Tropical power series and holomorphic approximation on the complex plane

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Let $\mathcal{D}$ denote $\mathbb{C}^d$ or the unit ball B^d of $\mathbb{C}^d$, $d \geq 1$.
For the unit disk B^1, we also use the symbol $\mathbb{D}$.

For $R = 1$ or $R = +\infty$, let $w : [0, R) \to (0, +\infty)$ be *a weight function*, that is, let w be non-decreasing, continuous and unbounded.
Setting $w(z) = w(|z|)$ for $z \in \mathcal{D}$, we extend w to *a radial weight* on $\mathcal{D}$. In what follows, we freely exchange a weight function and its extension to a radial weight.

Given a set X and functions $u, v : X \to (0, +\infty)$, we write $u \asymp v$ and we say that u and v are *equivalent* if $C_1 u(x) \leq v(x) \leq C_2 u(x)$, $x \in X$, for some constants $C_1, C_2 > 0$.

Let $H(\mathcal{D})$ denote the space of holomorphic functions on $\mathcal{D}$.

Definition.

A radial weight w on $\mathcal{D}$ is called approximable by a finite sum of moduli (in brief, $w \in \mathcal{D}_{\mathrm{mod}}$) if there exist $f_1, f_2, \dots, f_n \in H(\mathcal{D})$, $n \in \mathbb{N}$, such that

$$|f_1(z)| + |f_2(z)| + \cdots + |f_n(z)| \asymp w(z), \quad z \in \mathcal{D}.$$

Partial results on the unit disk have been obtained by several authors.

– W. Ramey and D. Ullrich (1991) for $w(t) = \frac{1}{1-t}$, $0 \leq t < 1$.

Similar results for explicit weight functions:

– B. R. Choe and K. S. Rim (1996);

– P. M. Gauthier and J. Xiao (2002);

– J. Xiao (2004);

– P. Galanopoulos (2008);

– S. G. Krantz and S. Stević (2009);

– D. Girela, J. Á. Peláez, F. Pérez-González and J. Rättyä (2008).

Certain classes of subexponential weight functions:

– E. G. Kwon and M. Pavlović (2011);

– J. Gröhn, J. Á. Peláez and J. Rättyä (2013).

By definition, a function $v : [0, R) \to (0, +\infty)$ is called *log-convex* if $\log v(t)$ is a convex function of $\log t$, $0 < t < R$. We write $w \in \mathcal{D}_{\log}$ if w is equivalent to a log-convex radial weight on $\mathcal{D}$.

Theorem (E. Abakumov & E.D., 2013–15)

Let w be a radial weight on B^d, $d \geq 1$. Then the following properties are equivalent:
– w is approximable by a finite sum of moduli on B^d ($w \in B^d_{\mathrm{mod}}$);
– w is equivalent to a log-convex weight function ($w \in B^1_{\log}$).

Trivial examples $w_n(t) = 1 + t^{n+\frac{1}{2}}$ show that the property $w \in \mathbb{C}_{\log}$ is not appropriate in the setting of the complex plane $\mathbb{C}$. To avoid such examples, in what follows we assume that $w : [0, +\infty) \to (0, +\infty)$ is *rapid*; this means, by definition, that $\lim_{t\to\infty} t^{-n} w(t) = \infty$ for all $n \in \mathbb{N}$. However, there exists a rapid radial weight w on $\mathbb{C}$ such that $w \in \mathbb{C}_{\log}$ but $w \notin \mathbb{C}_{\mathrm{mod}}$.

Assuming that w is rapid, we show that the equivalence to a log-convex function ($w \in \mathbb{C}_{\log}$) should be replaced by the equivalence to a *log-tropical* function.

Following C. O. Kiselman (2014), we say that

$$\Psi(x) = \sup_{j\in\mathbb{Z}_+} (a_j + jx), \quad x \in \mathbb{R},\ a_j \in \mathbb{R},\ j = 0, 1, \dots,$$

is a *tropical power series*.
Tropical geometry: $[-\infty, +\infty)$ with operations max and $+$.
Tropical polynomials: $j \in E$, E is finite.

Classical geometry: $[0, +\infty)$ with operations $+$ and $*$.
Power series: $\sum b_j t^j$
Polynomials: $j \in E$, E is finite

Appropriate logarithmic transformation: $\Phi_w(x) = \log w(e^x)$;
For $w(t) := b_j t^j$, we have $\Phi_w(x) = \log b_j + jx$.

Given a weight function $v : [0, +\infty) \to (0, +\infty)$, we say that v is *log-tropical* if Φ_v is a tropical power series.

We write $w \in \mathbb{C}_{\mathrm{trop}}$ if w is equivalent to a log-tropical function.

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- *w is approximable by power series with positive coefficients ($w \in \mathbb{C}_{\mathrm{pspc}}$);*
- *w is approximable from below by monomials ($w \in \mathbb{C}_{\mathrm{mon}}$);*
- *w is essential on $\mathbb{C}$ ($w \in \mathbb{C}_{\mathrm{ess}}$);*

Definition.

A weight function $w : [0, R) \to (0, +\infty)$ *is called approximable by the maximum of a holomorphic function modulus (in brief,* $w \in \mathcal{D}_{\max}$*) if there exists* $f \in H(\mathcal{D})$ *such that*

$$M_f(t) \asymp w(t),\ 0 \le t < R,\ \textit{where } M_f(t) = \max\{|f(z)| : |z| = t\}.$$

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Definition.

We say that a weight function $w : [0, R) \to (0, +\infty)$ *is approximable by power series with positive coefficients (in brief,* $w \in \mathcal{D}_{\mathrm{pspc}}$*) if there exist* $a_k \ge 0,\ k = 0, 1, \dots,$ *such that*

$$\sum_{k=0}^{\infty} a_k t^k \asymp w(t), \quad 0 \le t < R.$$

To investigate the above property, P. Erdös and T. Kövári (1956) and U. Schmid (1998) use the function

$$P_w(t) = \max\left\{ \frac{t^n}{u_n} : \; n = 0, 1, \dots \right\},$$

where

$$u_n = \sup\left\{ \frac{t^n}{w(t)} : \; 0 \le t < R \right\}, \quad n = 0, 1, \dots .$$

In other words, $P_w(t)$ is the pointwise maximum of the monomials $y_n(t) = a_n t^n$ such that $y_n(t) \le w(t)$ and $y_n(t)$ reaches $w(t)$ from below. Clearly, $P_w(t) \le w(t)$. So, the reverse inequality is of interest.

Definition.

We say that a weight function $w : [0, R) \to (0, +\infty)$ *is approximable from below by monomials (in brief,* $w \in \mathcal{D}_{\mathrm{mon}}$*) if*

$$w(t) \leq C_{\mathrm{mon}} P_w(t), \quad 0 \leq t < R,$$

for a constant $C_{\mathrm{mon}} > 1$*.*

Considering the logarithmic transformation

$$\Phi(x) = \Phi_w(x) = \log w(e^x), \quad -\infty < x < +\infty,$$

we conclude that the above property of w holds if and only if w is equivalent to a log-tropical weight function.

Given a radial weight w on $\mathcal{D}$, *the associated weight* $\widetilde{w}$ is defined as

$$\widetilde{w}(z) = \sup\left\{|f(z)| : f \in H(\mathcal{D}),\ |f| \leq w \text{ on } \mathcal{D}\right\}, \quad z \in \mathcal{D}.$$

The notion of associated weight naturally arises in the study of the growth space $\mathcal{A}^w(\mathcal{D})$ which consists of those $f \in H(\mathcal{D})$ for which $|f| \leq Cw$ on $\mathcal{D}$ with some constant $C > 0$. In fact, the growth space $\mathcal{A}^w(\mathcal{D})$ is equal to $\mathcal{A}^{\widetilde{w}}(\mathcal{D})$ isometrically.

Definition. (K. D. Bierstedt, J. Bonet and J. Taskinen, 1998)

A weight function $w : [0, R) \to (0, +\infty)$ *is called essential (in brief,* $w \in \mathcal{D}_{\mathrm{ess}}$*) if*

$$\widetilde{w}(t) \asymp w(t), \quad 0 \leq t < R.$$

Clearly, $\widetilde{w}$ is a radial weight, so the associated weight function $\widetilde{w} : [0, R) \to (0, +\infty)$ is correctly defined.

We suppose, without loss of generality, that w is $\mathcal{C}^2$-smooth and log-convex, that is, Φ_w is convex. Also, observe that the properties under consideration depend only on the behavior of Φ_w at $+\infty$.

Theorem

Let $w : [0, +\infty) \to (0, +\infty)$ be a rapid weight function. Assume that w is log-convex and $\mathcal{C}^2$-smooth.
(i) If $\liminf_{x\to+\infty} \Phi_w''(x) > 0$, then $w \in \mathbb{C}_{\mathrm{mod}}$.
(ii) If $\limsup_{x\to+\infty} \Phi_w''(x) = 0$, then $w \notin \mathbb{C}_{\mathrm{mod}}$.

We suppose, without loss of generality, that w is $\mathcal{C}^2$-smooth and log-convex, that is, Φ_w is convex. Also, observe that the properties under consideration depend only on the behavior of Φ_w at $+\infty$.

Theorem

Let $w : [0, +\infty) \to (0, +\infty)$ be a rapid weight function. Assume that w is log-convex and $\mathcal{C}^2$-smooth.
(i) If $\liminf_{x\to+\infty} \Phi''_w(x) > 0$, then $w \in \mathbb{C}_{\mathrm{mod}}$.
(ii) If $\limsup_{x\to+\infty} \Phi''_w(x) = 0$, then $w \notin \mathbb{C}_{\mathrm{mod}}$.

Remark.

Under assumptions of the above theorem, we have $\Phi_w \in \mathcal{C}^2(\mathbb{R})$ and $\Phi''_w(x) \geq 0$ for all $x \in \mathbb{R}$. So, if $\lim_{x\to+\infty} \Phi''_w(x)$ exists (finite or infinite), then either (i) or (ii) applies.

Examples.

Let $\alpha > 1$, and let w_α be a weight function such that $w_\alpha(t) = e^{(\log t)^\alpha}$, $t > e$. Then w_α, $\alpha \geq 2$, has the equivalent properties listed in the main theorem, and w_α, $1 < \alpha < 2$, does not have the properties under consideration. Indeed, we have $\Phi_{w_\alpha}(x) = x^\alpha, x > 1$.

Standard applications of the main theorem are related to various concrete operators from a growth space $\mathcal{A}^w(\mathbb{C}^d)$ to a (quasi) Banach lattice Y.

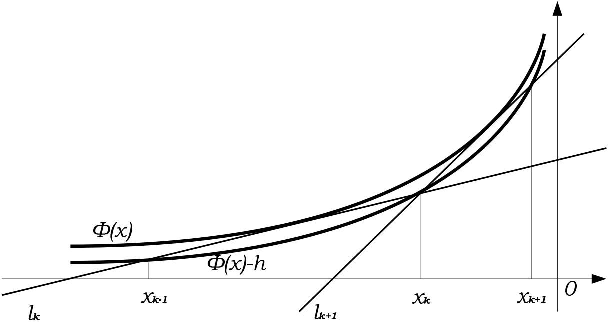
Here w is an *arbitrary* weight function (consider the *associated* weight function).

– Weighted composition operators.

– Generalized Volterra operators.

A mapping $f : \mathcal{D} \to \mathbb{C}^N$, $N \geq 2$, is called a *proper holomorphic immersion* if its preimage of every compact set is compact and the Jacobian of f is non-degenerate everywhere. By definition, a *proper holomorphic embedding* is a proper holomorphic immersion which is one to one. There are no simple examples of proper holomorphic embeddings $f : \mathbb{D} \to \mathbb{C}^2$. Nevertheless, J. Globevnik (2002) proves that a proper holomorphic embedding $f : \mathbb{D} \to \mathbb{C}^2$ may grow arbitrarily rapidly; also, he asks whether such an embedding may grow arbitrarily slowly.

Results related to approximation by finite sums of moduli provide quantitative assertions about possible growth of proper holomorphic immersions and embeddings of the unit disk $\mathbb{D}$. In particular, a proper holomorphic immersion $f : \mathbb{D} \to \mathbb{C}^2$ or a proper holomorphic embedding $f : \mathbb{D} \to \mathbb{C}^3$ may grow arbitrarily slowly.



Basic induction construction