Cosine families close to bounded scalar cosine families

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Cosine families, cosine functions, cosine sequences

Let G be an abelian group, and let $\mathcal{A}$ be a unital Banach algebra.
A $\mathcal{A}$-valued G-cosine family is a map $\mathbf{C} : G \to \mathcal{A}$ satisfying

$$C(0) = 1_{\mathcal{A}}, C(g_1 + g_2) + C(g_1 - g_2) = 2C(g_1)C(g_2) \text{ for } g_1, g_2 \in G.$$

A cosine family is obviously even.
$\mathbb{R}$-cosine families are called cosine functions.
$\mathbb{Z}$-cosine families are called cosine sequences.

Strongly continuous cosine functions

A cosine function $(C(t))_{t\in\mathbb{R}}$ with values in $\mathcal{B}(X)$, the algebra of bounded operators on a Banach space X, is said to be strongly continuous if the map $t \to C(t)x$ is continuous on $\mathbb{R}$ for every $x \in X$.
The (infinitesimal) generator A of a strongly continuous cosine function is defined by the formula

$$Ax = 2lim_{n\to+\infty}\frac{x - C(t)x}{t^2}$$

for $x \in D_A$, the domain of A, consisting in all $x \in X$ for which this limit exists.

The equation $Y''(t) + AY(t) = 0$, and norm continuity at 0.

Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine function with values in $\mathcal{B}(X)$, of generator A. Then if $y_0 \in D_{A^2} := \{y \in D_A \mid Ay \in D_A\}$, and if we set $Y(t) = C(t)y_0$, then the function $t \to Y(t)$ is twice differentiable on $\mathbb{R}$ and we have

$$Y''(t) + A^2 Y(t) = 0,\ Y(0) = y_0,\ Y'(0) = 0,$$

see for example "Vector-Valued Laplace Transforms and Cauchy Problems", Birkhäuser, Basel, 2001, by Arendt, Batty, Hieber and Neubrander.
There is thus an analogy between strongly continuous cosine functions and strongly continuous semigroups $\mathbf{T} = (T(t))_{t\geq 0}$ of bounded operators on X :
Let $B : x \to lim_{t\to 0^+}\frac{T(t)x-x}{t}$ be the generator of $\mathbf{T}$. If y_0 belongs to the domain of B, then if we set $Y(t) = T(t)y_0$, we have $Y'(t) - BY(t) = 0$, $Y(0) = y_0$.
Heuristically, we have $T(t) \approx exp(tB)$, and similarly $C(t) \approx cos(tA)$. If a cosine function with values in a Banach algebra $\mathcal{A}$ satisfies $lim_{t\to 0}\|C(t) - 1_{\mathcal{A}}\| = 0$, then there exists $A \in \mathcal{A}$ such that $C(t) = cos(tA)$ for $t \in \mathbb{R}$, where $cos(tA)$ is defined by the usual series. In particular if a strongly continuous cosine function is continuous at the origin with respect to the norm of $\mathcal{B}(X)$, then its generator A is bounded, and $C(t) = cos(tA)$.

Theorem: Let $(C(t))_{t\in\mathbb{R}}$ be a cosine function. If lim sup$_{t\to 0}\|C(t)-1_{\mathcal{A}}\| < 3/2$, then lim$_{t\to 0}\|C(t)-1_{\mathcal{A}}\| = 0$.

Arendt's three lines proof (Ulmer Seminare, 2012)

The usual formula $1-cos^2(a) = \frac{1-cos(2a)}{2}$ gives, writing $1_{\mathcal{A}} = 1$,

$$(1-C(t/2))(1+C(t/2)) = 1 - C(t/2)^2 = \frac{1-C(t)}{2}.$$

Let $k \in (0, 3/4)$ and $\eta > 0$ such that $\|1-C(t)\| \le 2k$ for $0 \le t \le \eta$. If $0 \le t \le \eta$, we have

$$1 + C(t/2) = 2\left(1 - \frac{1}{2}(1-C(t/2))\right), \left\|\frac{1}{2}(1-C(t/2))\right\| \le k.$$

So $1 + C(t/2)$ is invertible, $\|(1+C(t/2))^{-1}\| \le \frac{1}{2(1-k)}$,

$$\|1-C(t/2)\| \le \frac{1}{4(1-k)}\|1-C(t)\|,$$

which proves the result since $\frac{1}{4(1-k)} < 1$.

$\square$

Theorem: Let $(C(t))_{t\in\mathbb{R}}$ *be a cosine function. If lim sup*$_{t\to 0}\|C(t) - 1_{\mathcal{A}}\| < 2$, *then lim*$_{t\to 0}\|C(t) - 1_{\mathcal{A}}\| = 0$.

History of the zero-two law

This zero two law has been proved by

1. S. Fackler, for strongly continuous cosine functions on UMD spaces, *Regularity of semigroups via the asymptotic behaviour at zero,* Semi-group Forum, 2013,
2. F. Schwenninger and H. Zwart, for strongly continuous cosine functions on general Banach spaces, *Regularity of semigroups via the asymptotic behaviour at zero,* Journal of Evolution Equations, 2015,
3. W. Chojnacki, for general cosine functions, *Around Schwenninger and Zwart's zero-two law for cosine families*, submitted, 2015.

The proofs of Fackler and Schwenninger-Zwart rely on elaborate operator theoretical methods. To prove the general case, Chojnacki first shows, using rather sophisticated arguments, that the condition *sup*$_{t\in\mathbb{R}}\|C(t) - 1_{\mathcal{A}}\| < 2$ implies that $C(t) = 1_{\mathcal{A}}$ for $t \in \mathbb{R}$, and uses ultrapowers to deduce the zero-two law from this result.

Folklore result 1: Let $c(t)$ be a scalar cosine sequence. If lim sup$_{t\to 0}|c(t)-1|<2$, then lim sup$_{t\to 0}|c(t)-1|=0$.
Hence if lim sup $\rho(C(t)-1_{\mathcal{A}})<2$, ρ denoting the spectral radius, then for every character χ on $\mathcal{A}$ there exists $a_\chi\in\mathbb{C}$ such that $\chi(C(t))=cos(ta_\chi)$.
Folklore result 2: Let $(C(t))_{t\in\mathbb{R}}$ be a cosine function. If lim sup$_{t\to 0}\rho(C(t)-1_{\mathcal{A}})<2$, then lim sup$_{t\to 0}\rho(C(t)-1_{\mathcal{A}})=0$.

A short proof of the zero-two law (trick from the Long Beach Conference)

For $\|x\|\le 1$, define $\sqrt{1_{\mathcal{A}}-x}$ by the usual series, so that $\|1_{\mathcal{A}}-\sqrt{1_{\mathcal{A}}-x}\|\le 1-\sqrt{1-\|x\|}$ since the coefficients of the series giving $1_{\mathcal{A}}-\sqrt{1_{\mathcal{A}}-x}$ are positive. Since $C(t/2)^2=1_{\mathcal{A}}-\frac{1_{\mathcal{A}}-C(t)}{2}=\sqrt{1_{\mathcal{A}}-\frac{1_{\mathcal{A}}-C(t)}{2}}^{\,2}$,
$\left(C(t/2)-\sqrt{1_{\mathcal{A}}-\frac{1_{\mathcal{A}}-C(t)}{2}}\right)\left(C(t/2)+\sqrt{1_{\mathcal{A}}-\frac{1_{\mathcal{A}}-C(t)}{2}}\right)=0$ for $\|1_{\mathcal{A}}-C(t)\|\le 2$. The folklore result 2 shows that there exists $\eta>0$ such that the right hand factor is invertible for $0\le t\le\eta$, and so

$$1_{\mathcal{A}}-C(t/2)=1_{\mathcal{A}}-\sqrt{1_{\mathcal{A}}-\frac{1_{\mathcal{A}}-C(t)}{2}},\ \|1_{\mathcal{A}}-C(t/2)\|\le 1-\sqrt{1-\left\|\frac{1_{\mathcal{A}}-C(t)}{2}\right\|}.$$

Set $l=$lim sup$_{t\to 0}\|1_{\mathcal{A}}-C(t)\|$. Then $l=0$, since $l\le 1-\sqrt{1-l/2}\le 1$. $\square$

Continuous bounded scalar sequences are isolated

Let $\mathcal{C}(X)$ be the set of bounded strongly continuous cosine functions, equipped with the sup-norm associated to the norm of $\mathcal{B}(X)$. Bobrowski and Chojnacki proved that the isolated points of $\mathcal{C}(X)$ are exactly the bounded scalar continuous cosine functions, i.e. the cosine functions of the form $c(t) = cos(at)I_X$ for some $a \in \mathbb{R}$ (Studia Mathematica, 2013), and they showed that if a strongly continuous cosine function satisfies $sup_{t\in\mathbb{R}}\|C(t) - cos(at)I_X\| < 1/2$ for some $a \in \mathbb{R}$, then $C(t) = cos(at)$ for every $t \in \mathbb{R}$. Schwenninger and Zwart (arXiv:1310.6202, 2013) showed that this result remains true with the constant 1 instead of the constant 1/2, and A. Bobrowski, W. Chojnacki, and A. Gregosiewicz (J. Math. An. and Appl., 2015) showed that if a cosine family $(C(t))_{t\in\mathbb{R}}$ in unital Banach algebra satisfies $sup_{\in\mathbb{R}}\|C(t) - cos(at)1_{\mathcal{A}}\| < \frac{8}{3\sqrt{3}}$ fo some $a \notin 2\pi\mathbb{Z}$, then $C(t) = cos(at)1_{\mathcal{A}}$ for $t \in \mathbb{R}$.

For $a \in \geq 0$, $0 \leq m \leq 2$ set $\Omega(a,m) = \{b \geq 0 \mid sup_{t\in\mathbb{R}}|cos(at) - cos(bt)| \leq m\}$. Easy verifications using Kronecker's theorem on independent subsets of the unit circle show that $\Omega(a,m)$ is finite if $m < 2$, and $\Omega(a,m) = \{a\}$ if $m < \frac{8}{3\sqrt{3}}$. We have the following result, obtained before the work by Bobrowski, Chojnacki and Gregosiewicz appeared. Notice that when $a = 0$, (ii) holds for $m < 2$.

A result to appear in Int. Eq. Op. Th.

Theorem: Let $a > 0$, and let $(C(t))_{t\in\mathbb{R}}$ be a cosine function with values in a unital Banach algebra $\mathcal{A}$.
(i) If $m := sup_{t\in\mathbb{R}}\|C(t) - cos(at)1_{\mathcal{A}}\| < 2$, then the closed subalgebra of $\mathcal{A}$ generated by the cosine function $(C(t))_{t\in\mathbb{R}}$ is a finite dimensional algebra isomorphic to $\mathbb{C}^k$ for some $k \geq 1$, and there exists a family $p_1, \dots, p_k$ of pairwise orthogonal idempotents of $\mathcal{A}$ and a family $(b_1, \dots, b_k)$ of elements of $\Omega(a,m)$ such that we have

$$C(t) = \sum_{j=1}^{k} cos(b_j t)p_j \quad \forall t \in \mathbb{R}.$$

(ii) If $m < \frac{8}{3\sqrt{3}}$, then $C(t) = cos(at)1_{\mathcal{A}}$ for $t \in \mathbb{R}$.

We have, $\sum_{n=0}^{+\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2} = \frac{\pi}{2}$, and, for $x \in (-1, 1)$,

$$arccos(x) = \frac{\pi}{2} - \sum_{n=0}^{+\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2} x^{2n+1}.$$

So if an element x of a unital Banach algebra $\mathcal{A}$ is power-bounded we can define $arccos(x) \in \mathcal{A}$ by using the above series, and we have $cos(arccos(x)) = x$. If a cosine sequence $(c_n)_{n\in\mathbb{Z}}$ is bounded, then c_1 is power-bounded, and $c_n = cos(narccos(c_1))$ for $n \in \mathbb{Z}$.
The other Banach algebra ingredient in the proof of the general result about bounded cosine functions close to bounded continuous scalar ones is given by the following lemma

Cosine sequences and commutative radical Banach algebras

Lemma: Let $\mathcal{A}$ be a commutative unital Banach algebra, let x, y be two quasinilpotent elements of $\mathcal{A}$ and let $\lambda \in \mathbb{C}$ satisfying $cos(\lambda 1_{\mathcal{A}} + x) = cos(\lambda 1_{\mathcal{A}} + y)$.
(i) If $\lambda \notin \pi\mathbb{Z}$, then $x = y$.
(ii) If $\lambda \in \pi\mathbb{Z}$, then $x^2 = y^2$.

Set $\Delta(a,m) := \{b \in [0,\pi] \mid sup_{n\geq 1}|cos(an) - cos(bn)| \leq m\}$ for $a \in [0,\pi]$, $m \leq 2$. Then $\Delta(a,m)$ is finite for every $m < 2$, and there exists $\sigma(a) \in [0,\pi] \setminus \{a\}$ such that, for every $b \in [0,\pi] \setminus \{a\}$, $k(a) := sup_{n\geq 1}|cos(an) - cos(\sigma(a)n))| \leq sup_{n\geq 1}|cos(an) - cos(bn)|$

A general result

Theorem: Let $a \in [0,\pi]$, and let $C(n)$ be a cosine sequence in a unital Banach algebra $\mathcal{A}$.

(i) If $m := sup_{n\geq 1}\|C(n) - cos(an)1_{\mathcal{A}}\| < 2$, then the closed subalgebra of $\mathcal{A}$ generated by the cosine sequence $(C(n))_{n\in\mathbb{Z}}$ is a finite dimensional algebra isomorphic to $\mathbb{C}^k$ for some $k \geq 1$, and there exists a family $p_1, \ldots, p_k$ of pairwise orthogonal idempotents of $\mathcal{A}$ and a family $(b_1, \ldots, b_k)$ of elements of $\Delta(a,m)$ such that we have $C(n) = \sum_{j=1}^{k} cos(b_j n)p_j \quad \forall n \in \mathbb{Z}$.

(ii) If $m < k(a)$, then $C(n) = cos(an)1_{\mathcal{A}}$ for $n \in \mathbb{Z}$.

The closed subalgebra generated by a G-cosine family which s at a distance less than two from a scalar bounded G-cosine family is not in general finite-dimensional, as shown by the following example.

A counter-example

Proposition: Let $G := (\mathbb{Z}/3\mathbb{Z})^{\mathbb{N}}$, so that elements of G can be written under the form $g = (\overline{g}_m)_{m\geq 1}$, where $g_m \in \{0, 1, 2\}$. Set

$$C(g) := \left(\cos\left(\frac{2g_m\pi}{3}\right)\right)_{m\geq 1}.$$

Then $(C(g))_{g\in G}$ is a G-cosine family in l^∞ which satisfies the following conditions

(i) $\sup_{g\in G}\|1_{l^\infty} - C(g)\| = \frac{3}{2}$,

(ii) The algebra $\mathcal{U}$ generated by the family $(C(g))_{g\in G}$ is dense in l^∞.

Proof: Property (i) is obvious. Now let ϕ be an idempotent of l^∞. There exists $g \in G$ satisfying $C(0_G) - C(g) = 1_{l^\infty} - C(g) = \frac{3}{2}\phi$, and so $\phi \in \mathcal{U}$. We can identify l^∞ to $\mathcal{C}(\beta\mathbb{N})$, the algebra of continuous functions on the Stone-Cĕch compactification of $\mathbb{N}$, an extremely disconnected compact set. Hence the idempotents of l^∞ separate points of $\beta\mathbb{N}$, and it follows from the Stone-Weierstrass theorem that $\mathcal{U}$ is dense in l^∞, which proves (ii). $\square$

Let $a \notin \pi\mathbb{Q}$. Since the set $(e^{ian})_{n\geq 1}$ is dense in the unit circle, one can see that $k(a) = sup_{t\in\mathbb{R}}|cos(at) - cos(3at)| = \frac{8}{3\sqrt{3}}$. Schwenninger and Zwart observed that $k(0) = \frac{3}{2}$, and we have also $k(\pi/2) = k(\pi) = \frac{3}{2}$. For all other values of $a \in [0, \pi]$, we have $k(a) \leq sup_{n\geq 1}|cos(an) - cos(3an)|$.

A taste of Galois theory

Proposition: If $a \in \pi\mathbb{Q}$, *then* $k(a) < \frac{8}{3\sqrt{3}}$.

Outline of proof: We have $|cos(nx) - cos(3nx)| < \frac{8}{\pi\sqrt{3}}$ if $x \notin \pm arccos\left(\frac{1}{\sqrt{3}}\right) + \pi\mathbb{Z}$. If $na \in \pm arccos\left(\frac{1}{\sqrt{3}}\right) + \pi\mathbb{Z}$ for some $n \geq 1$, then $arccos\left(\frac{1}{\sqrt{3}}\right)/\pi$ would be rational, and $\alpha := \frac{1}{\sqrt{3}} + \frac{\sqrt{2}i}{\sqrt{3}}$ would be a root of unity. So $\beta = \alpha^2 = -\frac{1}{3} + \frac{2\sqrt{2}i}{3}$ would b a primitive n-th root of 1 for some $n \geq 2$. and the degree of the field $\mathbb{Q}(\beta)$ over $\mathbb{Q}$ would be equal to 2, since $3\beta^2 + 2\beta + 3 = 0$. On the other hand if β were a primitive n-th root of 1 the Galois group $Gal(\mathbb{Q}(\beta)/\mathbb{Q})$ would be isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{\times}$, the group of invertible elements of $\mathbb{Z}/n\mathbb{Z}$, and we would have $2 = deg(\mathbb{Q}(\beta)/\mathbb{Q})$ $= card\left((\mathbb{Z}/n\mathbb{Z})^{\times}\right)$. The only possibilities to get this are $n = 3$, $n = 4$, and $n = 6$. Since $\beta^3 \neq 1$, $\beta^4 \neq 1$, and $\beta^6 \neq 1$, we see that $arccos\left(\frac{1}{\sqrt{3}}\right)/\pi \notin \mathbb{Q}$. $\square$

An elementary argument of Chojnacki (J. Australian Math. Soc., 2015) shows that $k(a) \geq 1$ for every a, but this lower estimate is not optimal. Elementary, but nontrivial arguments, lead to the following general result.

A description of the values of $k(a)$

Theorem: Let $a \in [0, \pi]$.

- *If $a \in \left\{\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}\right\}$, then $k(a) = cos\left(\frac{\pi}{5}\right) + cos\left(\frac{2\pi}{5}\right) \approx 1,1180$.*
- *If $a \in \left\{\frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{5\pi}{4}, \frac{7\pi}{8}\right\}$, then $k(a) = \sqrt{2} \approx 1,4142$.*
- *If $a \in \left\{\frac{\pi}{11}, \frac{2\pi}{11}, \frac{3\pi}{11}, \frac{4\pi}{11}, \frac{5\pi}{11}, \frac{6\pi}{11}, \frac{7\pi}{11}, \frac{8\pi}{11}, \frac{9\pi}{11}, \frac{10\pi}{11}\right\}$, then $k(a) = cos\left(\frac{2\pi}{11}\right) + cos\left(\frac{3\pi}{11}\right) \approx 1,4961$.*
- *If $a \in \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi\right\} \cup \left\{\frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}\right\} \cup \left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}\right\} \cup \left\{\frac{\pi}{15}, \frac{2\pi}{15}, \frac{4\pi}{15}, \frac{7\pi}{15}, \frac{8\pi}{15}, \frac{11\pi}{15}, \frac{13\pi}{15}, \frac{14\pi}{15}\right\}$, then $k(a) = 1.5$.*
- *For all other values of $a \in \pi\mathbb{Q}$ we have $1.5 < k(a) < \frac{8}{3\sqrt{3}} \approx 1.5396$, and $k(a) = \frac{8}{3\sqrt{3}}$ if $a \notin \pi\mathbb{Q}$.*

Moreover the set $\{a \in [0, \pi] \mid k(a) \leq m\}$ is finite for every $m < \frac{8}{3\sqrt{3}}$.

It follows from the discussion of the values of $k(a)$ that $k(a) \geq \frac{\sqrt{5}}{2} = k(\pi/5)$ for every $a \in [0, \pi]$. We obtain the following result.

Cosine families close to bounded scalar cosine families

Theorem: Let G be an abelian group, let $(C(g))_{g \in G}$ be a G-cosine family in a unital Banach algebra $\mathcal{A}$, and let $(c(g))_{g \in G}$ be a bounded scalar G-cosine family. If $sup_{g \in G} \|C(g) - c(g)1_{\mathcal{A}}\| < \frac{\sqrt{5}}{2}$, then $C(g) = c(g)1_{\mathcal{A}}$ for every $g \in G$.

Since $sup_{n \geq 1} cos\left(\frac{n\pi}{5}\right) - cos\left(\frac{3n\pi}{5}\right) = \frac{\sqrt{5}}{2}$, the constant $\frac{\sqrt{5}}{2}$ is obviously optimal.

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THANK YOU