On zeros of holomorphic functions from certain classes and applications to Lieb-Thirring inequalities

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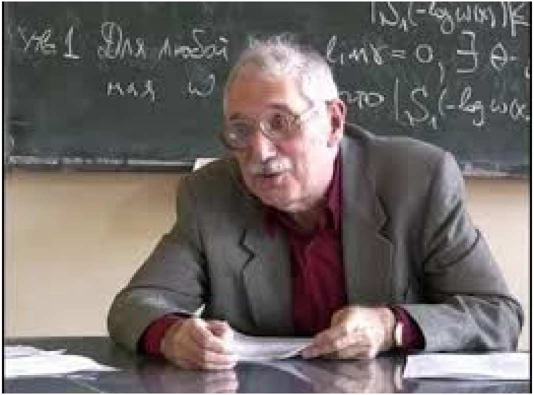
In Memoriam



Viktor Petrovich Havin, 1933 – 2015.

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4. Some open problems.

Some motivation

Let

$$J = J(\{a_k\}, \{b_k\}, \{c_k\}) = \begin{bmatrix} b_1 & c_1 & 0 & \dots \\ a_1 & b_2 & c_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

be a complex-valued Jacobi matrix.

Some motivation

Put

$$J_0 = J(\{1\},\{0\},\{1\}) = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that $\sigma(J_0) = [-2, 2]$.

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Note that $\sigma(J_0) = [-2, 2]$.

When $J - J_0 \in \mathcal{S}_\infty$

(i.e., $\lim_{k\to\infty}(|a_k - 1| + |b_k| + |c_k - 1|) = 0$),

one has $\sigma_{ess}(J) = \sigma_{ess}(J_0) = \sigma(J_0) = [-2, 2]$.

Some motivation

A fast reminder on Schatten-von Neumann ideals :

Let $A \in \mathcal{S}_\infty$, the class of compact operators. Let $s_n(A) = \lambda_n(A^*A)^{1/2}$, and

$$\mathcal{S}_p = \{A \in \mathcal{S}_\infty : ||A||^p_{\mathcal{S}_p} = \sum_n s_n(A)^p < \infty\}, \qquad p \geq 1.$$

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For $A \in \mathcal{S}_1$, on can consider

$$\det(I + \lambda A) = \det{}_1(I + \lambda A) = \prod_j (1 + \lambda\lambda_j(A)).$$

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Suppose now $J - J_0 \in \mathcal{S}_p$ and $p \in \mathbb{N}$ for simplicity. Consider

$$f(\lambda) = \det{}_p(J - \lambda)(J_0 - \lambda)^{-1} = \det{}_p(I + (J - J_0)(J_0 - \lambda)^{-1}).$$

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- $f \in Hol(\bar{\mathbb{C}} \backslash \sigma(J_0))$,
- $Z(f)$, the zero set of f, coincides with $\sigma_p(J)$, the discrete spectrum of J, up to multiplicities,
- $$\begin{aligned} |f(\lambda)| &\leq \exp\left(\Gamma_p \|(J - J_0)(J_0 - \lambda)^{-1}\|^p_{\mathcal{S}_p}\right) \\ &\leq \exp\left(\Gamma_p \|(J - J_0)\|^p_{\mathcal{S}_p} d(\lambda, \sigma(J_0))^{-p}\right). \end{aligned}$$

Some motivation

After uniformization, one gets

$$|f(z)| \le \exp\left(\Gamma_p \, \frac{||J - J_0||^p_{\mathcal{S}_p}}{(1-|z|)^p |1-z^2|^p}\right), \quad z \in \mathbb{D}.$$

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Results on zeros of holomorphic functions

- “Classical” Blaschke conditions.

Results on zeros of holomorphic functions

Theorem (Borichev-Golinskii-K' 2009)

Let $f \in \mathcal{A}(\mathbb{D})$, $|f(0)| = 1$, *satisfy the growth condition*

$$\log|f(z)| \le \frac{K}{(1-|z|)^p \, d^r(z,F)}$$

for $z \in \mathbb{D}$ *and* $p, r \ge 0$. *Then for each* $\tau > 0$ *there is a positive constant* C_1 *such that*

$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\tau} \, d^{(r-1+\tau)_+}(\zeta, F) \le C_1 \cdot K.$$

Results on zeros of holomorphic functions

Theorem (Borichev-Golinskii-K' 2015)

Let $f \in \mathcal{A}(\mathbb{D})$, $|f(0)| = 1$, satisfy the growth condition

$$\log|f(z)| \leq \frac{K}{(1-|z|)^p} \frac{d^q(z,E)}{d^r(z,F)},$$

where $z \in \mathbb{D}$ and $p, q, r \geq 0$. Then for all $0 \leq \tau' < \tau$, there is a positive constant C_2 such that

$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\tau} \frac{d^{(r-1+\tau)_+}(\zeta, F)}{d^{\min(p,q)+\tau'}(\zeta, E)} \leq C_2 \cdot K.$$

For some special p, q, one can replace τ' with τ.

Applications to Lieb-Thirring inequalities

Here are some references on recent results on properties of discrete spectrum of a complex perturbation of a “model” self-adjoint (normal) operator :

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- **from 2009** - A. Borichev, S. Favorov, L. Golinskii, SK,
  M. Demuth, M. Hansmann, G. Katriel,
  C. Dubuisson, D. Sambou, ...
  J.C. Cuenin, R. Frank, A. Laptev, J. Sabin, B. Simon, C. Tretter, ...

Lieb-Thirring inequalities

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One has $V(H_0 - \lambda)^{-1} \in \mathcal{S}_\infty$, the class of compact operators, and, by Weyl's theorem $\sigma_{ess}(H) = \sigma_{ess}(H_0) = \mathbb{R}_+$.

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Theorem (Lieb-Thirring)

Let $p > 0, d \geq 3$. Then

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^p \leq C_{p,d} \int_{\mathbb{R}^d} V_-(x)^{p+d/2}\, dx = C_{p,d} ||V_-||_{L^{p+d/2}}^{p+d/2}$$

where $V_- = \max\{-V, 0\}$.

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The above construction applies again.

To Lieb-Thirring inequalities

Using Theorem 1, one obtains the following

Theorem (Demuth-Hansmann-Katriel)

For $\tau > 0$ small enough, one has :

- *for* $d/2 < p < d$

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{(p+\tau)/2}(1+|\lambda|)^{(d-p)/2+3\tau/2}} \leq C_{\omega_0} ||V||_{L^p}^p,$$

- *for* $p \geq d$

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{d/2}(1+|\lambda|)^{2\tau}} \leq C_{\omega_0} ||V||_{L^p}^p.$$

To Lieb-Thirring inequalities

Using Theorem 2, one obtains slightly improved inequalities.

Theorem (Dubuisson-Golinskii-K)

For $\tau > 0$ small enough and $d/2 < p < d-1$, one has

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{(d-1+\tau)/2}(1+|\lambda|)^{1/2+3\tau/2}} \leq C_{\omega_0} ||V||_{L^p}^p.$$

For $p \geq d-1$, the obtained inequality is the same as in previous theorem.

Similar improvements for Dirac, Klein-Gordon, fractional Schrödinger operators, etc. etc.

Open problems

- One was interested in the case $H_0 = -\Delta$ and one used that

$$||g(x)\, f(i\nabla)||_{\mathcal{S}_p} \leq ||g||_{L^p}\, ||f|||_{L^p}.$$

Get similar bounds for H_0 = Stark operator, Schrödinger operator with Hardy, Bessel potentials, etc.

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