GDR AFHP 2015

Absolutely summing composition operators on H^p spaces

(joint work with Luis Rodríguez Piazza)

Pascal Lefèvre
Université d'Artois, France

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Computing the exact value of $\|C_\varphi\|$ is still an open problem !

Nevanlinna counting function

1. The Nevanlinna counting function N_φ:

$$N_\varphi(w) = \begin{cases} \displaystyle\sum_{\varphi(z)=w} \log \frac{1}{|z|} & \text{if } w \neq \varphi(0) \text{ and } w \in \varphi(\mathbb{D}) \\ 0 & \text{else.} \end{cases}$$

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Subordination principle (Littlewood)

The boundedness of C_φ is “equivalent” to

$$N_\varphi(w) \leq \log \left| \frac{1-\overline{\varphi(0)}w}{\varphi(0)-w} \right| = O\Big((1-|w|)\Big) \qquad \text{when } |w| \to 1^-$$

Composition operators on H^p via the pullback measure

With $\lambda_\varphi(E) = \lambda\big(\varphi^{*\,-1}(E)\big)$ where $E \subset \overline{\mathbb{D}}$ $\qquad \big\|f \circ \varphi\big\|_p^p = \displaystyle\int_{\overline{\mathbb{D}}} |f|^p \, d\lambda_\varphi$

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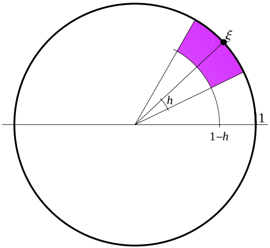

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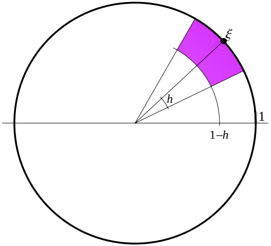

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Compactness

Theorem (Shapiro-Taylor 73)

- The problem reduces to the hilbertian case:

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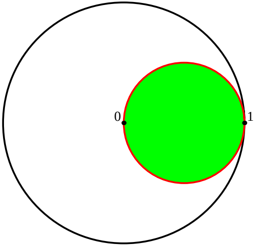
The converse is false in general: McCluer-Shapiro (86) constructed inner functions φ admitting no angular derivatives at any point of the circle.

Compactness on Hardy spaces: two examples

$$\varphi(z) = \frac{1+z}{2} \quad \text{(not compact)}$$



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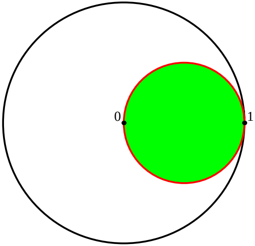
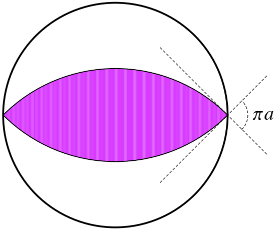


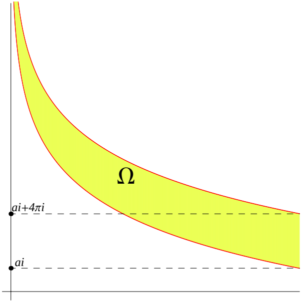


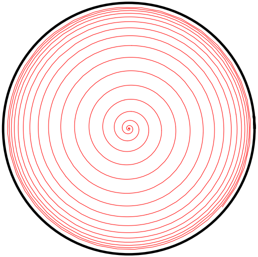
lens map $(0 < a < 1)$ very compact (nuclear)

BUT

Compactness on Hardy spaces: a third example

A surjective symbol inducing a compact composition operator:





where

$$f : \mathbb{D} \to \Omega \text{ is a conformal mapping}$$

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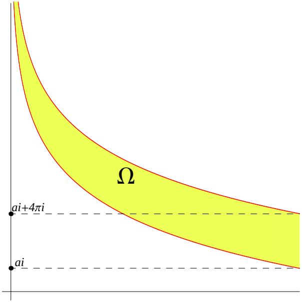

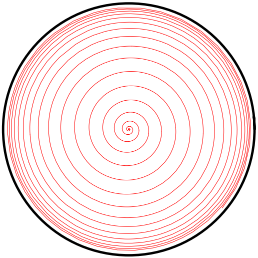

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$$\forall z \in \mathbb{D}, \qquad \varphi_1(z) = \exp\big(-f(z)\big) \quad \text{and} \quad \varphi = \Big(\frac{a-z}{1-\bar{a}z}\Big)^2 \circ \varphi_1 \text{ fits...}$$

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Theorem(Power 80, Mac-Cluer 85)

C_φ is compact *if and only if* λ_φ is a vanishing Carleson measure i.e.

$$\rho_\varphi(h) = \sup_{\xi\in\mathbb{T}} \lambda_\varphi\big(\mathcal{W}(\xi,h)\big) = o\,(h) \qquad \text{when } h \to 0$$

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(Ackeroyd '10)

$$\|C_\varphi\|_e = \limsup_{|a|\to 1^-} \left\| C_\varphi\left(\frac{k_a}{\|k_a\|_{H^2}}\right)\right\|_{H^2}$$

(k_a is the reproducing kernel)

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There are two kinds of viewpoints/characterizations:
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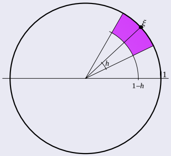

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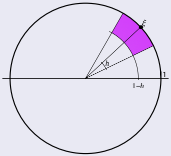

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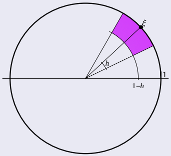

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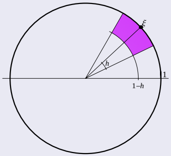

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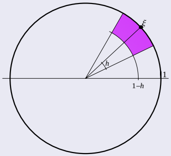

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In the sequel, we shall rather have a "Carleson" viewpoint....

Schatten Classes

Definition

Let H be a (separable) Hilbert spaces, and T a bounded (compact) operator on H.
For $p \geq 1$, define the Schatten p-norm of T as

$$\|T\|_{\mathcal{S}^p} := \Big(\sum_{n\geq 1} \lambda_n^p(|T|)\Big)^{1/p} = \Big(tr(|T|^p)\Big)^{1/p}$$

where
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Remark: T belongs to $\mathcal{S}^2$ *if and only if* T is Hilbert-Schmidt.

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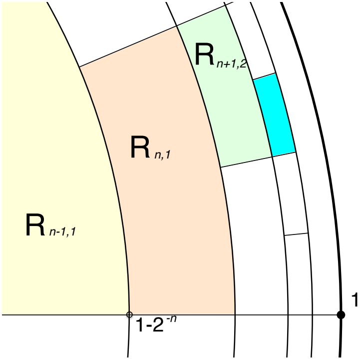
Let $n, j \geq 1$:

$$R_{n,j} = \left\{ z \in \mathbb{D}\,;\; 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1} \quad \text{and} \quad \frac{2(j-1)\pi}{2^n} \leq \arg z < \frac{2j\pi}{2^n} \right\}$$



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Let us mention too

(Luecking-Zhu '92)

$$C_\varphi \in \mathcal{S}_p \qquad \textit{if and only if} \qquad \int_{\mathbb{D}} \left(\frac{N_\varphi(z)}{\log(1/|z|)}\right)^{p/2} \frac{d\mathcal{A}}{(1-|z|^2)^2} < +\infty.$$

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- For every symbol φ, there exists $r \in (0,1)$ s.t. $a_n(C_\varphi) \gtrsim r^n$.
- If φ is the lens map (of index $\theta \in (0,1)$), then

$$e^{-\alpha_\theta \sqrt{n}} \lesssim a_n(C_\varphi) \lesssim e^{-\beta_\theta \sqrt{n}}$$

r-summing operators

Suppose $1 \leq r < +\infty$
and
let $T\colon X \to Y$ be a (bounded) operator between Banach spaces.

We say T is a *r-summing operator* if there exists $C > 0$ such that

$$\Big(\sum_{j=1}^{n} \|Tx_j\|^r\Big)^{1/r} \leq C \sup_{x^* \in B_{X^*}} \Big(\sum_{j=1}^{n} |\langle x^*, x_j\rangle|^r\Big)^{1/r} = C \sup_{a \in B_{\ell^{r'}}} \Big\|\sum_{j=1}^{n} a_j x_j\Big\|,$$

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- This forms an operator ideal.

a (generic) example

Let (K,ν) a probability space, where K is compact and consider

$$T: \left| \begin{array}{ccc} C(K) & \longrightarrow & L^r(K,\nu) \\ f & \longmapsto & f \end{array} \right.$$

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Actually, up to factorizations, any r-summing looks like this:

Pietsch Theorem

(Pietsch '67)

$T : X \to Y$ is a r-summing operator

if and only if

there exists a (probability) measure ν on the compact (B_{X^*}, w^*) s.t.

$$\forall x \in X \,, \qquad \|T(x)\| \lesssim \Big(\int_{B_{X^*}} |\xi(x)|^r \, d\nu(\xi)\Big)^{1/r}$$

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we have the following factorization

$$\begin{array}{ccccc} X & & \xrightarrow{T} & Y & \\ \Big\downarrow & & & \Big\uparrow & \widetilde{T} \\ \widetilde{X} \subset C(B_{X^*}) & & \xrightarrow{\text{“}id\text{”}} & X_r & \subset L^r(B_{X^*}, \nu) \end{array}$$

for some probability measure ν on B_{X^*}.

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Point out that r-summing operators on H^p (with $p > 1$) are compact.

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Hence we are interested in the following more general problem:

Assume from now on that μ is a Carleson measure, concentrated in the open disk $\mathbb{D}$:

When the Carleson embedding $j_\mu \colon H^p \hookrightarrow L^p(\mathbb{D}, \mu)$ is a r-summing operator ?

Known facts

(Shapiro-Taylor '73)

Let $p \geq 2$. The composition operator $C_\varphi : H^p \to H^p$ is p-summing *if and only if*

$$\int_{\mathbb{T}} \frac{1}{1 - |\varphi^*|^2} \, d\lambda < +\infty \, .$$

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But

(Domenig '99)

Let $p \in [1,2)$. There exist (p-)summing composition operator on H^p such that

$$\int_{\mathbb{D}} \frac{1}{1-|\varphi^*|^2}\, d\lambda = +\infty$$

First new results: the annulus case.

joint work with L. Rodríguez-Piazza

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Let n be an integer. We denote by μ_n the restriction of μ to the (dyadic) annulus

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Moreover, for $p > 1$:

$$H_n^p \sim \ell_{2^n}^p$$

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3. $\pi_r(D_a)$, where $D_a \colon \ell^p_{2^n} \to \ell^p_{2^n}$ is the diagonal operator whose multipliers are $a_j = \big(2^n \mu(R_{n,j})\big)^{1/p}$ (where $j = 1, 2, \ldots, 2^n$).

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But the summing norms of multipliers on sequence spaces are known, so:

1. $1 < p \leq 2$: $\quad \pi_r(j_n) \approx \left(\sum_{j=1}^{2^n} \left[2^n \mu(R_{n,j}) \right]^{2/p} \right)^{1/2}$

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   - if $p' \leq r \leq p$. $\quad \pi_r(j_n) \approx \left(\sum_{j=1}^{2^n} \left[2^n \mu(R_{n,j})\right]^{r/p} \right)^{1/r}$
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How to glue the pieces ?

First results: when $p \geq 2$

In some cases, we succeed in gluing.
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3. In the case $1 \leq r \leq p'$, the standard glue does not paste the norms...
But there might be a characterization of different nature *(result too fresh: put into quarantine)*

The case $p \leq 2$.

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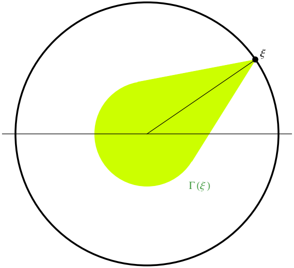
Theorem

Let $1 < p \le 2$. The Carleson embedding $j_\mu \colon H^p \to L^p(\mu)$ is absolutely summing *if and only if*

$$\int_{\mathbb{T}} \Big(\int_{\Gamma(\xi)} \frac{d\mu(z)}{(1-|z|)^{1+p/2}} \Big)^{2/p} d\lambda(\xi) < +\infty$$

where $\Gamma(\xi)$ is the Stolz domain in ξ:



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Step 1 (via Maurey factorization theorem)

Let $r > 1$ with $1/r + 1/2 = 1/p$ $\quad T : X \to L^p(\mu)$ a bounded operator.

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T is a 2-summing operator

if and only if

There exists $F \in L^r(\mu)$, with $F > 0$ μ-a.e., such that $T\colon X \to L^2(\nu)$ is well defined and 2-summing, where ν is the measure defined by

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Moreover, we have

$$\pi_2\big(T\colon X \to L^p(\mu)\big)$$

$$\approx$$

$$\inf\Big\{\pi_2\big(T\colon X \to L^2(\nu)\big) : d\nu = d\mu/F^2, F \geq 0, \int F^r\, d\mu \leq 1\Big\}.$$

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In fact we have

$$\pi_2\big(j\colon H^p \to L^2(\nu)\big) \approx \bigg(\int_{\mathbb{T}} \Big(\int_{\mathbb{D}} \frac{d\nu(z)}{|z-w|^2} \Big)^{p'/2} d\lambda(w) \bigg)^{1/p'} .$$

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Step 1 and Step 2 lead to

$j_\mu : H^p \to L^p(\mu)$ is 2-summing *if and only if*

$$\inf\left\{ \int_{\mathbb{T}} \left(\int_{\mathbb{D}} \frac{d\mu(z)}{|z-w|^2 \cdot F(z)^2} \right)^{p'/2} d\lambda(w) : F \geq 0, \int F^r \, d\mu \leq 1 \right\} \text{ is finite}$$

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By Ky Fan's lemma the order of taking the sup and the inf can be interchanged.

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Using Fubini and the following result:

Lemma

Let $h\colon \Omega \to [0, +\infty)$ be a measurable function on (Ω, Σ, μ) and $\alpha > 0$. Then

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Applying a result of Luecking, Blasco-Jarchow, we get the conclusion.

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Moreover

- As soon as $q < 2$, it is 1-summing
- When $2 \leq q < p$, it is (at least) q-summing.

Merci à vous !

Réclame

Mini-Cours du GDR AFHP à **Lens**
du 23 au 25 mai 2016

- Béatrice Vedel: *ondelettes*
- Valentin Ferenczi: *sous-groupes d'isométries*
- Eric Ricard: *espaces L^p non commutatifs*