Uniqueness for discrete Schrödinger evolutions

Eugenia Malinnikova
NTNU

joint work with Ph. Jaming, Yu. Lyubarskii, K.-M. Perfekt

Journées du GDR,
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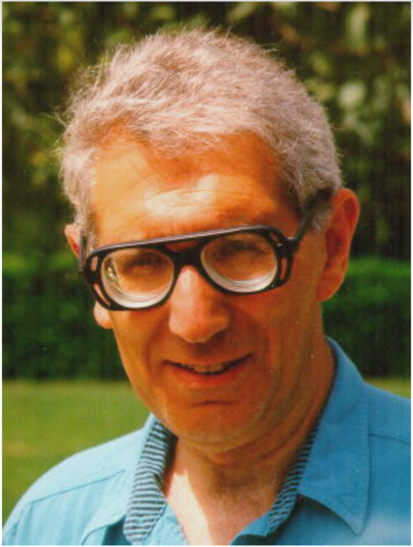

In memory of Victor Petrovich Havin

V.P. Havin (1933-2015)

Results in holomorphic and harmonic approximation

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- 1991-1998 (with A. Presa, E. Malinnikova, S. Smirnov), *Approximation by harmonic vector fields*, non-locality, topological obstacles, duality and construction.

Uniqueness, Normality, Approximation

The Cauchy problem for the Laplacian (with V.G. Maz'ya, 1974)
Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with flat boundary near zero, $v(x)$ is a regular function such that $\int_0 \log v(\rho) d\rho = -\infty$, $E \subset S^{n-1}$ of positive measure.

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- If $u \in Harm(\Omega)$, goes to zero faster than any power as $x \to 0$ along E and $\int_{|x|<r} |u_n| d\sigma \leq v(r)$ then then $u = 0$.

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- If $f_1, f_2 \in C(\partial\Omega)$ are arbitrary and $h \in C(\partial\Omega)$, $h \geq 0$, $h(0) = 0$ and goes to zero along E faster than any power, then for any $\epsilon > 0$ there is a harmonic function H such that $v(|x|)|f_1(x) - H(x)| + h(x)|f_2(x) - H_n(x)| < \epsilon$.

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- Let V be a regular function with $\int_0 \log V < \infty$. The family of harmonic in Ω functions with integrable normal derivative that satisfy $\int_{x \in \partial\Omega, |x|>r} |u| + |u_n| d\sigma \leq V(r)$ is normal.

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A vector field w in $\mathbb{R}^3$ is harmonic if $div(w) = 0$ and $curl(w) = 0$.

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- (with S.Smirnov) Approximation by harmonic functions and their gradients on metrically discontinuous sets.

Further related results by Havin and his students I

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- Uniqueness results for Riesz potentials:
  1982 V.Havin, The uncertainty principle for one-dimensional Riesz potentials. Dokl. Akad. Nauk SSSR
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  2007 K. Izyurov, On a uniqueness theorem for M. Riesz potentials. St.Petersburg Math. J.

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- Normal families of harmonic functions:
  2014 A. Logunov, On the higher dimensional harmonic analog of the Levinson loglog theorem. C. R. Math. Acad. Sci. Paris

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2015 M. Dubashinskiy, Interpolation by periods in planar domains, arXiv:1511.07015.

Carleman estimates

Carleman (1933)

$$\|e^{T\phi(x)}u\| \leq C_\phi \|e^{T\phi(x)}Pu\|$$

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$$\|e^{T\phi(x)}u\| \leq C_\phi \|e^{T\phi(x)}Pu\|$$

Implies unique continuation for a large class of second order elliptic PDE; can be use to show that an analytic function in the disk can not vanish on a set of positive measure on the boundary circle.

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Discrete Schrödinger evolutions

Equation

$$\partial_t u = i(\Delta_d u + Vu),$$

where $u : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{C}$ and Δ_d is the discrete Laplacian, that is, for a complex valued function $f : \mathbb{Z} \to \mathbb{C}$,

$$\Delta_d f(n) := f(n+1) + f(n-1) - 2f(n).$$

We assume that the potential $V = V(t, n)$ is a (real-valued) bounded function.

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We assume that the potential $V = V(t, n)$ is a (real-valued) bounded function.

Uniqueness

$$|u(0,n)| + |u(1,n)| \le Cm(n) \quad \Rightarrow \quad u \equiv 0.$$

Continuous case

L. Escauriaza, C. E. Kenig, G. Ponce, L. Vega (and M. Cowling) (2006-12)

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$$\partial_t u = i\Delta u, \quad |u(0,x)| + |u(1,x)| \leq C \exp(-x^2/4),$$
$$(*) \qquad \Rightarrow \quad u(0,x) = A \exp(-(1+i)x^2/4)$$

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For any bounded $V(x,t)$ and any $a > 1/4$

$$\partial_t u = i\Delta u + Vu, \quad |u(0,x)| + |u(1,x)| \leq C \exp(-ax^2),$$

$$(**) \qquad \Rightarrow \quad u(t,x) \equiv 0$$

Free evolution and uncertainty principles

Hardy's uncertainty principle:

$$|f(x)| \leq C \exp(-\pi|x|^2), \quad |\hat{f}(\xi)| \leq C \exp(-\pi|\xi|^2),$$

$$\Rightarrow \quad f(x) = A \exp(-\pi|x|^2)$$

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Heisenberg's uncertainty principle can be reformulated in terms of

$$h(t) = \|xu(t,x)\|_2$$

Logarithmic convexity for weighted norms of solutions to PDE

Elliptic PDE: S. Agmon (1966); Landis and others (1980s), Garofalo and Lin (1987), Brummelhuis (1995)

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$$H_R(t) = \|\phi_R(x)u(t,x)\|_2^2, \quad \phi_R(x) = \exp(\gamma|x + Rt(1-t)|^2)$$

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$$\exp(-R^2(16\gamma)^{-1})H_R(1/2) \leq H_R(0)^{1/2}H_R(1)^{1/2} = H(0)^{1/2}H(1)^{1/2}$$

Let $R \to \infty$ and get a contradiction when $\gamma > \gamma_0$.

Discrete case: previous results

Discrete potential theory: old and new

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Aim of our work: prove logarithmic convexity of the norms

$$H(t) = \|\psi(n)u(t,n)\|_2$$

for some appropriate ψ and derive uniqueness.

Toy example: Free discrete Schrödinger

Proposition

Let $\partial_t u = i\Delta_d u$*, and*

$$|u(0,n)|,\ |u(1,n)| \le C\frac{1}{\sqrt{|n|}}\left(\frac{e}{2|n|}\right)^{|n|} \sim J_n(1) \sim 2^{-n}(n!)^{-1}.$$

Then $u(t,n) = Ai^{-n}e^{-2it}J_n(1-2t)$ *for all* $n \in \mathbb{Z}$ *and* $0 \le t \le 1$*, for some constant* A*.*

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$$\Phi(t,\theta) = \sum_{k=-\infty}^{\infty} u(t,k)\theta^k \ \in L^2(\mathbb{T})$$

$$\partial_t\Phi(t,\theta) = i(\theta+\theta^{-1}-2)\Phi(t,\theta),\ \Phi(1,\theta) = e^{i(\theta+\theta^{-1}-2)}\Phi(0,\theta)$$

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$$\Phi_0(z) = A\exp(-i(z+z^{-1})/2)$$

Time-independent real potentials

Theorem

Let $u(t,n)$, $t>0$, $n\in\mathbb{Z}$ be a solution of $\partial_t u = i(\Delta u + Vu)$, where the potential $V = V_n$ does not depend on time is bounded and real-valued. If, for some $\varepsilon>0$,

$$|u(t,n)| \le C\left(\frac{e}{(2+\varepsilon)|n|}\right)^{|n|}, \quad n\in\mathbb{Z},\ t\in\{0;1\},$$

then $u=0$.

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Generalized eigenvectors $e_n(\lambda)$, $\Psi(\lambda, t) = \sum_n u(t,n) e_n(\lambda)$, Phragmén-Lindelöf theorem, spectral theorem: generalized eigenvectors are dense.

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Now we have our dream majorant/weight:
$m(n)=J_n(1)=\psi^{-1}(n)$.

Main result

Theorem (Jaming, Lyubarskii, M, Perfekt 2015; Fernandez-Bertolin, Vega, 2015)

If u is a solution of

$$\partial_t u = i(\Delta_d u + Vu)$$

where $V(t,n)$ is a bounded function, and

$$\|(1+|n|)^{\gamma(1+|n|)}u(0,n)\|_2, \|(1+|n|)^{\gamma(1+|n|)}u(1,n)\|_2 < +\infty,$$

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First energy estimate

$$\psi_\alpha(t) = \{\psi_\alpha(t,n)\}_{n\in\mathbb{Z}} = \{(1+|n|)^{\alpha|n|/(1+t)}\}_{n\in\mathbb{Z}}$$

Proposition

Let $V = V_1 + iV_2$, with $V_1, V_2 : [0,T]\times\mathbb{Z}\to\mathbb{R}$ and V_2 bounded and $F : [0,T]\times\mathbb{Z}\to\mathbb{C}$ bounded.

$$\partial_t u(t,n) = i(\Delta u(t,n) + V(t,n)u + F(t,n)).$$

Assume that $\{\psi_\alpha(0,n)u(0,n)\}\in\ell^2(\mathbb{Z})$ for some $\alpha\in(0,1]$. Then

$$\|\psi_\alpha(T,n)u(T,n)\|_2^2 \le e^{CT}\left(\|\psi_\alpha(0,n)u(0,n)\|_2^2 + \int_0^T \|\psi_\alpha(s,n)F(s,n)\|_2^2\,ds\right)$$

Formal computations

Let $f = \psi(t,n)u(t,n)$, $\partial_t f = \mathcal{S}f + \mathcal{A}f + iVf$, where $\mathcal{S}$ and $\mathcal{A}$ are symmetric and anti-symmetric operators, respectively. Explicitly

$$\mathcal{S}f = \frac{i}{2}\left(\psi\Delta(\psi^{-1}f) - \psi^{-1}\Delta(\psi f)\right) + \partial_t \kappa f,$$
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Then $\partial_t H(t) = 2\langle \mathcal{S}f, f\rangle$, since V is real-valued, and thus

$$\begin{aligned}
\partial_t^2 H(t) &= 2\langle \mathcal{S}_t f, f\rangle + 4\Re\langle \mathcal{S}f, f_t\rangle \\
&= 2\langle \mathcal{S}_t f, f\rangle + 4\|\mathcal{S}f\|^2 + 2\langle [\mathcal{S},\mathcal{A}]f, f\rangle + 4\Re\langle \mathcal{S}f, iVf\rangle \\
&= 2\langle \mathcal{S}_t f, f\rangle + 2\langle [\mathcal{S},\mathcal{A}]f, f\rangle + 4\Re\langle \mathcal{S}f + iVf, \mathcal{S}f\rangle \\
&= 2\langle \mathcal{S}_t f, f\rangle + 2\langle [\mathcal{S},\mathcal{A}]f, f\rangle + \|2\mathcal{S}f + iVf\|^2 - \|Vf\|^2.
\end{aligned}$$

Estimates with an auxiliary weight

Proposition

Let $\gamma > 0$. *Assume that* u *is a strong solution of*

$$\partial_t u = i(\Delta_d u + Vu)$$

where the potential V *is a bounded real-valued function. Let also*

$$\|(1+|n|)^{\gamma(1+|n|)} u(t,n)\|_2 < +\infty, \quad t \in \{0;1\}.$$

Then, for all $t \in [0,1]$, $\|(1+|n|)^{\gamma(1+|n|)} u(t,n)\|_2 < +\infty$.

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Auxiliary weight and logarithmic convexity

$$\psi(n) = e^{\kappa_b(n)}, \ \kappa_b(n) = \gamma(1+|n|)\ln^b\big(1+|n|\big),$$

where $1/2 < b < 1$, then $b \to 1$.

Final convexity estimates with a parameter

$\psi(t,n) = e^{\kappa(t,n)}$, where $\kappa(t,n) = \gamma(|n| + R(t)) \ln(|n| + R(t))$, and $R(t) = C_0 + R_0 t(1-t)$, $R_0 > 0$, C_0 being large enough. As before we define $H(t) = \|u(t,n)\psi(t,n)\|_2^2$.

Lemma

For every $\gamma > (3+\sqrt{3})/2$ there exists $C(\gamma)$ such that for $C_0 > C(\gamma)$ and $R(t) = C_0 + R_0 t(1-t)$ we have

$$\partial_t^2(\log H(t)) \geq -\frac{4\gamma}{2\gamma - 3} R_0 \log R_0 - C_1 R_0 - C_2,$$

where C_1 and C_2 depend on γ and $\|V\|_\infty$ only.

THANK YOU!