Some results on the structure of Lipschitz-free spaces

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Introduction

Let $(M, d, 0)$ be a pointed metric space. Consider the linear space $\mathrm{Lip}_0(M)$ of all real-valued Lipschitz functions on M that vanish at 0. The minimal Lipschitz constant is a norm that makes $\mathrm{Lip}_0(M)$ a Banach space.

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There is a natural embedding $\delta : M \hookrightarrow \mathrm{Lip}_0(M)^*$ that sends each $x \in M$ to the evaluation functional δ_x, i.e. $\delta_x(f) := f(x)$ for every $f \in \mathrm{Lip}_0(M)$. It is an isometric embedding and we define the Lipschitz-free space over M, $F(M)$, to be the Banach subspace $\overline{\mathrm{span}\{\delta_x : x \in M\}}$.

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From that we get that for any $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in M$ we have $\|\alpha_1\delta_{x_1} + \ldots + \alpha_n\delta_{x_n}\| = \sup\{\alpha_1 f(x_1) + \ldots + \alpha_n f(x_n) : f \in \mathrm{Lip}_0(M), \|f\|_{\mathrm{Lip}_0} \leq 1\}$.

Alternative definition

We again start with $(M, d, 0)$. Consider the real vector space with the Hamel basis $M \setminus \{0\}$. For any $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in M$ set

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| := \inf\{|\beta_1| d(a_1, b_1) + \ldots + |\beta_m| d(a_m, b_m) :$$

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = \beta_1(a_1 - b_1) + \ldots + \beta_m(a_m - b_m); \beta_i \in \mathbb{R}, a_i, b_i \in M \forall i\}.$$

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One can check that this formula actually gives a norm. We shall consider the completion.
It is easy to show that both Banach spaces satisfy the following universal property:

Universal property of Lipschitz-free spaces

Let X be a Banach space and suppose $L : M \to X$ is a Lipschitz map such that $L(0) = 0$. Then there exists a unique linear map $\widehat{L} : F(M) \to X$ extending L, i.e., the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "L"] \arrow[d, "\delta_M"'] & X \arrow[d, "\mathrm{id}_X"] \\ F(M) \arrow[r, dashed, "\widehat{L}"] & X \end{tikzcd}$$

and $\|\widehat{L}\| = \|L\|_{\mathrm{Lip}_0}$.

Universal property

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The universal property clearly defines $F(M)$ uniquely up to linear isometry, thus we conclude that both spaces are one and the same.

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Since for every $f \in F(M)^*$ we have that $f \upharpoonright M \in \mathrm{Lip}_0(M)$ and $\|f\| = \|f \upharpoonright M\|_{\mathrm{Lip}_0}$, and conversely every $f \in \mathrm{Lip}_0(M)$ uniquely extends to $\hat{f} \in F(M)^*$ with $\|f\|_{\mathrm{Lip}_0} = \|\hat{f}\|$, we conclude that the space $\mathrm{Lip}_0(M)$ is isometric to $F(M)^*$.

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Remark

From both definitions, perhaps easier from the first one, one can show that for a metric subspace $N \subseteq M$, $F(N)$ is a subspace, resp. linearly isometric to a subspace, of $F(M)$.
That is not true, though, in the complex case.

ℓ_1 in Lipschitz-free spaces

A lot of "small/simple" metric spaces give rise to a Lipschitz-free space that is isomorphic to ℓ_1; e.g. measure zero sets containing all the branching points in $\mathbb{R}$-trees (Godard), ultrametric spaces. We also mention the following result:

Theorem[P. Kaufmann]

For X a Banach space we have $F(X) \cong (\bigoplus_{n\in\mathbb{N}} F(X))_{\ell_1}$.

Perhaps it is possible to find ℓ_1 in every Lipschitz-free space.

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Corrolaries

1. $\mathrm{Lip}_0(M)$ contains a subspace isomorphic to ℓ_∞ (update: M. Cúth and M. Johanis proved that it contains ℓ_∞ isometrically).
2. No Lipschitz-free space is complemented in a $C(K)$ space.
3. $\mathrm{Lip}_0(M)$ is not weakly sequentially complete.
4. $F(M)$ is projectively universal separable Banach space.

Separable Lipschitz-free spaces among separable Banach spaces

Fact[Godefroy, Kalton]

If M is Lipschitz universal separable metric space, i.e. it contains a bi-Lipschitz copy of every separable metric space, then $F(M)$ will be a linearly universal separable Banach space.

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Fact

For every separable ultrametric space M, $F(M)$ is isomorphic to ℓ_1.

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As mentioned, Lipschitz-free spaces over measure zero sets containing branching points in real trees and over ultrametric spaces are isometric to ℓ_1.

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Theorem[A. Dalet]

If K is a countable compact metric space, then $F(K)$ is a dual space with the metric approximation property.

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Theorem[A. Dalet]

If K is a countable compact metric space, then $F(K)$ is a dual space with the metric approximation property.

So we conjectured:

Conjecture

For K countable compact metric, $F(K)$ is isomorphic to ℓ_1.

Some answers

Theorem

There is a countable compact metric space K which is just one convergent sequence such that $F(K)$ does not embed into L_1.

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Remark

As Tony Procházka observed, this convergent sequence can be taken from $\mathbb{R}^2$. And as Gilles Lancien observed, $F(K)$ does not even bi-Lipschitz-embed into L_1.

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Theorem

For every n, $F(\mathbb{R}^n)$ is weakly sequentially complete. In particular, it is not universal.
Moreover, it has been known (attributed to Naor and Schechtman) that $F(\mathbb{R}^2)$ does not embed into L_1.

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Proof ideas

We use the results of Bourgain that $C^1([0,1]^n)^*$ is weakly sequentially complete.
Then we define a mapping from $[0,1]^n$ into $C^1([0,1]^m)^*$ by sending $x \in [0,1]^n$ to its evaluation functional. We check that this is Lipschitz, thus extends to a linear operator on $F([0,1]^n)$, and then we check that this operator is actually an isomorphic embedding.
$F([0,1]^n)$ is isomorphic to $F(\mathbb{R}^n)$.

Open problems

Dutrieux and Ferenczi asked whether $F(M)$ is linearly universal if and only M is bi-Lipschitz universal.

For M just a metric space, the answer is known to be 'no'.

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There exists a compact metric space K such that every Banach space X that contains an isometric copy of K is universal. In particular, $F(K)$ is universal.

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Theorem[Kaufmann]

For every Banach space X, $F(X) \cong F(B_X)$. In particular, for a universal Banach space X, $F(B_X)$ is universal.

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How many universal Lipschitz-free space are there?

Let $\mathbb{P}$ be the Pełczyński universal basis space. Then $F(\mathbb{P})$ and $\mathbb{P}$ are isomorphic.

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It was proved by Aharoni that c_0 is bi-Lipschitz universal separable metric space.

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Is, for instance, $F(c_0)$ isomorphic to some known universal space? Is it isomorphic to $\mathbb{P}$ or $F(\mathbb{U})$?

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Does the Kaufmann's theorem hold true even for general metric space?

Question

Does there exist for every metric space M a *bounded* metric space B such that $F(M)$ and $F(B)$ are isomorphic?
For separable M, can B be even taken compact?