $M_{p,q}$ is γ-Hölder for some $0 < \gamma < 1$. If $0 < p, q \leq 1$ this holds for $\gamma < \frac{1}{2q}\left(\frac{p}{3^k}\right)^2$ where $k \geq 0$ is the smallest integer such that $\frac{p}{q} < 3^k$.

Thank you !

All preprints are on arxiv

Reference: Gilles Pisier and Éric Ricard. The non-commutative Khintchine inequalities for $0 < p < 1$, to appear in J.Inst.Math.Jussieu