Upper frequent hypercyclicity, and related notions

joint work with Antonio Bonilla

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Linear dynamics

Linear dynamics studies the behaviour of orbits of (linear) operators.

Throughout the talk, let

$$T : X \to X$$

be an operator on a separable Banach space (Fréchet space, F-space) X.

T is hypercyclic if it has a dense orbit. That is, there is some $x \in X$ such that, for any non-empty open set $U \subset X$,

$$\{n \geq 0 : T^n x \in U\} \neq \varnothing.$$

The vector x is then called a hypercyclic vector.

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T is frequently hypercyclic (Bayart, Grivaux 2004) if there is some $x \in X$ such that, for any non-empty open set $U \subset X$,

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T is upper frequently hypercyclic (Shkarin 2009) if there is some $x \in X$ such that, for any non-empty open set $U \subset X$,

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Fact

If T is hypercyclic then the set $HC(T)$ of hypercyclic vectors is a dense G_δ-set. (Consequence of the Birkhoff transitivity theorem, using Baire).

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Baire is back! How about Birkhoff?

Also: How to explain this differing behaviour?

Upper frequent hypercyclicity

There is indeed a Birkhoff-type theorem for upper frequent hypercyclicity.

Theorem (Bonilla, G-E 2015)

An operator T is upper frequently hypercyclic if and only if, for any non-empty open subset V of X, there is some $\delta > 0$ such that for any non-empty open subset U of X there is some $x \in U$ such that

$$\overline{dens}\,\{n \geq 0 : T^n x \in V\} > \delta.$$

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Theorem (Upper Frequent Hypercyclicity Criterion, Bonilla, G-E 2015)

Suppose that there are dense subsets X_0, Y_0 of X and mappings $S_n : Y_0 \to X$ with the following property: For any $y \in Y_0$ and $\varepsilon > 0$, there exists a set $E \subset \mathbb{N}_0$, $\overline{dens}\, E > 0$, such that

- *for any $x \in X_0$*
$$T^n x \to 0 \text{ as } n \to \infty, n \in E;$$
- $$\sum_{n\in E} S_n y \text{ converges;}$$
- *for any $m \in E$*
$$\Big\| T^m \sum_{n\in E} S_n y - y \Big\| < \varepsilon.$$

Then the operator T is upper frequently hypercyclic.

Bès, Menet, Peris and Puig (2015) also have a criterion for upper frequent hypercyclicity. But the two results are incomparable: they are useful in different situations. Their result works nicely for bilateral shifts, ours for unilateral shifts...

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So, let's look at (unilateral) weighted shift operators

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For ℓ^p $(1 \leq p < \infty)$ Bayart and Ruzsa (2015) already settled the problem: the following are equivalent:

- B_w is frequently hypercyclic;
- B_w is upper frequently hypercyclic;
- B_w is chaotic $(\Longleftrightarrow \sum_{n=1}^{\infty} \frac{1}{(w_1 w_2 \cdots w_n)^p} < \infty)$.

For c_0, the situation is much more complicated. Since Bayart and Grivaux (2007) we know that there are frequently hypercyclic shifts that are not chaotic.

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Theorem (Bayart, Ruzsa 2015)

Let $B_w : c_0 \to c_0$ be a (unilateral) weighted backward shift. Then B_w is upper frequently hypercyclic if and only if there are $M_p > 0$, $M_p \to \infty$, and sets $E_p \subset \mathbb{N}_0$, $\overline{\text{dens}}\, E_p > 0$, such that

- *for any $p, q \geq 1$, if $p \neq q$ then $(E_p + [0,p]) \cap (E_q + [0,q]) = \varnothing$;*
- *for any $p \geq 1$, $\lim_{n \in E_p + [0,p]} w_1 w_2 \cdots w_n = \infty$;*
- *for any $p, q \geq 1$, for any $n \in E_p$, $m \in E_q$ with $m < n$, for any $j \in [0,p]$,*

$$w_1 w_2 \cdots w_{n-m+j} > M_p M_q.$$

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The Bès-Menet-Peris-Puig criterion for upper frequent hypercyclicity leads to the same result. A similar result holds for invertible bilateral shifts.

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Theorem (Bonilla, G-E 2015)

Let $B_w : c_0 \to c_0$ be a weighted backward shift. Then B_w is upper frequently hypercyclic if and only if, for any $M > 0$ and $p \geq 0$, there is a set $E \subset \mathbb{N}_0$, $\overline{\text{dens}}\, E > 0$, such that

- $w_1 w_2 \cdots w_n \to \infty$ *as* $n \to \infty, n \in E$;
- *for any* $n, m \in E, n > m$,

$$w_1 w_2 \cdots w_{n-m+p} > M.$$

Note that we need no condition that involves two different sets E_p.

Application to counter-examples:

Example (Bayart, Grivaux 2007)

$$\text{frequent hypercyclicity} \not\Longrightarrow \text{chaos}$$

Indeed this can be achieved by a suitable weighted shift on c_0.

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$\mathcal{A}$-hypercyclicity

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Let $\mathcal{A}$ be a Furstenberg family, that is, a family of subsets of $\mathbb{N}_0$ such that

$$A \in \mathcal{A}, B \supset A \Longrightarrow B \in \mathcal{A}.$$

Then T is $\mathcal{A}$-hypercyclic (Shkarin 2009) if there is some $x \in X$ such that, for any non-empty open set $U \subset X$,

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The vector x is then called an $\mathcal{A}$-hypercyclic vector.

Bès, Menet, Peris and Puig (2015) have undertaken the first systematic study of this notion.

Do our main results

- Birkhoff transitivity theorem for $\mathcal{A}$-hypercyclicity
- the set of $\mathcal{A}$-hypercyclic vectors is either empty or residual
- $\mathcal{A}$-Hypercyclicity Criterion
- an easy characterization for $\mathcal{A}$-hypercyclic weighted shifts

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Examples (of upper Furstenberg families)

- infinite sets (ordinary hypercyclicity)
- sets of positive upper density (upper frequent hypercyclity)
- sets of positive upper Banach density/uniform density (reiterative hypercyclicity - Bès, Menet, Peris, Puig 2015)
- sets of positive upper weighted density
- sets of positive upper φ-density (Shkarin 2009)
- sets of positive upper exponential density
- ...

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