Non-commutativity of the exponential spectrum

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Joint work with Thomas Ransford

Let $(\mathcal{A}, +, \times)$ be a complex unital Banach algebra
(i.e. $(\mathcal{A}, +)$ is a Banach space and $\|a \times b\| \leq \|a\| \, \|b\|$).
Denote by

- $\mathbf{1}$ the unit of $\mathcal{A}$,
- $\mathrm{Inv}\mathcal{A}$ the group of invertible elements in $\mathcal{A}$,
- $\sigma(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathrm{Inv}\mathcal{A}\}$, $a \in \mathcal{A}$,
- H a complex Hilbert space,
- E a complex Banach space,
- $\mathcal{B}(E)$ the bounded linear operators acting on E,
- $\mathcal{K}(E)$ the compact linear operators acting on E,
- $\mathcal{C}(X, Y)$ the continuous functions $f : X \to Y$.

The exponential spectrum

Denote by $\mathrm{Exp}\mathcal{A}$ the connected component of $\mathrm{Inv}\mathcal{A}$ which contains $\mathbf{1}$.

Theorem

$$\mathrm{Exp}(\mathcal{A}) = \{e^{a_1} \dots e^{a_i} : a_1, \dots, a_i \in \mathcal{A}\}.$$

Definition/Theorem

Let $\mathcal{A}$ be a complex unital Banach algebra. Then $\mathrm{Exp}\mathcal{A}$ is an open and closed normal subbgroup of $\mathrm{Inv}\mathcal{A}$.
The *abstract index group* of $\mathcal{A}$ is $\frac{\mathrm{Inv}\mathcal{A}}{\mathrm{Exp}\mathcal{A}}$.
The *abstract index map* is the natural homomorphism $\gamma_{\mathcal{A}} : \mathrm{Inv}\mathcal{A} \to \frac{\mathrm{Inv}\mathcal{A}}{\mathrm{Exp}\mathcal{A}}$.
The *abstract index* of $a \in \mathcal{A}$ is $\gamma_{\mathcal{A}}(a)$.

Definition

An operator $T \in \mathcal{B}(H)$ is *Fredholm*, if $n(T) = \dim(\mathrm{Ker}(T)) < \infty$, $d(T) = \dim(\mathrm{Ker}(T^*)) < \infty$ and $\mathrm{Ran}(T)$ is closed.

The (classical) Fredholm index of a Fredholm operator T $\mathrm{Ind}(T) := n(T) - d(T)$. Denote by $\mathcal{F}$ (respectively $\mathcal{F}_n$) the set of all Fredholm operators (respectively of index n).

Theorem

There exists a group isomorphism $\alpha : \dfrac{\mathrm{Inv}\,\frac{\mathcal{B}(H)}{\mathcal{K}(H)}}{\mathrm{Exp}\,\frac{\mathcal{B}(H)}{\mathcal{K}(H)}} \to \mathbb{Z}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\mathrm{Ind}"] \arrow[d, "\Theta"'] & & \mathbb{Z} \\
\mathrm{Inv}\,\frac{\mathcal{B}(H)}{\mathcal{K}(H)} \arrow[rr, "\gamma_{\frac{\mathcal{B}(H)}{\mathcal{K}(H)}}"] & & \dfrac{\mathrm{Inv}\,\frac{\mathcal{B}(H)}{\mathcal{K}(H)}}{\mathrm{Exp}\,\frac{\mathcal{B}(H)}{\mathcal{K}(H)}} \arrow[u, "\alpha"']
\end{tikzcd}$$

We have that $\mathrm{Inv}\mathcal{C}(\mathbb{T},\mathbb{C}) = \mathcal{C}(\mathbb{T},\mathbb{C}^*)$. For every $f \in \mathcal{C}(\mathbb{T},\mathbb{C}^*)$ denote by $\mathrm{wn}(f)$ the winding number of f.

Theorem

There exists a group isomorphism $\alpha : \frac{\mathrm{Inv}\mathcal{C}(\mathbb{T},\mathbb{C})}{\mathrm{Exp}\mathcal{C}(\mathbb{T},\mathbb{C})} \to \mathbb{Z}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Inv}\mathcal{C}(\mathbb{T},\mathbb{C}) \arrow[rr, "\mathrm{wn}"] \arrow[d, "\gamma_{\mathcal{C}(\mathbb{T},\mathbb{C})}"'] & & \mathbb{Z} \\
\frac{\mathrm{Inv}\mathcal{C}(\mathbb{T},\mathbb{C})}{\mathrm{Exp}\mathcal{C}(\mathbb{T},\mathbb{C})} \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$

Harte (1976)

Let $a \in \mathcal{A}$. The exponential spectrum of a in $\mathcal{A}$ is

$$\varepsilon(a) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathrm{Exp}\mathcal{A}\}.$$

Harte (1976)

$$\partial\varepsilon(a) \subset \sigma(a) \subset \varepsilon(a).$$

For example, if $\sigma(a) = \mathbb{T}$, then $\varepsilon(a) = \mathbb{T}$ or $\varepsilon(a) = \overline{\mathbb{D}}$.

$$\varepsilon_{\frac{\mathcal{B}(H)}{\mathcal{K}(H)}}(T+\mathcal{K}(H)) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not Fredholm or } \mathrm{Ind}(\lambda - T) \neq 0\}.$$

$$\varepsilon_{\mathcal{C}(\mathbb{T},\mathbb{C})}(\phi) = \phi(\mathbb{T}) \cup \{\lambda \in \mathbb{C} \setminus \phi(\mathbb{T}) : \mathrm{wn}(\lambda - \phi) \neq 0\}.$$

Let $\phi \in \mathcal{C}(\mathbb{T},\mathbb{C})$ and T_ϕ be the Toeplitz operator of symbol ϕ, then

$$\sigma_{\mathcal{B}(H^2)}(T_\phi) = \varepsilon_{\mathcal{C}(\mathbb{T},\mathbb{C})}(\phi).$$

Theorem

Let $\Theta : \mathcal{A} \to \mathcal{B}$ be a continuous homomorphism of Banach algebra onto. We have that:

$$\Theta(\mathrm{Inv}\mathcal{A}) \subset \mathrm{Inv}\mathcal{B},$$
$$\Theta(\mathrm{Exp}\mathcal{A}) = \mathrm{Exp}\mathcal{B}.$$

Denote by $\Theta : \mathcal{B}(H) \to \frac{\mathcal{B}(H)}{\mathcal{K}(H)}$ the projection on the Calkin algebra. We have that

$$\Theta(\mathrm{Inv}\mathcal{B}(H)) = \mathcal{F}_0 + \mathcal{K}(H) \neq \mathcal{F} + \mathcal{K}(H) = \mathrm{Inv}\left(\frac{\mathcal{B}(H)}{\mathcal{K}(H)}\right),$$

$$\Theta(\mathrm{Exp}\mathcal{B}(H)) = \mathcal{F}_0 + \mathcal{K}(H) = \mathrm{Exp}\left(\frac{\mathcal{B}(H)}{\mathcal{K}(H)}\right).$$

Theorem

Let $a, b \in \mathcal{A}$. Then

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}.$$

Recall that if $\lambda\mathbf{1} - ab$ is invertible and $\lambda \neq 0$, then $(\lambda\mathbf{1} - ba)^{-1} = \frac{1}{\lambda}(\mathbf{1} - b(\lambda\mathbf{1} - ab)^{-1}a)$.

Question Murphy (1992)

Let $a, b \in \mathcal{A}$. Do we always have that

$$\varepsilon(ab) \setminus \{0\} = \varepsilon(ba) \setminus \{0\}?$$

The exponential spectrum is commutative if

- $\mathcal{A}$ is commutative ($\mathcal{C}(X, \mathbb{C}), \mathcal{A}(\mathbb{D}), \dots$),
- $\mathrm{Exp}\mathcal{A} = \mathrm{Inv}\mathcal{A}$ ($\mathcal{B}(H), \dots$),
- $\overline{\mathrm{Inv}(\mathcal{A})} = \mathcal{A}$ ($\mathcal{M}_n(\mathbb{C}), \dots$).

This straightaway rules out many candidates for a counterexample. The Calkin algebra $\frac{\mathcal{B}(H)}{\mathcal{K}(H)}$ does not satisfies those conditions. However, for every $A, B \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}^*$ such that $\lambda - AB \in \mathcal{F}$, we have that

$$\mathrm{Ind}(\lambda - AB) = \mathrm{Ind}(\lambda - BA).$$

In other words the exponential spectrum is commutative in the Calkin algebra.

Let $k \in \mathbb{N}$.

$$\mathcal{S}^{2k+1} := \left\{ (z_0, \dots, z_k) \in \mathbb{C}^{k+1} : \sum_{i=0}^{k} |z_i|^2 = 1 \right\},$$

$$\mathcal{S}^{2k} := \left\{ (z_0, \dots, z_k) \in \mathbb{C}^{k+1} : \sum_{i=0}^{k} |z_i|^2 = 1, \operatorname{Im}(z_k) = 0 \right\}.$$

The equators of $\mathcal{S}^{2k+1}$ and $\mathcal{S}^{2k}$ are

$$\left\{ (z_0, \dots, z_k) \in \mathcal{S}^{2k+1} : \operatorname{Im}(z_k) = 0 \right\},$$

$$\left\{ (z_0, \dots, z_k) \in \mathcal{S}^{2k} : \operatorname{Re}(z_k) = 0 \right\}.$$

The upper hemisphere of $\mathcal{S}^{2k+1}$ and $\mathcal{S}^{2k}$ are

$$\left\{ (z_0, \dots, z_k) \in \mathcal{S}^{2k+1} : \operatorname{Im}(z_k) > 0 \right\},$$

$$\left\{ (z_0, \dots, z_k) \in \mathcal{S}^{2k} : \operatorname{Re}(z_k) > 0 \right\}.$$

K., Ransford (2015)

There exists $a, b \in \mathcal{C}(\mathcal{S}^4, \mathcal{M}_2(\mathbb{C}))$ such that

$$\varepsilon(ab) \setminus \{0\} \neq \varepsilon(ba) \setminus \{0\}.$$

K., Ransford (2015)

There exists $a, b \in \mathcal{C}(\mathcal{S}^4, \mathcal{M}_2(\mathbb{C}))$ such that

$$\varepsilon(ab) \setminus \{0\} \neq \varepsilon(ba) \setminus \{0\}.$$

Fact

The homotopy group $\pi_4(\mathrm{GL}_2(\mathbb{C}))$ of the continuous functions $\mathcal{C}(\mathcal{S}^4, \mathrm{GL}_2(\mathbb{C}))$ is non trivial. Moreover

$$\begin{aligned} c : \mathcal{S}^4 &\to \mathrm{GL}_2(\mathbb{C}) \\ (z_0, z_1, z_2) &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{(1 + \mathrm{i} z_2)^2} \begin{pmatrix} z_0\overline{z_0} & z_0\overline{z_1} \\ z_1\overline{z_0} & z_1\overline{z_1} \end{pmatrix} \end{aligned}$$

is not homotopic to $\mathbf{1}$ in $\mathcal{C}(\mathcal{S}^4, \mathrm{GL}_2(\mathbb{C}))$.

Let

$$
\begin{aligned}
a : \mathcal{S}^4 &\to \mathcal{M}_2(\mathbb{C}) \\
(z_0, z_1, z_2) &\mapsto \frac{1}{1 + \mathrm{i} z_2} \left(\begin{array}{cc} z_0 & 0 \\ z_1 & 0 \end{array} \right) \\
b : \mathcal{S}^4 &\to \mathcal{M}_2(\mathbb{C}) \\
(z_0, z_1, z_2) &\mapsto \frac{1}{1 + \mathrm{i} z_2} \left(\begin{array}{cc} \overline{z_0} & \overline{z_1} \\ 0 & 0 \end{array} \right).
\end{aligned}
$$

We have that

$$
\begin{aligned}
(\mathbf{1} - 2ab)(z_0, z_1, z_2) &= \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - \frac{2}{(1 + \mathrm{i} z_2)^2} \left(\begin{array}{cc} z_0 \overline{z_0} & z_0 \overline{z_1} \\ z_1 \overline{z_0} & z_1 \overline{z_1} \end{array} \right) \\
&= c(z_0, z_1, z_2).
\end{aligned}
$$

So $\mathbf{1} - 2ab$ is not homotopic to $\mathbf{1}$ in $\mathrm{Inv}(\mathcal{C}(\mathcal{S}^4, \mathcal{M}_2(\mathbb{C})))$, and $\mathbf{1} - 2ab \notin \mathrm{Exp}(\mathcal{C}(\mathcal{S}^4, \mathcal{M}_2(\mathbb{C})))$.

We have that

$$
\begin{aligned}
(\mathbf{1} - 2ba)(z_0, z_1, z_2) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2(|z_0|^2 + |z_1|^2)}{(1 + +\mathrm{i}z_2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2(1 - |z_2|^2)}{(1 + \mathrm{i}z_2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \phi(z_2) & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}
$$

with $\phi(z_2) = 1 - \frac{2(1-|z_2|^2)}{(1+\mathrm{i}z_2)^2}$. We have that $(\mathbf{1} - 2ba)$ factors as:

$$
\begin{array}{ccccccc}
\mathcal{S}^4 & \longrightarrow & [-1,1] & \to & \mathbb{T} & \longrightarrow & \mathrm{GL}_2(\mathbb{C}) \\
(z_0, z_1, z_2) & \mapsto & z_2 & \mapsto & \phi(z_2) & \mapsto & \begin{pmatrix} \phi(z_2) & 0 \\ 0 & 1 \end{pmatrix}.
\end{array}
$$

So $(\mathbf{1} - 2ba)$ is homotopic to $\mathbf{1}$ in $\mathrm{Inv}(\mathcal{C}(\mathcal{S}^4, \mathcal{M}_2(\mathbb{C})))$, and $\mathbf{1} - 2ba \in \mathrm{Exp}(\mathcal{C}(\mathcal{S}^4, \mathcal{M}_2(\mathbb{C})))$.
In other words we have that $\frac{1}{2} \notin \varepsilon(ba)$ and $\frac{1}{2} \in \varepsilon(ab)$.

The Hopf map is

$$h : \mathcal{S}^3 \to \mathcal{S}^2$$
$$(z_0, z_1) \mapsto (-2z_0\overline{z_1}, |z_0|^2 - |z_1|^2).$$

Definition

Let $f : \mathcal{S}^k \to \mathcal{S}^n$. The suspension of f

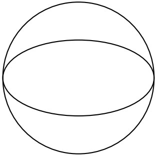
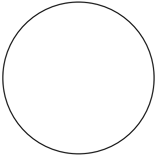
$$Ef : \mathcal{S}^{k+1} \to \mathcal{S}^{n+1}$$

is the map which reduce to f on the equators and send the upper (respectively lower) hemisphere of $\mathcal{S}^{k+1}$ to the upper (respectively lower) hemisphere of $\mathcal{S}^{n+1}$. This is done by mapping the center of each hemisphere into the center of the corresponding hemisphere and extending radially.

$$f : \mathcal{S}^1 \to \mathcal{S}^0$$

 $\mapsto$ 

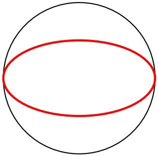
$$Ef : \mathcal{S}^2 \to \mathcal{S}^1$$

 $\mapsto$ 

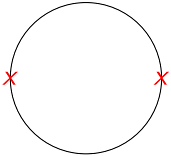
$f : \mathcal{S}^1 \to \mathcal{S}^0$

$\mapsto$

$Ef : \mathcal{S}^2 \to \mathcal{S}^1$



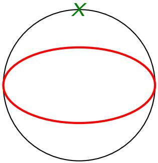
$\mapsto$



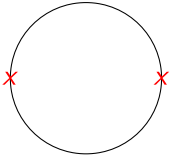
$$f : \mathcal{S}^1 \to \mathcal{S}^0$$

$\mapsto$

$$Ef : \mathcal{S}^2 \to \mathcal{S}^1$$



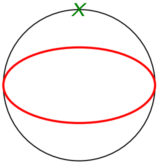
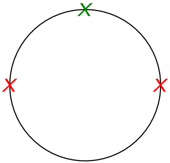
$\mapsto$



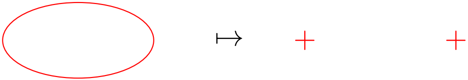
$$f : \quad \mathcal{S}^1 \quad \rightarrow \quad \mathcal{S}^0$$

 $\mapsto$ 

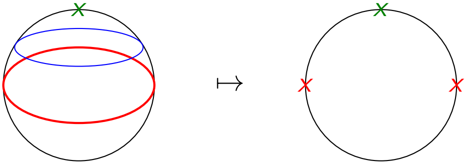
$$Ef : \quad \mathcal{S}^2 \quad \rightarrow \quad \mathcal{S}^1$$

 $\mapsto$ 

$f: \quad \mathcal{S}^1 \quad \to \quad \mathcal{S}^0$



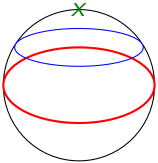
$Ef: \quad \mathcal{S}^2 \quad \to \quad \mathcal{S}^1$



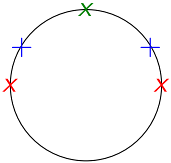
$$f: \quad \mathcal{S}^1 \quad \rightarrow \quad \mathcal{S}^0$$

$$\mapsto$$

$$Ef: \quad \mathcal{S}^2 \quad \rightarrow \quad \mathcal{S}^1$$



$$\mapsto$$



The suspension of the Hopf map is given by

$$
\begin{aligned}
& Eh : \mathcal{S}^4 \to \mathcal{S}^3 \\
& (z_0, z_1, z_2) \mapsto \left(\frac{-2z_0\overline{z_1}}{\sqrt{1-|z_2|^2}}, \frac{|z_0|^2 - |z_1|^2}{\sqrt{1-|z_2|^2}} + \mathrm{i}z_2 \right).
\end{aligned}
$$

Hopf/ Freudenthal

The Hopf map $h : \mathcal{S}^3 \to \mathcal{S}^2$ and its suspension $Eh : \mathcal{S}^4 \to \mathcal{S}^3$ are not null-homotopic.

K., Ransford (2015)

There exists $a, b \in \mathcal{C}(\mathcal{S}^4, \mathcal{M}_2(\mathbb{C}))$ such that

$$\varepsilon(ab) \setminus \{0\} \neq \varepsilon(ba) \setminus \{0\}.$$

Fact

The homotopy group $\pi_4(\mathrm{GL}_2(\mathbb{C}))$ of the continuous functions $\mathcal{C}(\mathcal{S}^4, \mathrm{GL}_2(\mathbb{C}))$ is non trivial. Moreover

$$\begin{aligned} c : \mathcal{S}^4 &\to \mathrm{GL}_2(\mathbb{C}) \\ (z_0, z_1, z_2) &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{(1 + iz_2)^2} \begin{pmatrix} z_0\overline{z_0} & z_0\overline{z_1} \\ z_1\overline{z_0} & z_1\overline{z_1} \end{pmatrix} \end{aligned}$$

is not homotopic to $\mathbf{1}$ in $\mathcal{C}(\mathcal{S}^4, \mathrm{GL}_2(\mathbb{C}))$.

K., Ransford (2015)

There exists $a, b \in \mathcal{C}(\mathcal{S}^4, \mathcal{M}_2(\mathbb{C}))$ such that

$$\varepsilon(ab) \setminus \{0\} \neq \varepsilon(ba) \setminus \{0\}.$$

Fact

The homotopy group $\pi_4(\mathrm{GL}_2(\mathbb{C}))$ of the continuous functions $\mathcal{C}(\mathcal{S}^4, \mathrm{GL}_2(\mathbb{C}))$ is non trivial. Moreover

$$c : \mathcal{S}^4 \to \mathrm{GL}_2(\mathbb{C})$$
$$(z_0, z_1, z_2) \mapsto \begin{pmatrix} 1 - \frac{2z_0\overline{z_0}}{(1+iz_2)^2} & -\frac{2z_0\overline{z_1}}{(1+iz_2)^2} \\ -\frac{2z_1\overline{z_0}}{(1+iz_2)^2} & 1 - \frac{2z_1\overline{z_1}}{(1+iz_2)^2} \end{pmatrix}$$

is not homotopic to $\mathbf{1}$ in $\mathcal{C}(\mathcal{S}^4, \mathrm{GL}_2(\mathbb{C}))$.

Let

$$p : \mathcal{M}_2(\mathbb{C}) \to \mathbb{C}^2$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (b, d).$$

We have that

$$pc(z_0, z_1, z_2) = \left(\frac{-2z_0\overline{z_1}}{(1 + \imath z_2)^2}, 1 - \frac{2\left|z_1\right|^2}{(1 + \imath z_2)^2} \right).$$

Denote by $f := \frac{pc}{|pc|}$. On the equator of $\mathcal{S}^4$ we have that $|pc| = 1$ and $f = pc$. We have that

$$Eh(z_0, z_1, z_2) = \left(\frac{-2z_0\overline{z_1}}{\sqrt{1 - \left|z_2\right|^2}}, \frac{\left|z_0\right|^2 - \left|z_1\right|^2}{\sqrt{1 - \left|z_2\right|^2}} + \imath z_2 \right).$$

Eh and f coincide on the equator of $\mathcal{S}^4$ and send the upper (respectively lower) hemisphere of $\mathcal{S}^4$ to the upper (respectively lower) hemisphere of $\mathcal{S}^3$.
So $Eh(z_0, z_1, z_2)$ and $f(z_0, z_1, z_2)$ are not antipodals points of $\mathcal{S}^3$.
So

$$t \mapsto \frac{(1-t)f - t(Eh)}{|(1-t)f - t(Eh)|},$$

is an homotopy between Eh and f.
So f is not homotopic to a constant map.
If c is Homotopic to a constant map, then pc would also be homotopic to a constant map, so be f. But f isn't homotopic to a constant map, and neither c.

K. Ransford (2015)

Let $n \in \mathbb{N}$ such that $n \geq 2$. Then there exist $a, b \in \mathcal{C}(\mathcal{S}^{2n}, \mathcal{M}_n(\mathbb{C}))$ such that

$$\varepsilon(ab) \setminus \{0\} \neq \varepsilon(ba) \setminus \{0\}.$$

Moreover, we have that $\frac{\mathrm{Inv}\mathcal{C}(\mathcal{S}^{2n}, \mathcal{M}_n(\mathbb{C}))}{\mathrm{Exp}\mathcal{C}(\mathcal{S}^{2n}, \mathcal{M}_n(\mathbb{C}))} \sim \frac{\mathbb{Z}}{n!\mathbb{Z}}$.

Let SM_n be the unreduced suspension of $\mathcal{M}_n(\mathbb{C})$, i.e.

$$\mathrm{SM}_n := \{f \in \mathcal{C}([0,1], \mathcal{M}_n(\mathbb{C})) : f(0) = \alpha I_n, f(1) = \beta I_n, \alpha, \beta \in \mathbb{C}\}.$$

Paulsen (1982)

$$\frac{\mathrm{InvSM}_n}{\mathrm{ExpSM}_n} \sim \frac{\mathbb{Z}}{n\mathbb{Z}}.$$

One can prove that for every $a, b \in \mathrm{SM}_n$

$$\varepsilon(ab) \setminus \{0\} = \varepsilon(ba) \setminus \{0\}.$$

Question

Let $\mathcal{A}$ be a complex unital Banach algebra such that the exist $a, b \in \mathcal{A}$ such that

$$\varepsilon(ab) \setminus \{0\} \neq \varepsilon(ba) \setminus \{0\}.$$

Should $\frac{\mathrm{Inv}(\mathcal{A})}{\mathrm{Exp}(\mathcal{A})}$ posses a non trivial element of finite order?

Question

Does there exists a Banach space E and operators $S, T \in \mathcal{B}(E)$ such that

$$\varepsilon(ST) \setminus \{0\} \neq \varepsilon(TS) \setminus \{0\}?$$