Fourier multipliers of the homogeneous Sobolev space $\dot{W}^{1,1}$

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What is a Fourier multiplier?

Let B a Banach space of tempered distributions such that $\mathcal{S}(\mathbb{R}^d)$ is dense in B. Fourier multipliers of B are Fourier transforms of tempered distributions S such that for $f \in \mathcal{S}(\mathbb{R}^d)$ the function $f * S$ is in B and

$$\|m\| := \sup_{\|f\|_B = 1} \|\|S * f\|_B < \infty.$$

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Well-known:
Fourier multipliers of $L^1(\mathbb{R}^d)$ are Fourier transforms of bounded measures.

For $p > 1$, many sufficient conditions (Mihlin, Hörmander, Marcinkiewicz theorems). In particular any homogeneous function of degree 0, which is smooth outside of the origin, is a Fourier multiplier of $L^p(\mathbb{R}^d)$, $p > 1$, and also of $\mathcal{H}^1(\mathbb{R}^d)$.

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For $p > 1$ isomorphism between $\dot{W}^{1,p}$ and L^p given by

$$-(-\Delta)^{1/2} = \sum_{j=1}^{d} \left(\partial_{x_j} (-\Delta)^{-1/2} \right) \partial_{x_j} = \sum_{j=1}^{d} R_j \partial_{x_j}.$$

Recall: Riesz transforms are bounded on L^p for $p > 1$, not for $p = 1$.

Links between Fourier multipliers

Lemma 1. *For $p > 1$ Fourier multipliers of $\dot{W}^{1,p}(\mathbb{R}^d)$ coincide with Fourier multipliers of $L^p(\mathbb{R}^d)$.*
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Lemma 2. *A Fourier multiplier of $W^{1,p}(\mathbb{R}^d)$ is a Fourier multiplier of $\dot{W}^{1,p}(\mathbb{R}^d)$.*
Proof: Write the inequality

$$\|\nabla(S * f)\|_p \leq C(\|f\|_p + \|\nabla f\|_p)$$

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Corollary. *For $p > 1$ Fourier multipliers of $W^{1,p}(\mathbb{R}^d)$ also coincide with Fourier multipliers of $L^p(\mathbb{R}^d)$.*

The case $p = 1$.

Lemma 3. *Fourier multipliers of* $W^{1,1}(\mathbb{R}^d)$ *which have compact support are Fourier transforms of bounded measures.*

Proof: It is sufficient to test it on functions such that $\widehat{f}$ has compact support. By Bernstein Inequality

$$\|\nabla f\|_1 \leq C\|f\|_1.$$

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Lemma 4. *Fourier multipliers of $\dot{W}^{1,1}(\mathbb{R}^d)$ which vanish in a neighborhood of the origin are Fourier multipliers of $W^{1,1}(\mathbb{R}^d)$.*

Proof: It suffices to test it on f such that $\widehat{f}$ vanishes on a ball centered at 0. Then $\|f\|_1 \leq C\|\nabla f\|_1$. Indeed,

$$\widehat{f}(\xi) = \psi(\xi)\sum_{j=1}^{d} \frac{\xi_j}{|\xi|^2}\widehat{\partial_{x_j} f}(\xi),$$

with ψ smooth,vanishing in a ball centered at 0 and equal to 1 in the doubled ball. But $\psi(\xi)\frac{\xi_j}{|\xi|^2}$ is the Fourier transform of a bounded measure.

The case $p = 1$–Second Part.

$$\mathcal{H}^1(\mathbb{R}^d) \subsetneqq (-\Delta)^{1/2}\dot{W}^{1,1}(\mathbb{R}^d).$$

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Lemma 5. *For $d = 1$: Fourier multipliers of $W^{1,1}(\mathbb{R})$ and $\dot{W}^{1,1}(\mathbb{R})$ also coincide with Fourier multipliers of $L^1(\mathbb{R})$.*

Proof. For $W^{1,1}(\mathbb{R})$ use the fact that $I + \frac{d}{dx} : W^{1,1}(\mathbb{R}) \mapsto L^1(\mathbb{R})$ is an isomorphism. For $\dot{W}^{1,1}(\mathbb{R})$, use homogeneity.

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So what interests us is the study of Fourier multipliers of $\dot{W}^{1,p}(\mathbb{R}^d)$ for $p = 1$ and $d > 1$.

New Fourier multipliers for $d > 1$

If μ is a bounded measure, $\widehat{\mu}$ a Fourier multiplier of $\dot{W}^{1,p}(\mathbb{R}^d)$:

$$\partial_{x_j}(\mu * f) = \mu * (\partial_{x_j} f).$$

Let $d > 1$. Assume that

$$\partial_{x_j} S = \sum_{k=1}^{d} \partial_{x_k} \mu_{k,j}.$$

Then $\partial_{x_j}(S * f) = \sum(\mu_{k,j} * \partial_{x_k} f)$, so $\mathcal{F}S$ is a Fourier multiplier of $\dot{W}^{1,1}(\mathbb{R}^d)$.

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For $d > 1$, there exists such distributions S that are not bounded measures.

Theorem [Poornima 1983]. *For $d > 1$ there exists Fourier multipliers of $\dot{W}^{1,1}(\mathbb{R}^d)$ which are not Fourier transforms of bounded measures.*

Ornstein's non-inequalities for differential operators.

There exists a function f on $\mathbb{R}^2$ such that $\frac{\partial^2}{\partial x_1^2} f$ and $\frac{\partial^2}{\partial x_2^2} f$ are integrable but $\frac{\partial^2}{\partial x_1 \partial x_2} f$ is not.

Proved by Ornstein in 1962. New proof by Conti, Faraco & Maggi, 2004, generalized by Kirchheim & Kristensen 2015. Extended by Kazaniecki, Stolyarov and Wojciechowski, 2015.

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Ornstein: more generally, non L^1-inequality for $P_1(D), \cdots, P_L(D)$ and $Q(D)$ linearly independent, homogeneous polynomials of same degree.

No non constant homogeneous Fourier multipliers.

Theorem. [B. & Poornima 1987] *For* $d > 1$ *non constant homogeneous functions of degree* 0 *are not Fourier multipliers of* $\dot{W}^{1,1}(\mathbb{R}^d)$.

Proof for even spherical harmonics $Q(\xi)/|\xi|^{2N}$:
Start from counterexamples of Ornstein with the polynomials $\xi_j|\xi|^{2N}, j = 1, \cdots, d$ and $Q_k(\xi) = \xi_k Q(\xi)$. There exists g with

$$\|\nabla(Q(D)g)\|_1 = \infty \qquad \|\nabla(\Delta^N g)\|_1 = 1.$$

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Theorem [Kazaniecki & Wojciechowski 2014.] *Fourier multipliers of $\dot{W}^{1,1}(\mathbb{R}^d)$ are continuous functions on $\mathbb{R}^d$.*

De Leeuw type theorems.

Theorem. *The restriction of a Fourier multiplier of* $\dot{W}^{1,1}(\mathbb{R}^d)$ *to any affine subspace of dimension* k *identifies with a Fourier multiplier of* $\dot{W}^{1,1}(\mathbb{R}^k)$.

Sufficient to consider $\xi'' = a$, with $\xi'' = \xi - \xi'$ and ξ' the projection of ξ on the subspace generated by the k first coordinates.

Sufficient to restrict to $a = 0$ and $a = (0, 0, \cdots, 1)$.

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Hint of the proof, $k = 1$, $a = 0$. One considers integrals

$$\int_{\mathbb{R}} m(\xi_1, 0)\xi_1 \widehat{f}(\xi_1)\widehat{h}(\xi_1) d\xi_1$$

as the limit of

$$\frac{1}{\lambda^{d-1}} \int_{\mathbb{R}^d} m(\xi_1, \xi')\xi_1 \widehat{f}(\xi_1)\widehat{h}(\xi_1)\varphi^2(\xi'/\lambda) d\xi_1 d\xi'.$$

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For $a \neq 0$, the restriction is a Fourier multiplier of the non homogeneous space $W^{1,1}$. Surjectivity?

Restriction theorem on $\mathbb{Z}^d$

Theorem. *The restriction of a Fourier multiplier of* $\dot{W}^{1,1}(\mathbb{R}^d)$ *to* $\mathbb{Z}^d$ *identifies with a Fourier multiplier of* $\dot{W}^{1,1}(\mathbb{T}^d)$*. Moreover there is surjectivity.*

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This, with fractional integration, gives for $\mathbb{T}^d$the corollary

$$\sum_{n\neq 0} \frac{|\widehat{f}(n)|^2}{|n|^{d-2}} \leq C\|f\|^2_{W^{1,1}(\mathbb{T}^d)}.$$

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$$\| \sum_{n\neq 0} \frac{n_j \varepsilon_n \widehat{f}(n)}{|n|^{d/2}} e^{in.x} \|_1^2 \leq \sum_{n\neq 0} \frac{|\widehat{f}(n)|^2}{|n|^{d-2}} \leq C\|f\|^2_{W^{1,1}(\mathbb{T}^d)}.$$

New Fourier multipliers.

Theorem. *There are Fourier multipliers of* $\dot{W}^{1,1}(\mathbb{R}^d)$*, which are not Fourier transforms of distributions* S *such that one can write* $\partial_{x_j} S = \sum_{k=1}^{d} \partial_{x_k} \mu_{k,j}$.

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Recall that Poornima's multipliers are such that $\partial_{x_j} S = \sum_{k=1}^{d} \partial_{x_k} \mu_{k,j}$ or, equivalently, that $\partial_{x_j} S$ belongs to the dual of

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Proof on $\mathbb{T}^d$. For every choice of ε_n of modulus 1 the sequence $\varepsilon_n |n|^{-d/2}$ is uniformly a Fourier multiplier of $W^{1,1}(\mathbb{T}^d)$. Assume it is uniformly a Poornima's multiplier on $\mathbb{T}^d$. Then $\varepsilon_n n_j |n|^{-d/2}$ are Fourier coefficients of an element of the dual of $\mathcal{C}^1$, so that

$$|\sum n_j \varepsilon_n |n|^{-d/2} \widehat{f}(n)| \leq C \|f\|_{\mathcal{C}^1}.$$

Contradiction with the existence of $\mathcal{C}^1$ functions such that $\sum |n|^{-d/2+1} |\widehat{f}(n)| = \infty$.

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Also, for almost every choice of $\varepsilon_n = \pm 1$ the multiplier is not the sequence of Fourier coefficients of a measure.

Continuity of Fourier multipliers.

Theorem [Kazaniecki & Wojciechowski 2014.] *Fourier multipliers of $\dot{W}^{1,1}(\mathbb{R}^d)$ are continuous functions on $\mathbb{R}^d$.*

Proof.

- First easy step: they are continuous outside 0 and bounded by the norm of the operator.
  Indeed, if m is a multiplier and ψ is supported in a shell $r \leq |\xi| \leq R$, then $m\psi$ is the Fourier transform of a measure, thus continuous.
- They are continuous on each line and all values at 0 are the same.
- By contradiction, we would find a sequence of points ξ_j with rational coordinates such that $\xi_j/|\xi|$ converges and values for even and odd indices do not converge to the same limit.

The main point.

For c_k a lacunary sequence, the (eventually infinite) Riesz product $\prod(1 + \cos c_k \cdot x)$ is a positive measure of mass 1 on $\mathbb{T}^d$.

Lemma (Latala 2013)

For a sequence $c_j \in \mathbb{T}^d$ sufficiently lacunary,

$$\sum_{j=0}^{J} (-1)^j \cos c_j \cdot x \prod_{1<k<j} (1 + \cos c_k \cdot x)$$

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In fact result on the torus $\mathbb{T}^\infty$, where we replace $c_k.x$ by independent variables.

Condition $|c_{k+1}| > M|c_k|$ and $\sum \left(\frac{|c_{k+1}|}{|c_k|}\right)^2 < \infty$.

The context of Uchiyama's Theorem

Theorem.[Uchiyama 1982] *Let* $m_1, m_2, \cdots, m_L$ *homogenous of degree* 0 *functions which are smooth outside* 0. *Then the space of distributions* f *such that* $m_j\widehat{f} \in L^1$ *for all* j *coincides with* $\mathcal{H}^1$ *if and only if, for all* $\xi \neq 0$, *the two vectors*

$$(m_j(\xi))_{j=1}^L \qquad \textit{and} \qquad (m_j(-\xi))_{j=1}^L$$

are not colinear.

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In Ornstein's counter-example, $m_j(\xi) = \frac{\xi_j^2}{|\xi|^2}$. Then other m, which are not linearly dependent, do not lead to L^1 functions.
Same question with different choices, such as the two multipliers 1 and $\xi_1/|\xi|$ in two dimensions?

Higher order Sobolev spaces

All Fourier multipliers of $\dot{W}^{k,1}(\mathbb{R}^d)$ are Fourier multipliers of $\dot{W}^{k+1,1}(\mathbb{R}^d)$. Counter-example to prove that the two spaces are not the same?