Spectral multipliers for abstract self-adjoint operators: a review of some recent results

El Maati Ouhabaz, Univ. Bordeaux

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Our aim: suppose minimal regularity conditions on F: spectral multiplier results.

Theorem (Duong-Ou-Sikora, 2002, also Hebisch)

Suppose that (X, ρ, μ) is an open subset of a space of homogeneous type with homogeneous "dimension" d. Suppose that the heat kernel $p(t, x, y)$ of L satisfies the Gaussian upper bound

$$|p(t, x, y)| \le \frac{Ce^{-c\frac{\rho^2(x,y)}{t}}}{v(x, \sqrt{t})}, \; t > 0, x, y \in X.$$

If $\sup_{t>0} \|F(t\cdot)\varphi(\cdot)\|_{W^{\beta,\infty}} < \infty$ for some non-trivial $\varphi \in C_c^\infty(0, \infty)$ and some $\beta > d/2$, then $F(L)$ is of weak type $(1, 1)$ and extends to a bounded operator on L^p for all $p \in (1, \infty)$.

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Theorem (Hörmander)

If $\sup_{t>0} \|F(t\cdot)\varphi(\cdot)\|_{W^{\beta,2}} < \infty$ for some non-trivial $\varphi \in C_c^\infty(0,\infty)$ and some $\beta > d/2$, then the Fourier multiplier $F(-\Delta)$ is of weak type $(1,1)$ and extends to a bounded operator on $L^p(\mathbb{R}^d)$ for all $p \in (1,\infty)$.

Extension to Lie group settings by Christ, Mauceri-Meda, Alexopoulos....

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- **Schrödinger operators**: $L = -\Delta + V$ where $0 \leq V \in L^1_{loc}$. Possible Gaussian upper bound if V has a "small" negative part.
- **Laplace-Beltrami** on manifolds M:

$$|p(t,x,y)| \leq \frac{Ce^{-c\frac{\rho^2(x,y)}{t}}}{v(x,\sqrt{t})}$$

holds in many cases: non-negative Ricci curvature (Li-Yau), global Sobolev inequality or Faber-Krahn inequality...results by Grigory'an, Coulhon, Saloff-Coste, Varopoulous....

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$$R_\lambda f(\omega) := \hat{f}(\lambda\omega), \ \ \omega \in \mathbb{S}^{d-1}, \lambda > 0$$

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For this reason, we call $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ a *Stein-Tomas restriction type condition* and (R_p) the *Stein-Tomas* $(p, 2)$ *restriction condition.*

In order to state some of our results for F with less regularity, we recall that L satisfies the finite speed propagation property if the kernel of $\cos(t\sqrt{L})$ satisfies:

$$\text{(FS)} \qquad \operatorname{Supp} K_{\cos(t\sqrt{L})} \subseteq \{(x, y) \in X \times X : \rho(x, y) \leq t\} \quad \forall t > 0\,.$$

Property (FS) holds for most of second order self-adjoint operators and is equivalent to Davies-Gaffney estimates.

Theorem (P.Chen, E.M. Ou, A. Sikora, L. Yan, 2012)

Suppose that L satisfies the finite speed propagation property and the restriction estimate $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ *for some* p, s, q *such that* $1 \leq p < s \leq \infty$ *and* $1 \leq q \leq \infty$.

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(i) **Compactly supported multipliers:** *Let F be an even function such that* $\mathrm{Supp} F \subseteq [-1, 1]$ *and* $F \in W^{\beta,q}$ *for some* $\beta > d(1/p - 1/s)$. *Then* $F(\sqrt{L})$ *is bounded on* $L^p(X)$, *and*

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(ii) **General multipliers:** *Suppose* $s = 2$ *and F satisfies*

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For operators not satisfying (FS), a similar result but slightly weaker, was proved recently by Kunstmann and Uhl (J. Op. Theory 2015).

As a consequence (under $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$):
for all $\delta > \max\{d(1/p - 1/2) - 1/q, 0\} =: \delta_q(p)$ the Bochner-Riesz mean of order δ, $S^\delta_R(L)$ with

$$S^\delta_R(\lambda) = \begin{cases} \left(1 - \frac{\lambda}{R^2}\right)^\delta & \text{for } \lambda \le R^2 \\ 0 & \text{for } \lambda > R^2. \end{cases}$$

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Theorem

Assume that L satisfies the finite speed propagation property and the restriction condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ for some p, q satisfying $1 \leq p < 2$ and $1 \leq q \leq \infty$. Then the Bochner Riesz mean $S_R^{\delta_q(p)}(L)$ is of weak-type (p, p) uniformly in R.

In the Euclidean case, Christ and Sogge proved weak-type $(1, 1)$ for $S_R^{\delta_2(1)}(-\Delta)$. Weak-type (p, p) estimates of $S_R^{\delta_2(p)}(-\Delta)$ are proved by Christ when $p < \frac{2d+2}{d+3}$. The endpoint estimates for $p = \frac{2d+2}{d+3}$ are proved by Tao. Bochner-Riesz summability for $-\Delta$ on $\mathbb{R}^d$ holds $p \leq \frac{2d+4}{d+4}$ (due to S. Lee 2004) and improved recently by Bourgain-Guth '2011.

Links to dispersive or Strichartz estimates

Strichartz estimates for the Schrödinger equation associated to L:

$$\partial_t u + iLu = 0, \;\; u(0) = f \in L^2$$

read as follows:

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Theorem

(i) *Suppose that L satisfies the Strichartz estimate and the classical smoothing property*

$$\|\exp(-tL)\|_{p \to \frac{2d}{d+2}} \leq K t^{-\frac{d}{2}(\frac{1}{p} - \frac{d+2}{2d})},$$

for all $p \in [1, \frac{2d}{d+2}]$. *Then the restriction estimate* (R_p) *is satisfied.*

(ii) *Fix* $p \in [1, \frac{2d}{d+2}]$. *Suppose that* $V(x, r) \sim r^d$. *Assume that L satisfies the finite speed propagation property together with Strichartz and smoothing estimates as in* (i). *Then the previous sharp spectral multiplier results hold with regularity* $W^{\beta,2}$ *for* $\beta > d(1/p - 1/2)$.

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$$\|e^{itL}\|_{1\to\infty} \leq C|t|^{-d/2}, \quad t \in \mathbb{R},\ t \neq 0$$

implies the Strichartz estimate (due to Keel and Tao, 1998). Therefore for L satisfying the dispersive estimate one has sharp spectral multiplier results for $p \in [1, \frac{2d}{d+2}]$.

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- The Stein-Tomas type restriction estimate $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ does not hold for operators having discrete spectrum. We have a different formulation for these operators (examples: the harmonic oscillator $-\Delta + |x|^2$, the Laplacian on a compact manifold...). In this setting the corresponding "restriction estimate" (R_p) is the *Sogge's spectral cluster condition*

$$\left\|E_{\sqrt{L}}[\lambda, \lambda+1)\right\|_{p\to p'} \le C(1+\lambda)^{d(\frac{1}{p}-\frac{1}{p'})-1}.$$

Remarks

- The dispersive estimate

$$\|e^{itL}\|_{1\to\infty} \le C|t|^{-d/2}, \quad t \in \mathbb{R},\ t \neq 0$$

implies the Strichartz estimate (due to Keel and Tao, 1998). Therefore for L satisfying the dispersive estimate one has sharp spectral multiplier results for $p \in [1, \frac{2d}{d+2}]$.
- The Euclidean Laplacian satisfies the dispersive estimate.
- The Schrödinger operator $-\Delta - \frac{c}{|x|^2}$ satisfies Strichartz estimates (due to Burq, Planchon, Stalker and Tahvildar-Zadeh, 2003).
- The Stein-Tomas type restriction estimate $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ does not hold for operators having discrete spectrum. We have a different formulation for these operators (examples: the harmonic oscillator $-\Delta + |x|^2$, the Laplacian on a compact manifold...). In this setting the corresponding "restriction estimate" (R_p) is the *Sogge's spectral cluster condition*

$$\left\|E_{\sqrt{L}}[\lambda, \lambda+1)\right\|_{p\to p'} \le C(1+\lambda)^{d(\frac{1}{p}-\frac{1}{p'})-1}.$$

- Is the limitation in p, i.e. $p \le \frac{2d}{d+2}$ optimal in our abstract setting ? could one push this to $\frac{2d+2}{d+3}$?

Links to the semigroup

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Theorem (Bernicot-Ou 2013)

Let d be a positive constant and fix $p \in [1, 2)$. The following assertions are equivalent.
1) The restriction estimate (R_p) holds for every $\lambda > 0$;
2) There exists a positive constant C such that

$$\|L^N e^{-tL}\|_{L^p \to L^{p'}} \leq C(N-1)! N^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})} t^{-N-\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})},$$

for all $t > 0$ and all $N \in \mathbb{N}$.

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The main ingredient for the proof is the following formula for the functional calculus:

$$\lim_{N\to\infty} \frac{1}{(N-1)!} \int_0^\infty \phi(s^{-1}) \langle ((N-1)sL)^N e^{-s(N-1)L} f, g \rangle \frac{ds}{s} = \langle \phi(L) f, g \rangle.$$

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Hence

$$dE_L(\lambda) = \lim_{N\to\infty} \frac{1}{N!} \left[\lambda^{-1} (N\lambda^{-1} L)^{N+1} e^{-N\lambda^{-1} L} \right].$$

Assume that the heat semigroup $(e^{-tL})_{t>0}$ satisfies the classical $L^p - L^2$ estimates

$$\|e^{-tL}\|_{L^p\to L^2} \le Ct^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \qquad \text{for every } t>0 \text{ and some } p\in[1,2].$$

Then we observe that for every integer $N \ge 3$

$$\begin{aligned}
\|L^N e^{-tL}\|_{L^p\to L^{p'}} &\le \|e^{-\frac{t}{N}L}\|_{L^2\to L^{p'}} \|L^N e^{-t(1-\frac{2}{N})L}\|_{L^2\to L^2} \|e^{-\frac{t}{N}L}\|_{L^p\to L^2} \\
&\le C\left(\frac{t}{N}\right)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} \left(\frac{N}{t(1-\frac{2}{N})}\right)^{N} e^{-N} \\
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(use Stirling's formula for the last inequality).

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Therefore we see that the gap between this very general estimate and the one required in the previous theorem is an extra term of order $N^{\frac{1}{2}}$.

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Therefore we see that the gap between this very general estimate and the one required in the previous theorem is an extra term of order $N^{\frac{1}{2}}$.
Example: if L satisfies the dispersive estimate

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then it satisfies assertion 2) of the previous theorem, namely

$$\|L^N e^{-tL}\|_{L^p \to L^{p'}} \le C(N-1)! N^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})} t^{-N-\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}.$$