Multiple sampling and interpolation in the Fock space

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Plan

Interpolation and sampling

Let H be a reproducing kernel Hilbert space of analytic functions on a domain, k_λ the reproducing kernel in λ.

A sequence $\Lambda \subset \mathbb{C}$ is called *sampling* for H if

$$\|f\|^2 \asymp \sum_{\lambda\in\Lambda} \frac{|f(\lambda)|^2}{k_\lambda(\lambda)}, \quad f \in H,$$

and *interpolating* if for every $v = (v_\lambda)_{\lambda\in\Lambda} \in \ell^2$, there exists $f \in H$ such that

$$\frac{f(\lambda)}{\|k_\lambda\|} = v_\lambda, \qquad \lambda \in \Lambda.$$

Fock space

$$\mathcal{F}^2 = \{f \in \mathrm{Hol}(\mathbb{C}) : \|f\|_2^2 := \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dm(z) < \infty\}.$$

Scalar product :

$$\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} dm(z)$$

Orthonormal basis :

$$e_k(z) = \frac{z^k}{\sqrt{k!}}, \quad k \geq 0$$

Reproducing kernel :

$$k_z(\zeta) = \sum_{k\geq 0} e_k(\zeta)\overline{e_k(z)} = e^{\bar{z}\zeta}$$

with norm $\|k_z\|^2 = k_z(z) = e^{|z|^2}$.

Isometric translation :

$$T_z f(\zeta) = e^{\bar{z}\zeta - \frac{1}{2}|z|^2} f(\zeta - z), \qquad f \in \mathcal{F}^2.$$

Classical interpolation and sampling

Define lower and upper densities by

$$D^-(\Lambda) = \liminf_{r\to\infty} \frac{n^-(r)}{r^2}, \qquad D^+(\Lambda) = \limsup_{r\to\infty} \frac{n^+(r)}{r^2},$$

where $n^-(r)$ and $n^+(r)$ denote the smallest and the largest number of points of Λ in a disk $D(z,r)$, $z \in \mathbb{C}$.

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Theorem 1 (Seip, Seip-Wallstén, 1992)

A discrete set Λ is a set of sampling for $\mathcal{F}^2$ if and only if it can be expressed as a finite union of uniformly discrete sets and contains a uniformly discrete subset Λ' for which $D^-(\Lambda') > 1$.

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Theorem 2 (Seip, Seip-Wallstén, 1992)

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Observation : **no** simultaneous interpolation and sampling.

Multiple interpolation and sampling

Idea : Hermite type interpolation — more information in the nodes, less nodes required.

Observe : if $f \in \mathcal{F}^2$ then $f(z) = \sum_{k\geq 0} a_k e_k(z) = \sum_{k\geq 0} a_k \frac{z^k}{\sqrt{k!}}$, $\|f\|^2 = \sum_{k\geq 0} |a_k|^2$.

Equating with Taylor series $\langle f, e_k\rangle = a_k = \frac{f^{(k)}(0)}{\sqrt{k!}}$

Note $T_\lambda e_0(z) = T_\lambda 1(z) = e^{\bar{\lambda} z - \frac{1}{2}|\lambda|^2} = \frac{k_\lambda}{\|k_\lambda\|}$, and $\langle f, T_\lambda e_0\rangle = \frac{f(\lambda)}{\|k_\lambda\|}$.

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Generalized restriction operator
Let $X = \{(\lambda, m_\lambda)\}$ a *divisor* ($\lambda \in \Lambda$, $m_\lambda \in \mathbb{N}^*$).
$\mathcal{R} : \mathcal{F}^2 \longrightarrow l^2$, $f \longmapsto (\langle f, T_\lambda e_k \rangle)_{\lambda\in\Lambda, 0\leq k \leq m_\lambda - 1}$

New definition (Seip, Brekke-Seip)

X is called *sampling* for $\mathcal{F}^2$ if $\mathcal{R}$ is bounded and left-invertible :

$$\|f\|_2^2 \asymp \sum_{\lambda\in\Lambda}\sum_{k=0}^{m_\lambda-1} |\langle f, T_\lambda e_k\rangle|^2 = \sum_{\lambda\in\Lambda} \|f\|^2_{\mathcal{F}^2/N_\lambda^2}, \qquad f\in\mathcal{F}^2,$$

where $N_\lambda = \{f\in\mathcal{F}^2 : f(\lambda)=\cdots=f^{(m_\lambda-1)}(\lambda)=0\}$.

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X is called *interpolating* for $\mathcal{F}^2$ if $\mathcal{R}$ is onto : for every sequence $v = \{v_\lambda^{(k)}\}_{\substack{\lambda\in\Lambda \\ k<m_\lambda}}$ such that

$$\|v\|_2^2 := \sum_{\lambda\in\Lambda} \sum_{k=0}^{m_\lambda - 1} |v_\lambda^{(k)}|^2 < \infty$$

there exists a function $f \in \mathcal{F}^2$ such that

$$\langle f, T_\lambda e_k\rangle = v_\lambda^{(k)}, \qquad 0 \le k < m_\lambda, \quad \lambda \in \Lambda.$$

Uniformly bounded multiplicities

Recall :

- X sampling if $\|f\|_2^2 \asymp \sum_{\lambda\in\Lambda}\sum_{k=0}^{m_\lambda-1} |\langle f, T_\lambda e_k\rangle|^2$,
- X interpolating if $\sum_{\lambda\in\Lambda}\sum_{k=0}^{m_\lambda-1} |v_\lambda^{(k)}|^2 < \infty$ implies $\exists f \in \mathcal{F}^2$, $\langle f, T_\lambda e_k\rangle = v_\lambda^{(k)}$.
- Densities: $D^-(X) = \liminf_{r\to\infty} \frac{n^-(r)}{r^2}$, $D^+(X) = \liminf_{r\to\infty} \frac{n^+(r)}{r^2}$

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Again : **no** simultaneous interpolation & sampling.

Unbounded multiplicities : a motivation

Probably the only Hilbert space where the situation of unbounded multiplicities is completely understood : the Hardy space H^2.

Let $\Theta = \prod_{n\geq 1} \theta_n$ be an inner function.

Define $\ell^2(H^2/\theta_n H^2) = \{(f_n)_n : f_n \in H^2, \sum_{n\geq 1} \|f_n\|^2_{H^2/\theta_n H^2} < +\infty\}$.

The sequence $(\theta_n)_n$ satisfies the generalized Carleson condition (or Carleson-Vasyunin condition) if $|\Theta(z)| \geq \delta \inf_n |\theta_n(z)|$, for some $\delta > 0$.

Theorem 5 (Nikolski, Vasyunin, 1978 ($p = 2$))

In the notation above, the following assertions are equivalent :

- *The operator* $\mathcal{R} : H^2 \longrightarrow \ell^2(H^2/\theta_n H^2)$, $\mathcal{R}(f) = (f + \theta_n H^2)$ *is bounded and onto,*
- $(\theta_n)_n$ *satisfies the generalized Carleson condition.*

Note $H^2/\theta_n H^2 = K_{\theta_n} := H^2 \ominus \theta_n H^2$, and the isometry is given by the orthogonal projection $P_{\theta_n} : H^2 \longrightarrow K_{\theta_n}$.

Special case : $\theta_n = b_{\lambda_n}^{m_n}$ which gives interpolation with unbounded multiplicities.

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Question : (Brekke-Seip) Are their simultaneous interpolating/sampling sequences, when $\sup_\lambda m_\lambda = \infty$?

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Consider covering/separation conditions :

- Separation : $D(\lambda, \sqrt{m_\lambda} + C)$ are separated ($C \in \mathbb{R}$ suitable).
- Covering : $\bigcup_\lambda D(\lambda, \sqrt{m_\lambda} + C) = \mathbb{C} \setminus K$ ($C \in \mathbb{R}$ suitable, K compact).
- Finite overlap condition : $\sup_{z\in\mathbb{C}} \sum_{\lambda\in\Lambda} \chi_{D(\lambda,\sqrt{m_\lambda})}(z) < \infty$.

(A Carleson embedding type condition.)

Main results

Sampling results

Theorem 6 (Borichev-H-Kellay-Massaneda, 2015)

(a) *If* $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ *is sampling for* $\mathcal{F}^2$*, then* X *satisfies the finite overlap condition and there exists* $C > 0$ *such that*

$$\bigcup_{\lambda\in\Lambda} D(\lambda, \sqrt{m_\lambda} + C) = \mathbb{C}.$$

(b) *Conversely, if* $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ *satisfies the finite overlap condition and there exists* $C = C(X) > 0$*,* $K \subset \mathbb{C}$ *compact such that*

$$\bigcup_{\lambda\in\Lambda\,:\,m_\lambda > \alpha C^2} D(\lambda, \sqrt{m_\lambda} - C) = \mathbb{C} \setminus K,$$

then X *is sampling for* $\mathcal{F}^2$*.*

Main results — continued

Interpolation

Theorem 7 (Borichev-H-Kellay-Massaneda, 2015)

(a) *If* $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ *is an interpolating divisor for* $\mathcal{F}^2$, *then there exists* $C > 0$ *such that the discs* $\{D(\lambda, \sqrt{m_\lambda} - C)\}_{\lambda\in\Lambda}$ *are pairwise disjoint.*

(b) *Conversely, if the disks* $\{D(\lambda, \sqrt{m_\lambda} + C)\}_{\lambda\in\Lambda}$ *are pairwise disjoint for some* $C > 0$, *then* X *is an interpolating divisor for* $\mathcal{F}^2$.

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Consequence : The results on interpolation and sampling allow to deduce a partial answer to the question by Brekke and Seip :

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Corollary 1 (Borichev-H-Kellay-Massaneda, 2015)

Suppose $\lim_{|\lambda|\to\infty} m_\lambda = +\infty$*. Then* X *cannot be simultaneously interpolating and sampling for* $\mathcal{F}^2$*.*

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- estimates on the incomplete Gamma-function :
  $\lambda_k(x) = \frac{1}{k!}\int_0^x y^k e^{-y}dy$, $\omega_k(x) = e^{-x}\sum_{s=0}^{k}\frac{x^s}{s!}$.

  $\lambda_k(k - t\sqrt{k}) \geq \varepsilon$, $\omega_k(k + t\sqrt{k}) \geq \varepsilon$, $\lambda_k(m - t\sqrt{m}) \leq \varepsilon\lambda_k(m)$ $(t^2 \leq m \leq k)$

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- Local norm $\sum_{k=0}^{m-1}|\langle f, T_\lambda e_k\rangle|^2 \leq C\int_{D(\lambda,\sqrt{m}-A)}|f(z)|^2 e^{-|z|^2}dm(z)$
- The above implies

Lemma 1

If $\sum_{k=0}^{m-1}|\langle f, T_\lambda e_k\rangle|^2$ *small and* $\int_{D(\sqrt{m})}|f(z)|^2 e^{-|z|^2}dm \leq 1$, *then* $\int_{D(\sqrt{m}-C)}|f(z)|^2 e^{-|z|^2}dm$ *small.*

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- The above implies

Lemma 1

If $\sum_{k=0}^{m-1}|\langle f, T_\lambda e_k\rangle|^2$ *small and* $\int_{D(\sqrt{m})}|f(z)|^2 e^{-|z|^2}dm \leq 1$, *then* $\int_{D(\sqrt{m}-C)}|f(z)|^2 e^{-|z|^2}dm$ *small.*

- Zero-sets :

Theorem 8 (Borichev-H-Kellay-Massaneda, 2015)

If $\mathbb{C}\setminus\bigcup_{\lambda\in\Lambda}D(\lambda,\sqrt{m_\lambda})$ *is compact, then* X *is not a zero-divisor of* $\mathcal{F}^\infty$

Ideas of proof

Key ingredients :

- estimates on the incomplete Gamma-function :
  $\lambda_k(x) = \frac{1}{k!}\int_0^x y^k e^{-y}dy$, $\omega_k(x) = e^{-x}\sum_{s=0}^{k}\frac{x^s}{s!}$.

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Necessary condition : $\bigcup_{\lambda\in\Lambda} D(\lambda, \sqrt{m_\lambda} + C) = \mathbb{C}$ (*).

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Sufficient condition : the disks $D(\lambda, \sqrt{m_\lambda} + C)$ are pairwise disjoint

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- Smooth interpolating function $F(z) = \sum_{\lambda\in\Lambda} Q(z)\eta(|z-\lambda| - r_\lambda)$, $r_\lambda = \sqrt{m_\lambda} + C$, $\operatorname{supp}\eta \subset (-\infty, 0)$, $\eta \equiv 1$ on $(-\infty, -C)$, $|\eta'| \lesssim 1$,

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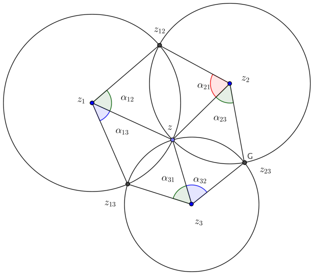
- But $\int_{D(w_k,1)} |T_{w_k}1|^2 e^{|z|^2} dm = c > 0$.

Proof of non existence of simultaneous sampling & interpolation

Here is a simple geometric argument.



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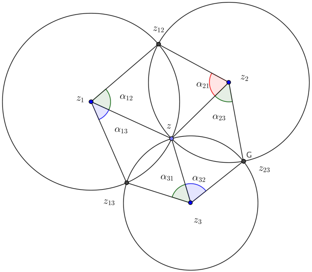

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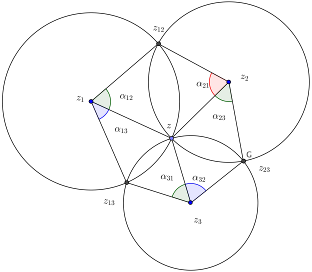

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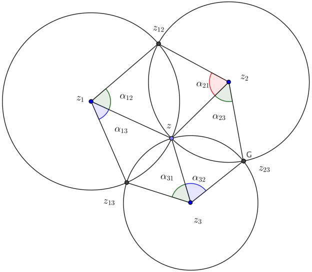

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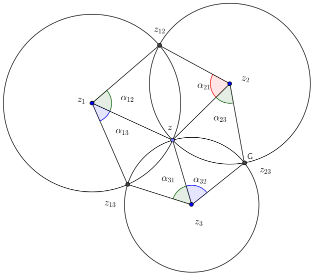

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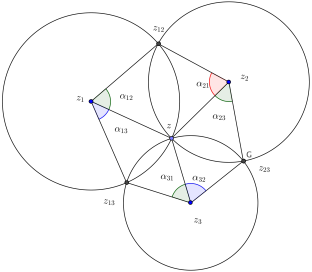

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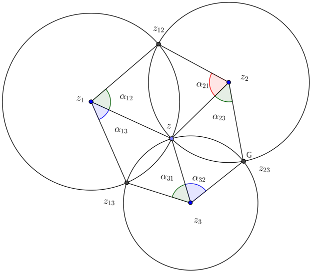

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So the disks cannot be separated by an additive constant.

Final remark : Uniform norm

Let

$$\mathcal{F}^\infty = \{f \in \mathrm{Hol}(\mathbb{C}) : \|f\|_\infty := \sup |f(z)|e^{-|z|^2/2} < \infty\}.$$

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Most of the proofs adapt to $p = +\infty$.

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Again, there are **no** simultaneously interpolating and sampling sequences in that case.