Approximative properties of polyanalytic polynomial modules

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The talk is based on joint works with **Anton Baranov** (St. Petersburg State University) and **Joan Carmona** (Universitat Autònoma de Barcelona)

Problem statement

For integers $m > 0$ and $0 < k_1 < k_2 < \cdots < k_m$ let

$$\mathscr{P}(\bar{z}^{k_1}, \dots, \bar{z}^{k_m}) = \{p_0 + \bar{z}^{k_1} p_1 + \cdots + \bar{z}^{k_m} p_m \colon p_0, \dots, p_m \in \mathbb{C}[z]\};$$

and let X be a compact set in $\mathbb{C}$.

Question (A. G. O'Farrell (in slightly different form), J. Carmona)

For which X the module $\mathscr{P}(\bar{z}^{k_1}, \dots, \bar{z}^{k_m})$ is dense in $C(X)$?

- The roots of this question can be traced back to 1970–1980th, when several problems on density of rational modules were considered (O'Farrell, Verdera, Carmona, Trent, Wang);
- In the case $k_j = j$, for $j = 1, \dots, m$ and integer $n \geqslant 2$, one has the question about density in $C(X)$ of the system of *polyanalytic polynomials* (of order $m + 1$);

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- In the most general form this question is states as the question of density in $C(X)$ of the module

  $$\{p_0(z) + w_1(z)p_1(z) + \cdots + w_m(z)p_m(z) \colon p_0, p_1, \dots, p_m \in \mathbb{C}[z]\},$$

  where $w_1, \dots, w_m$ are given (sufficiently regular) functions (generators of the module under consideration).

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- The similar questions are also stated for the spaces of smooth and integrable functions on X instead of $C(X)$.
- This question has very interesting relations with certain questions in the theory of model spaces $K_\theta = H^2 \ominus \theta H^2$:
  - existence of univalent functions in K_θ;
  - boundary behavior of univalent functions in K_θ;
  - taking roots in K_θ.

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and let X be a compact set in $\mathbb{C}$.

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For which X the module $\mathscr{P}(\bar{z}^{k_1}, \ldots, \bar{z}^{k_m})$ is dense in $C(X)$?

Let $\mathscr{R}_E(\bar{z}^{k_1}, \ldots, \bar{z}^{k_m}) = \left\{g_0 + \bar{z}^{k_1} g_1 + \cdots + \bar{z}^{k_m} g_m \colon g_0, \ldots, g_m \text{ are rational functions with poles outside a given set } E \subset \mathbb{C}\right\}$.

Question

For which X the module $\mathscr{R}_E(\bar{z}^{k_1}, \ldots, \bar{z}^{k_m})$ is dense in $C(X)$?

Examples: $E = X$, or $E = \overline{G}$ and $X = \partial G$, where G is a bounded simply connected domain in $\mathbb{C}$.

The case of modules, generated by single function $\bar{z}^d$, $d \geqslant 1$

- $A(X, \bar{z}^d) = \{f \in C(X)\colon f|_{X^\circ} = \bar{z}^d f_1 + f_0,\ f_0, f_1 \in Hol(X^\circ)\}$;
- $P(X, \bar{z}^d) = C(X)$-closure of $\{p|_X : p \in \mathscr{P}(\bar{z}^d)\}$;
- $R(X, \bar{z}^d) = C(X)$-closure of $\{g|_X : g \in \mathscr{R}_X(\bar{z}^d)\}$.

Let $U \subset \mathbb{C}$ be on open set with $0 \notin U$.

If $f \in C(U)$ satisfy the elliptic PDE

$$\bar{\partial}\left(\frac{1}{\bar{z}^{d-1}}\bar{\partial} f\right) = 0, \qquad (*)$$

where $\bar{\partial}$ be the standard Cauchy–Riemann operator,

then f has the form $f = \bar{z}^d f_1 + f_0$ with $f_0, f_1 \in Hol(U)$.

For $d = 1$ one has bianalytic functions $\bar{z} f_1(z) + f_0(z)$.

One has $P(X, \bar{z}^d) \subset R(X, \bar{z}^d) \subset A(X, \bar{z}^d)$.

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Theorem (Baranov–Carmona–F., J. Approx. Theor. 2016)

For any compact set $X \subset \mathbb{C}$ and for any integer $d \geqslant 1$ one has $A(X, \bar{z}^d) = R(X, \bar{z}^d)$.

For $d = 1$ it was proved by M. Mazalov [Mazalov, Sb. Math. 2004].

The proof of this theorem in the general case may be obtained following the same scheme, as in the proof of Mazalov's theorem.

Main difficulty: $(*)$ is not an equation with constant coefficients.

But one can define Vitushkin localization operator for solutions of $(*)$, and the properties of this operator, which are important for the proof are similar to the bianalytic case.

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X is a Carathéodory compact set if $\partial X = \partial \widehat{X}$, where $\widehat{X}$ us the union of X and all bounded connected components of $\mathbb{C} \setminus X$.

Theorem (Baranov–Carmona–F., J. Approx. Theor. 2016)

Let X be a Carathéodory compact set and $d \geqslant 2$ be an integer. Then $A(X, \bar{z}^d) = P(X, \bar{z}^d)$ if and only if each bounded connected component of the set $\mathbb{C} \setminus X$ is not a d-Nevanlinna domain.

For $d = 1$ it was proved in [Carmona–F.–Paramonov, Sb. Math. 2002].

d-Nevanlinna domain: this concept is the special analytic characteristic of bounded simply connected domains.

d-Nevanlinna domains: definition and examples

Definition

A bounded simply connected domain G in $\mathbb{C}$ is called d-Nevanlinna domain if there exist two functions $u, v \in H^\infty(G)$ such that $v \not\equiv 0$ and $\overline{z}^d = u/v$ *a.e. on* ∂G *in the sense of conformal mappings*.

It means that the equality of angular boundary values

$$\overline{\varphi(\zeta)}^d = \frac{(u \circ \varphi)(\zeta)}{(v \circ \varphi)(\zeta)}$$

holds a.e. on the unit circle $\mathbb{T}$, where φ is some conformal mapping from the unit disk $\mathbb{D}$ onto G.

Classes ND_d and $ND := ND_1$.

It is clear that $ND \subset ND_d \subset ND_{kd}$ for any integer $k \geqslant 1$.

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For $d = 1$ one has the concept of a Nevanlinna domain. See:
[F., Math. Notes 1996],
[Carmona–F.–Paramonov, Sb. Math 2002],
[F., Proc. Steklov Inst. Math. 2006],
[Baranov–F., Sb. Math. 2011],
[Mazalov–F.–Paramonov, Russian Math. Surveys 2012]
[Mazalov, Algebra i Analiz 2015; St. Petersburg Math. J. 2016]

ND_d-domains may have very irregular (nowhere analytic, non smooth and, even, non rectifiable) boundaries.

$ND_d = \{G = f(\mathbb{D})\colon f^d$ admits a pseudocontinuation$\}$
$ND_d = \{G = f(\mathbb{D})\colon f^d \in K_\Theta$ and f univalent in $\mathbb{D}\}$.

Examples:

Let $G_{a,b}$ be the domain bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let $G'_{a,b}$ be the image of $D_{a,b}$ under the inversion $z \mapsto 1/z$

- $\mathbb{D}$, $G'_{a,b} \in ND$; therefore, for any $d \geqslant 1$,
  $$P(\partial\mathbb{D}, \overline{z}^d) \neq C(\partial\mathbb{D}) \text{ and } P(\partial G'_{a,b}, \overline{z}^d) \neq C(\partial G'_{a,b});$$
- $G_{a,b} \notin ND_d$ for any integer $d \geqslant 1$, and hence
  $$P(\partial G_{a,b}, \overline{z}^d) = C(\partial G_{a,b});$$
- Any bounded simply connected domain bounded by *polygonal line* does not belong to ND_d for any $d \geqslant 1$.

Let $\psi_k(z) = \sqrt[k]{a - z}$, $a > 1$, and $B_k = \psi_k(\mathbb{D})$.

- $B_k \notin ND$, but $B_k \in ND_k$;

  $B_k \in ND_m$ for any $m \in k\mathbb{Z}$, but $B_k \notin ND_m$ for any $m \notin k\mathbb{Z}$.
  $$P(\partial B_k, \overline{z}^m) \neq C(\partial B_k) \text{ for } m \in k\mathbb{Z}, \text{ but}$$
  $$P(\partial B_k, \overline{z}^m) = C(\partial B_k) \text{ for } m \notin k\mathbb{Z}.$$

Approximation by $\mathscr{P}(\bar{z}^{k_1}, \dots, \bar{z}^{k_m})$ and $\mathscr{R}(\bar{z}^{k_1}, \dots, \bar{z}^{k_m})$

$R(X, Y, \bar{z}^{k_1}, \dots, \bar{z}^{k_m}) := C(X)\text{-clos.}\ \{g|_X : g \in \mathscr{R}_Y(\bar{z}^{k_1}, \dots, \bar{z}^{k_m})\}$
$P(X, \bar{z}^{k_1}, \dots, \bar{z}^{k_m}) := C(X)\text{-clos.}\ \{g|_X : g \in \mathscr{P}(\bar{z}^{k_1}, \dots, \bar{z}^{k_m})\}$

Recall, that a bounded simply connected domain G is called a Carathéodory domain, if $\partial G = \partial G_\infty$, where G_∞ is unbounded connected component of the set $\mathbb{C} \setminus \overline{G}$.

Theorem (Baranov–Carmona–F., J. Approx. Theor. 2016)

Let G be a Carathéodory domain, let $k_1 < \dots < k_m$ are positive integers, and let $d = \gcd(k_1, \dots, k_m)$. TFAE:

1. $R(\partial G, \overline{G}, \bar{z}^{k_1}, \dots, \bar{z}^{k_n}) = C(\partial G)$;
2. $R(\partial G, \overline{G}, \bar{z}^{d}) = C(\partial G)$;
3. G is not a d-Nevanlinna domain.

If $\overline{G}$ does not separate the plane (i.e. if the set $\mathbb{C} \setminus \overline{G}$ is connected), then $R(\partial G, \overline{G}, \dots) = P(\partial G, \dots)$ and $R(\partial G, \overline{G}, \bar{z}^d) = P(\partial G, \bar{z}^d)$.

Approximation by $\mathscr{P}(\bar{z}^{k_1}, \dots, \bar{z}^{k_m})$ and $\mathscr{R}(\bar{z}^{k_1}, \dots, \bar{z}^{k_m})$

$$R(X, Y, \bar{z}^{k_1}, \dots, \bar{z}^{k_m}) := C(X)\text{-clos. } \{g|_X : g \in \mathscr{R}_Y(\bar{z}^{k_1}, \dots, \bar{z}^{k_m})\}$$
$$P(X, \bar{z}^{k_1}, \dots, \bar{z}^{k_m}) := C(X)\text{-clos. } \{g|_X : g \in \mathscr{P}(\bar{z}^{k_1}, \dots, \bar{z}^{k_m})\}$$

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Proposition

Let Γ be a rect. simply closed curve, and $k_1, \dots, k_m, d$ be as before.

If $\widehat{\Gamma} \in ND_d$, then there exists a measure ν on Γ such that
i) $\nu \perp P(\Gamma, \bar{z}^{sd})$ for all positive integers s with $sd < k_m$, but
ii) $\nu \not\perp P(\Gamma, \bar{z}^{k_m})$ (and hence $\nu \not\perp P(\Gamma, \bar{z}^{k_1}, \dots, \bar{z}^{k_m})$).

Approximation by $\mathscr{P}(\overline{z}^{k_1}, \ldots, \overline{z}^{k_m})$ and $\mathscr{R}(\overline{z}^{k_1}, \ldots, \overline{z}^{k_m})$

Let $R(X, \overline{z}^{k_1}, \ldots, \overline{z}^{k_m}) = R(X, X, \overline{z}^{k_1}, \ldots, \overline{z}^{k_m})$.

Proposition

Let X be a Carathéodory compact set in $\mathbb{C}$. If $G \notin ND_d$ for any bounded connected component G of $\mathbb{C} \setminus X$, then

$$R(X, \overline{z}^{k_1}, \ldots, \overline{z}^{k_m}) = P(X, \overline{z}^{k_1}, \ldots, \overline{z}^{k_m}).$$

Conversely, if there exists some bounded connected component G of the set $\mathbb{C} \setminus X$ such that $G \in ND_d$, then

$$R(X, \overline{z}^{k_1}, \ldots, \overline{z}^{k_m}) \neq P(X, \overline{z}^{k_1}, \ldots, \overline{z}^{k_m}).$$

Question

Is it true, that

$$R(X, \overline{z}^{k_1}, \ldots, \overline{z}^{k_m}) = A(X, \overline{z}^{k_1}, \ldots, \overline{z}^{k_m})$$

at least for Carathéodory compact sets?

$m = 1$: the answer is affirmative [Carmona, J. Approx. Theor. 1985].