Rational approximation to functions with polar singular set

Laurent Baratchart

INRIA Sophia-Antipolis-Méditerrannée France

About the title wording

About the title wording

- "Rational approximation" means approximation by rational functions in the uniform norm to $f \in \mathcal{H}(\Omega)$ on a compact set $K \subset \Omega \subset \mathbb{C}$.

About the title wording

- "Rational approximation" means approximation by rational functions in the uniform norm to $f \in \mathcal{H}(\Omega)$ on a compact set $K \subset \Omega \subset \mathbb{C}$.
- The "singular set" is the set over which the initial branch (f, Ω) cannot be continued analytically.

About the title wording

- "Rational approximation" means approximation by rational functions in the uniform norm to $f \in \mathcal{H}(\Omega)$ on a compact set $K \subset \Omega \subset \mathbb{C}$.
- The "singular set" is the set over which the initial branch (f, Ω) cannot be continued analytically.
- "Polar" refers to a measure of smallness which is defined in potential-theoretic terms.

About the title wording

- "Rational approximation" means approximation by rational functions in the uniform norm to $f \in \mathcal{H}(\Omega)$ on a compact set $K \subset \Omega \subset \mathbb{C}$.
- The "singular set" is the set over which the initial branch (f, Ω) cannot be continued analytically.
- "Polar" refers to a measure of smallness which is defined in potential-theoretic terms.
- Example: f has branchpoints and countably many essential singularities.

The possibility of rational approximation

The possibility of rational approximation

- In 1885, Runge proved that holomorphic functions of one complex variable can be approximated by rational functions, locally uniformly on their domain of holomorphy.

The possibility of rational approximation

- In 1885, Runge proved that holomorphic functions of one complex variable can be approximated by rational functions, locally uniformly on their domain of holomorphy.
- **Theorem**[Runge, 1885]

The possibility of rational approximation

- In 1885, Runge proved that holomorphic functions of one complex variable can be approximated by rational functions, locally uniformly on their domain of holomorphy.
- **Theorem**[Runge, 1885]
  Let $K \subset \Omega \subset \mathbb{C}$ with K compact and Ω open. If $f \in \mathrm{Hol}(\Omega)$ and $\varepsilon > 0$, there is a rational function R such that

$$|f(z) - R(z)| < \varepsilon, \qquad z \in K.$$

The possibility of rational approximation

- In 1885, Runge proved that holomorphic functions of one complex variable can be approximated by rational functions, locally uniformly on their domain of holomorphy.
- **Theorem**[Runge, 1885]
  Let $K \subset \Omega \subset \mathbb{C}$ with K compact and Ω open. If $f \in \mathrm{Hol}(\Omega)$ and $\varepsilon > 0$, there is a rational function R such that

$$|f(z) - R(z)| < \varepsilon, \qquad z \in K.$$

- This is a simple consequence of the duality between complex measures and continuous functions with compact support.

The possibility of rational approximation

- In 1885, Runge proved that holomorphic functions of one complex variable can be approximated by rational functions, locally uniformly on their domain of holomorphy.
- **Theorem**[Runge, 1885]
  Let $K \subset \Omega \subset \mathbb{C}$ with K compact and Ω open. If $f \in \mathrm{Hol}(\Omega)$ and $\varepsilon > 0$, there is a rational function R such that

$$|f(z) - R(z)| < \varepsilon, \qquad z \in K.$$

- This is a simple consequence of the duality between complex measures and continuous functions with compact support.
- Runge's proof rests on his "pole shifting technique".

The possibility of rational approximation

- In 1885, Runge proved that holomorphic functions of one complex variable can be approximated by rational functions, locally uniformly on their domain of holomorphy.
- **Theorem**[Runge, 1885]
  Let $K \subset \Omega \subset \mathbb{C}$ with K compact and Ω open. If $f \in \mathrm{Hol}(\Omega)$ and $\varepsilon > 0$, there is a rational function R such that

$$|f(z) - R(z)| < \varepsilon, \qquad z \in K.$$

- This is a simple consequence of the duality between complex measures and continuous functions with compact support.
- Runge's proof rests on his "pole shifting technique".
  Useful in other contexts (*e.g.* density of gradients of harmonic polynomials in vector fields with gradient tangential component on proper regular compact subsets of the sphere [J. Leblond, J.Partington, L.B., 2009].

Subsequent developments

Subsequent developments

- Approximability on K of continuous functions analytic in $\overset{\circ}{K}$ [Bishop 60, Mergelyan 62, Vitushkin 66],

Subsequent developments

- Approximability on K of continuous functions analytic in $\overset{\circ}{K}$ [Bishop 60, Mergelyan 62, Vitushkin 66], approximability on noncompact sets [Roth, 1976].

Subsequent developments

- Approximability on K of continuous functions analytic in $\mathring{K}$ [Bishop 60, Mergelyan 62, Vitushkin 66], approximability on noncompact sets [Roth, 1976].
- Characterization of smoothness from the rate of approximation [Dolzhenko 68, Pekarskii 83, Peller 86].

Subsequent developments

- Approximability on K of continuous functions analytic in $\mathring{K}$ [Bishop 60, Mergelyan 62, Vitushkin 66], approximability on noncompact sets [Roth, 1976].
- Characterization of smoothness from the rate of approximation [Dolzhenko 68, Pekarskii 83, Peller 86].
- optimal rate of convergence as the degree of the approximant goes large [Walsh 62, Gonchar 78, Parfenov 86, Prokhorov 93].

Subsequent developments

- Approximability on K of continuous functions analytic in $\mathring{K}$ [Bishop 60, Mergelyan 62, Vitushkin 66], approximability on noncompact sets [Roth, 1976].
- Characterization of smoothness from the rate of approximation [Dolzhenko 68, Pekarskii 83, Peller 86].
- optimal rate of convergence as the degree of the approximant goes large [Walsh 62, Gonchar 78, Parfenov 86, Prokhorov 93].
- Constructive approximation (applications to number theory, numerical analysis, modeling and engineering ...).

Remarks

Remarks

- Determining where the poles of an optimal approximant of given degree should lie is the non-convex and most difficult part of the approximation problem.

Remarks

- Determining where the poles of an optimal approximant of given degree should lie is the non-convex and most difficult part of the approximation problem.
- Understanding how the poles distribute asymptotically is a key to obtain error rates of concrete sequences of approximants.

Remarks

- Determining where the poles of an optimal approximant of given degree should lie is the non-convex and most difficult part of the approximation problem.
- Understanding how the poles distribute asymptotically is a key to obtain error rates of concrete sequences of approximants.
- The talk is concerned with asymptotic error rates and pole distribution. The fundamental feature of our situation is that f extends analytically beyond the compact set K on which approximation takes place.

Some notation

Some notation

- f is holomorphic on a domain $\Omega \subset \mathbb{C}$.

Some notation

- f is holomorphic on a domain $\Omega \subset \mathbb{C}$.
- K is a compact subset of Ω.

Some notation

- f is holomorphic on a domain $\Omega \subset \mathbb{C}$.
- K is a compact subset of Ω.
- $\mathcal{R}_n$ denotes the set of rational functions of degree n:

$$\mathcal{R}_n = \{\frac{p_n}{q_n};\ p_n, q_n \text{ complex polynomials of degree at most } n\}.$$

Some notation

- f is holomorphic on a domain $\Omega \subset \mathbb{C}$.
- K is a compact subset of Ω.
- $\mathcal{R}_n$ denotes the set of rational functions of degree n:

$$\mathcal{R}_n = \{\frac{p_n}{q_n};\ p_n, q_n \text{ complex polynomials of degree at most } n\}.$$

- We set

$$e_n = e_n(f, K) := \inf_{r_n \in \mathcal{R}_n} \|f - r_n\|_{L^\infty(K)}.$$

Rates in approximation

Rates in approximation

- Strong asymptotics are estimates of $e_n(f, K)$ as n goes large, with respect to some scale depending on n.

Rates in approximation

- Strong asymptotics are estimates of $e_n(f, K)$ as n goes large, with respect to some scale depending on n.
- Strong asymptotics can usually be derived for specific functions f only.

Rates in approximation

- Strong asymptotics are estimates of $e_n(f, K)$ as n goes large, with respect to some scale depending on n.
- Strong asymptotics can usually be derived for specific functions f only.
- Weak or n-th root asymptotics are estimates of $e_n^{1/n}$ as n goes large.

Rates in approximation

- Strong asymptotics are estimates of $e_n(f, K)$ as n goes large, with respect to some scale depending on n.
- Strong asymptotics can usually be derived for specific functions f only.
- Weak or n-th root asymptotics are estimates of $e_n^{1/n}$ as n goes large.
- n-th root rates only estimate the geometric decay of the error.

Rates in approximation

- Strong asymptotics are estimates of $e_n(f, K)$ as n goes large, with respect to some scale depending on n.
- Strong asymptotics can usually be derived for specific functions f only.
- Weak or n-th root asymptotics are estimates of $e_n^{1/n}$ as n goes large.
- n-th root rates only estimate the geometric decay of the error.
- They make contact with logarithmic potential theory.

Some potential theory

Some potential theory

- The logarithmic potential of a positive measure μ with compact support in $\mathbb{C}$ is

$$V^{\mu}(z) := \int \log \left| \frac{1}{z-t} \right| d\mu(t)$$

Some potential theory

- The logarithmic potential of a positive measure μ with compact support in $\mathbb{C}$ is

$$V^{\mu}(z) := \int \log \left| \frac{1}{z-t} \right| d\mu(t)$$

- This is a superharmonic function valued in $\mathbb{R} \cup \{+\infty\}$, the solution to $\Delta u = -\mu$ which is smallest in modulus at ∞.

Some potential theory

- The logarithmic potential of a positive measure μ with compact support in $\mathbb{C}$ is

$$V^{\mu}(z) := \int \log \left| \frac{1}{z-t} \right| \, d\mu(t)$$

- This is a superharmonic function valued in $\mathbb{R} \cup \{+\infty\}$, the solution to $\Delta u = -\mu$ which is smallest in modulus at ∞.
- The logarithmic energy of μ is

$$I(\mu) := \int \int \log \left| \frac{1}{z-t} \right| \, d\mu(t) d\mu(z).$$

Some potential theory

- The logarithmic potential of a positive measure μ with compact support in $\mathbb{C}$ is

$$V^{\mu}(z) := \int \log \left| \frac{1}{z-t} \right| \, d\mu(t)$$

- This is a superharmonic function valued in $\mathbb{R} \cup \{+\infty\}$, the solution to $\Delta u = -\mu$ which is smallest in modulus at ∞.
- The logarithmic energy of μ is

$$I(\mu) := \int \int \log \left| \frac{1}{z-t} \right| \, d\mu(t) d\mu(z).$$

- The energy lies in $\mathbb{R} \cup \{+\infty\}$.

Potential theory cont'd

- The logarithmic capacity of K is $C(K) = e^{-I_K}$ where

$$I_K := \inf_{\mu \in \mathcal{P}_K} \int \int \log \left| \frac{1}{z-t} \right| d\mu(t) d\mu(x)$$

and $\mathcal{P}_K$ is the set of probability measures on K.

Potential theory cont'd

- The logarithmic capacity of K is $C(K) = e^{-I_K}$ where

$$I_K := \inf_{\mu \in \mathcal{P}_K} \int \int \log \left| \frac{1}{z-t} \right| d\mu(t) d\mu(x)$$

  and $\mathcal{P}_K$ is the set of probability measures on K.
- If $C(K) > 0$, there is a unique measure $\omega_K \in \mathcal{P}_K$ to meet the above infimum. It is called the equilibrium distribution on K.

Potential theory cont'd

- The logarithmic capacity of K is $C(K) = e^{-I_K}$ where

$$I_K := \inf_{\mu \in \mathcal{P}_K} \int \int \log \left| \frac{1}{z-t} \right| \, d\mu(t) d\mu(x)$$

  and $\mathcal{P}_K$ is the set of probability measures on K.
- If $C(K) > 0$, there is a unique measure $\omega_K \in \mathcal{P}_K$ to meet the above infimum. It is called the equilibrium distribution on K.
- If $C(K) = 0$ one says K is polar. Polar sets are very small and look very bad (totally disconnected, H^1-dimension zero...).

Potential theory cont'd

- The logarithmic capacity of K is $C(K) = e^{-I_K}$ where

$$I_K := \inf_{\mu \in \mathcal{P}_K} \int \int \log \left| \frac{1}{z-t} \right| d\mu(t) d\mu(x)$$

  and $\mathcal{P}_K$ is the set of probability measures on K.
- If $C(K) > 0$, there is a unique measure $\omega_K \in \mathcal{P}_K$ to meet the above infimum. It is called the equilibrium distribution on K.
- If $C(K) = 0$ one says K is polar. Polar sets are very small and look very bad (totally disconnected, H^1-dimension zero...).
- A property valid outside a polar set is said to hold quasi-everywhere.

Potential theory cont'd

- The logarithmic capacity of K is $C(K) = e^{-I_K}$ where

$$I_K := \inf_{\mu \in \mathcal{P}_K} \int \int \log \left| \frac{1}{z-t} \right| d\mu(t) d\mu(x)$$

  and $\mathcal{P}_K$ is the set of probability measures on K.
- If $C(K) > 0$, there is a unique measure $\omega_K \in \mathcal{P}_K$ to meet the above infimum. It is called the equilibrium distribution on K.
- If $C(K) = 0$ one says K is polar. Polar sets are very small and look very bad (totally disconnected, H^1-dimension zero...).
- A property valid outside a polar set is said to hold quasi-everywhere.
- ω_K is characterized by V^{ω_K} being constant q.e. on K (Frostman theorem).

Potential theory cont'd

Potential theory cont'd

- Capacity is a measure of size.

Potential theory cont'd

- Capacity is a measure of size.
- Example 1: the capacity of a disk is its radius and the equilibrium distribution is normalized arclength on the circumference.

Potential theory cont'd

- Capacity is a measure of size.
- Example 1: the capacity of a disk is its radius and the equilibrium distribution is normalized arclength on the circumference.
- Example 2: the capacity of a segment is $C_{[a,b]} = (b-a)/4$ and the equilibrium distribution is

$$\frac{dt}{\pi\sqrt{(t-a)(b-t)}}.$$

Potential theory cont'd

- Capacity is a measure of size.
- Example 1: the capacity of a disk is its radius and the equilibrium distribution is normalized arclength on the circumference.
- Example 2: the capacity of a segment is $C_{[a,b]} = (b-a)/4$ and the equilibrium distribution is

$$\frac{dt}{\pi\sqrt{(t-a)(b-t)}}.$$

- The equilibrium distribution is always supported on the outer boundary of K.

Potential theory cont'd

- Capacity is a measure of size.
- Example 1: the capacity of a disk is its radius and the equilibrium distribution is normalized arclength on the circumference.
- Example 2: the capacity of a segment is $C_{[a,b]} = (b-a)/4$ and the equilibrium distribution is

$$\frac{dt}{\pi\sqrt{(t-a)(b-t)}}.$$

- The equilibrium distribution is always supported on the outer boundary of K.
- The capacity of a set E is the supremum of C_K over all compact $K \subset E$.

Potential theory cont'd

Potential theory cont'd

- The weighted capacity of a non polar compact set K in the field ψ, assumed to be lower semi-continuous and not infinite q.e. on K, is $C_\psi(K) = e^{-I_\psi}$ where

$$I_\psi := \inf_{\mu \in \mathcal{P}_K} \int \int \log \frac{1}{|z-t|} d\mu(t) d\mu(z) + 2 \int \psi(t) d\mu(t).$$

Potential theory cont'd

- The weighted capacity of a non polar compact set K in the field ψ, assumed to be lower semi-continuous and not infinite q.e. on K, is $C_\psi(K) = e^{-I_\psi}$ where

$$I_\psi := \inf_{\mu \in \mathcal{P}_K} \int \int \log \frac{1}{|z-t|} d\mu(t) d\mu(z) + 2 \int \psi(t) d\mu(t).$$

- There is a unique measure $\omega_{K,\psi} \in \mathcal{P}_K$ to meet the infimum; it is called the weighted equilibrium distribution on K (w.r.t.ψ).

Potential theory cont'd

- The weighted capacity of a non polar compact set K in the field ψ, assumed to be lower semi-continuous and not infinite q.e. on K, is $C_\psi(K) = e^{-I_\psi}$ where

$$I_\psi := \inf_{\mu \in \mathcal{P}_K} \int \int \log \frac{1}{|z-t|} d\mu(t) d\mu(z) + 2 \int \psi(t) d\mu(t).$$

- There is a unique measure $\omega_{K,\psi} \in \mathcal{P}_K$ to meet the infimum; it is called the weighted equilibrium distribution on K (w.r.t.ψ).
- $\omega_{K,\psi}$ is characterized by the fact that $V^{\omega_{K,\psi}} + \psi$ is constant q.e. on $\mathrm{supp}(\omega_{K,\psi})$ and at least as large as this constant q.e. on K.

Potential theory cont'd

- The weighted capacity of a non polar compact set K in the field ψ, assumed to be lower semi-continuous and not infinite q.e. on K, is $C_\psi(K) = e^{-I_\psi}$ where

$$I_\psi := \inf_{\mu \in \mathcal{P}_K} \int \int \log \frac{1}{|z-t|} d\mu(t) d\mu(z) + 2 \int \psi(t) d\mu(t).$$

- There is a unique measure $\omega_{K,\psi} \in \mathcal{P}_K$ to meet the infimum; it is called the weighted equilibrium distribution on K (w.r.t.ψ).
- $\omega_{K,\psi}$ is characterized by the fact that $V^{\omega_{K,\psi}} + \psi$ is constant q.e. on $\mathrm{supp}(\omega_{K,\psi})$ and at least as large as this constant q.e. on K.
- It is the equilibrium distribution on a conductor K of a unit electric charge in the electric field ψ.

Potential theory cont'd

- The weighted capacity of a non polar compact set K in the field ψ, assumed to be lower semi-continuous and not infinite q.e. on K, is $C_\psi(K) = e^{-I_\psi}$ where

$$I_\psi := \inf_{\mu \in \mathcal{P}_K} \int \int \log \frac{1}{|z-t|} d\mu(t) d\mu(z) + 2 \int \psi(t) d\mu(t).$$

- There is a unique measure $\omega_{K,\psi} \in \mathcal{P}_K$ to meet the infimum; it is called the weighted equilibrium distribution on K (w.r.t.ψ).
- $\omega_{K,\psi}$ is characterized by the fact that $V^{\omega_{K,\psi}} + \psi$ is constant q.e. on $\mathrm{supp}(\omega_{K,\psi})$ and at least as large as this constant q.e. on K.
- It is the equilibrium distribution on a conductor K of a unit electric charge in the electric field ψ.
- When $\psi \equiv 0$ one recovers the usual capacity.

Green functions

Green functions

- Let Ω open have non-polar boundary $\partial\Omega$.

Green functions

- Let Ω open have non-polar boundary $\partial\Omega$.
- The Green function of Ω with pole at $z \in \Omega$ is the function $G_\Omega(z, .)$ such that

Green functions

- Let Ω open have non-polar boundary $\partial\Omega$.
- The Green function of Ω with pole at $z \in \Omega$ is the function $G_\Omega(z, .)$ such that
  - $t \mapsto G_\Omega(z, t) + \log|z - t|$ is bounded and harmonic in Ω,

Green functions

- Let Ω open have non-polar boundary $\partial\Omega$.
- The Green function of Ω with pole at $z \in \Omega$ is the function $G_\Omega(z, .)$ such that
  - $t \mapsto G_\Omega(z,t) + \log|z-t|$ is bounded and harmonic in Ω,
  - $$\lim_{t\to\xi} G_\Omega(z,t) = 0, \quad \text{q.e. } \xi \in \partial\Omega.$$

Green functions

- Let Ω open have non-polar boundary $\partial\Omega$.
- The Green function of Ω with pole at $z \in \Omega$ is the function $G_\Omega(z,.)$ such that
  - $t \mapsto G_\Omega(z,t) + \log|z-t|$ is bounded and harmonic in Ω,
  - $$\lim_{t \to \xi} G_\Omega(z,t) = 0, \quad \text{q.e. } \xi \in \partial\Omega.$$
- Equivalently, $G_\Omega(z,.)$ is the smallest positive solution to
$$\Delta u = -\delta_z \quad \text{in } \Omega.$$

Green functions

- Let Ω open have non-polar boundary $\partial\Omega$.
- The Green function of Ω with pole at $z \in \Omega$ is the function $G_\Omega(z, .)$ such that
  - $t \mapsto G_\Omega(z,t) + \log|z-t|$ is bounded and harmonic in Ω,
  - $$\lim_{t\to\xi} G_\Omega(z,t) = 0, \quad \text{q.e. } \xi \in \partial\Omega.$$
- Equivalently, $G_\Omega(z, .)$ is the smallest positive solution to

$$\Delta u = -\delta_z \quad \text{in } \Omega.$$

- Example: if $\mathbb{D}$ is the unit disk, then

$$G_{\mathbb{D}}(z,t) = \log\left|\frac{1-z\bar{t}}{z-t}\right|.$$

Potential theory cont'd

Potential theory cont'd

- Let $\partial\Omega$ be non-polar.

Potential theory cont'd

- Let $\partial\Omega$ be non-polar.
- The Green potential of a positive measure μ with compact support in Ω is

$$V_\Omega^\mu(z) := \int G_\Omega(z,t)\,d\mu(t).$$

Potential theory cont'd

- Let $\partial\Omega$ be non-polar.
- The Green potential of a positive measure μ with compact support in Ω is

$$V_\Omega^\mu(z) := \int G_\Omega(z,t)\, d\mu(t).$$

- It is the smallest solution to $\Delta u = -\mu$ in Ω.

Potential theory cont'd

- Let $\partial\Omega$ be non-polar.
- The Green potential of a positive measure μ with compact support in Ω is

$$V_{\Omega}^{\mu}(z) := \int G_{\Omega}(z,t)\,d\mu(t).$$

- It is the smallest solution to $\Delta u = -\mu$ in Ω.
- The Green energy of μ is

$$I^{G}(\mu) := \int\int G_{\Omega}(z,t)\,d\mu(t)d\mu(z).$$

Potential theory cont'd

- Let $\partial\Omega$ be non-polar.
- The Green potential of a positive measure μ with compact support in Ω is

$$V_\Omega^\mu(z) := \int G_\Omega(z,t)\, d\mu(t).$$

- It is the smallest solution to $\Delta u = -\mu$ in Ω.
- The Green energy of μ is

$$I^G(\mu) := \int\int G_\Omega(z,t)\, d\mu(t) d\mu(z).$$

$$\left(= \|\nabla V_\Omega^\mu\|^2_{L^2(\Omega)} \text{ in smooth case } \right)$$

Potential theory cont'd

Potential theory cont'd

- The Green capacity of K is $C(K, \Omega) = 1/\mathcal{I}_K$ where

$$\mathcal{I}_K := \inf_{\mu \in \mathcal{P}_K} I_G(\mu) = \inf_{\mu \in \mathcal{P}_K} \int \int G_\Omega(z, t)\, d\mu(t) d\mu(z).$$

Potential theory cont'd

- The Green capacity of K is $C(K,\Omega) = 1/\mathcal{I}_K$ where

$$\mathcal{I}_K := \inf_{\mu\in\mathcal{P}_K} I_G(\mu) = \inf_{\mu\in\mathcal{P}_K} \int\int G_\Omega(z,t)\, d\mu(t) d\mu(z).$$

- If K, is non polar, there is a unique measure $\omega^G_{K,\Omega} \in \mathcal{P}_K$ to meet the above infimum. It is called the Green equilibrium distribution of K in Ω.

Potential theory cont'd

- The Green capacity of K is $C(K,\Omega) = 1/\mathcal{I}_K$ where

$$\mathcal{I}_K := \inf_{\mu \in \mathcal{P}_K} I_G(\mu) = \inf_{\mu \in \mathcal{P}_K} \int\int G_\Omega(z,t)\, d\mu(t) d\mu(z).$$

- If K, is non polar, there is a unique measure $\omega^G_{K,\Omega} \in \mathcal{P}_K$ to meet the above infimum. It is called the Green equilibrium distribution of K in Ω.
- $\omega^G_{K,\Omega}$ is characterized by the fact that $V_G^{\omega^G_{K,\Omega}}$ is constant q.e. on K.

Potential theory cont'd

- The Green capacity of K is $C(K,\Omega) = 1/\mathcal{I}_K$ where

$$\mathcal{I}_K := \inf_{\mu \in \mathcal{P}_K} I_G(\mu) = \inf_{\mu \in \mathcal{P}_K} \int \int G_\Omega(z,t)\, d\mu(t) d\mu(z).$$

- If K, is non polar, there is a unique measure $\omega^G_{K,\Omega} \in \mathcal{P}_K$ to meet the above infimum. It is called the Green equilibrium distribution of K in Ω.
- $\omega^G_{K,\Omega}$ is characterized by the fact that $V_G^{\omega^G_{K,\Omega}}$ is constant q.e. on K.
- Green capacities and Green equilibrium distributions are conformally invariant.

Condensers

Condensers

Condensers

- The Green capacity also has a more symmetric definition as follows.

Condensers

- The Green capacity also has a more symmetric definition as follows.
- A pair of compact sets K_1, K_2 each of which is contained in a single component of the complement of the other is called a condenser with plates K_1, K_2.

Condensers

- The Green capacity also has a more symmetric definition as follows.
- A pair of compact sets $K_1, \ K_2$ each of which is contained in a single component of the complement of the other is called a condenser with plates $K_1, \ K_2$.
- The capacity of the condenser is $\mathfrak{C}(K_1, K_2)$ such that

$$\frac{1}{\mathfrak{C}(K_1, K_2)} = \inf_{\nu_1 \in \mathcal{P}_{K_1}, \ \nu_2 \in \mathcal{P}_{K_2}} \int \log \left| \frac{(x-y)(u-v)}{x-u)(y-v)} \right| d\nu_1(x) d\nu_1(y) d\nu_2$$

Condensers

- The Green capacity also has a more symmetric definition as follows.
- A pair of compact sets $K_1,\ K_2$ each of which is contained in a single component of the complement of the other is called a condenser with plates $K_1,\ K_2$.
- The capacity of the condenser is $\mathfrak{C}(K_1, K_2)$ such that

$$\frac{1}{\mathfrak{C}(K_1, K_2)} = \inf_{\nu_1 \in \mathcal{P}_{K_1},\ \nu_2 \in \mathcal{P}_{K_2}} \int \log \left| \frac{(x-y)(u-v)}{x-u)(y-v)} \right| d\nu_1(x) d\nu_1(y) d\nu_2$$

- It holds that

$$C(K, \Omega) = \mathfrak{C}(K, \Omega^c) = \mathfrak{C}(K, \partial\Omega) = \mathfrak{C}(\partial K, \partial\Omega)$$

Condensers

- The Green capacity also has a more symmetric definition as follows.
- A pair of compact sets $K_1,\ K_2$ each of which is contained in a single component of the complement of the other is called a condenser with plates $K_1,\ K_2$.
- The capacity of the condenser is $\mathfrak{C}(K_1, K_2)$ such that

$$\frac{1}{\mathfrak{C}(K_1, K_2)} = \inf_{\nu_1 \in \mathcal{P}_{K_1},\ \nu_2 \in \mathcal{P}_{K_2}} \int \log \left| \frac{(x-y)(u-v)}{x-u)(y-v)} \right| d\nu_1(x) d\nu_1(y) d\nu_2$$

- It holds that

$$C(K, \Omega) = \mathfrak{C}(K, \Omega^c) = \mathfrak{C}(K, \partial\Omega) = \mathfrak{C}(\partial K, \partial\Omega)$$

where boundaries are with respect to the component of the complement containing the other plate.

Condensers

- The Green capacity also has a more symmetric definition as follows.
- A pair of compact sets $K_1,\ K_2$ each of which is contained in a single component of the complement of the other is called a condenser with plates $K_1,\ K_2$.
- The capacity of the condenser is $\mathfrak{C}(K_1, K_2)$ such that

$$\frac{1}{\mathfrak{C}(K_1, K_2)} = \inf_{\nu_1 \in \mathcal{P}_{K_1},\ \nu_2 \in \mathcal{P}_{K_2}} \int \log \left| \frac{(x-y)(u-v)}{x-u)(y-v)} \right| d\nu_1(x) d\nu_1(y) d\nu_2$$

- It holds that

$$C(K, \Omega) = \mathfrak{C}(K, \Omega^c) = \mathfrak{C}(K, \partial\Omega) = \mathfrak{C}(\partial K, \partial\Omega)$$

  where boundaries are with respect to the component of the complement containing the other plate.
- The measure on K to realize the infimum is $\omega^G_{K,\Omega}$, and it is also the weighted equilibrium distribution in the field generated by minus the potential of the other plate.

Regularity

Regularity

- Points $\xi \in \partial\Omega$ where $\overline{\lim}_{t \to \xi} G_\Omega(z, t) > 0$ are independent of $z \in \Omega$ and are called irregular points of Ω^c.

Regularity

- Points $\xi \in \partial\Omega$ where $\overline{\lim}_{t\to\xi} G_\Omega(z,t) > 0$ are independent of $z \in \Omega$ and are called irregular points of Ω^c.
- Irregular points form a polar set. A closed set having no irregular points is called regular.

Regularity

- Points $\xi \in \partial\Omega$ where $\overline{\lim}_{t\to\xi} G_\Omega(z,t) > 0$ are independent of $z \in \Omega$ and are called irregular points of Ω^c.
- Irregular points form a polar set. A closed set having no irregular points is called regular.
- Irregular points admit the following characterization (Wiener criterion).
    - For $\xi \in \partial\Omega$ and $0 < \gamma < 1$, set

$$F_n = \{z \notin \Omega;\ \gamma^n < |z - \xi| \leq \gamma^{n-1}\}.$$

Regularity

- Points $\xi \in \partial\Omega$ where $\overline{\lim}_{t \to \xi} G_\Omega(z, t) > 0$ are independent of $z \in \Omega$ and are called irregular points of Ω^c.
- Irregular points form a polar set. A closed set having no irregular points is called regular.
- Irregular points admit the following characterization (Wiener criterion).
  - For $\xi \in \partial\Omega$ and $0 < \gamma < 1$, set

$$F_n = \{z \notin \Omega;\ \gamma^n < |z - \xi| \leq \gamma^{n-1}\}.$$

  - Then ξ is irregular iff

$$\Sigma_{n \geq 1} \frac{n}{\log(2/C_{F_n})} < \infty.$$

n-th root estimates: upper bound

n-th root estimates: upper bound

- J.L. Walsh was perhaps first to connect weak asymptotics in rational approximation with Green potentials in the late 40's.

n-th root estimates: upper bound

- J.L. Walsh was perhaps first to connect weak asymptotics in rational approximation with Green potentials in the late 40's. He proved the following:

n-th root estimates: upper bound

- J.L. Walsh was perhaps first to connect weak asymptotics in rational approximation with Green potentials in the late 40's. He proved the following:
- **Theorem**[Walsh]
  Let f be holomorphic on a domain Ω and $K \subset \Omega$ be compact;

n-th root estimates: upper bound

- J.L. Walsh was perhaps first to connect weak asymptotics in rational approximation with Green potentials in the late 40's. He proved the following:
- **Theorem**[Walsh]
  Let f be holomorphic on a domain Ω and $K \subset \Omega$ be compact; Put
  $$e_n = inf_{r_n \in \mathcal{R}_n} \|f - p_n/q_n\|_{L^\infty(K)}.$$

n-th root estimates: upper bound

- J.L. Walsh was perhaps first to connect weak asymptotics in rational approximation with Green potentials in the late 40's. He proved the following:
- **Theorem**[Walsh]
  Let f be holomorphic on a domain Ω and $K \subset \Omega$ be compact; Put
  $$e_n = inf_{r_n \in \mathcal{R}_n} \|f - p_n/q_n\|_{L^\infty(K)}.$$
  Then
  $$\limsup_{n\to\infty} e_n^{1/n} \leq \exp\left(-\frac{1}{C(K,\Omega)}\right).$$

n-th root estimates: upper bound

- J.L. Walsh was perhaps first to connect weak asymptotics in rational approximation with Green potentials in the late 40's. He proved the following:
- **Theorem**[Walsh]
  Let f be holomorphic on a domain Ω and $K \subset \Omega$ be compact; Put
  $$e_n = inf_{r_n \in \mathcal{R}_n} \|f - p_n/q_n\|_{L^\infty(K)}.$$
  Then
  $$\limsup_{n\to\infty} e_n^{1/n} \leq \exp\left(-\frac{1}{C(K,\Omega)}\right).$$
- There are functions for which this bound is sharp (Tikhomirov).

Proof on the disk

Proof on the disk

- By outer continuity of the Green capacity, we may assume that f is bounded on $\mathbb{D}$, say $\|f\|_{H^\infty(\mathbb{D})} = 1$.

Proof on the disk

- By outer continuity of the Green capacity, we may assume that f is bounded on $\mathbb{D}$, say $\|f\|_{H^\infty(\mathbb{D})} = 1$.
- For B_n a Blaschke product with zeros at $z_1, \cdots, z_n \in K$, projection of f onto $H^2 \ominus BH^2$ yields $r_n \in \mathcal{R}_n$ interpolating f at those points, $\|r_n\|_{H^2} \leq 1$. By a Bernstein-type estimate $\|r_n'\|_{H^\infty} \leq cn$ [Baranov-Zarouf, 2014] so that $\|r_n\|_{H^\infty} \leq Cn$.

Proof on the disk

- By outer continuity of the Green capacity, we may assume that f is bounded on $\mathbb{D}$, say $\|f\|_{H^\infty(\mathbb{D})} = 1$.
- For B_n a Blaschke product with zeros at $z_1, \cdots, z_n \in K$, projection of f onto $H^2 \ominus BH^2$ yields $r_n \in \mathcal{R}_n$ interpolating f at those points, $\|r_n\|_{H^2} \leq 1$. By a Bernstein-type estimate $\|r_n'\|_{H^\infty} \leq cn$ [Baranov-Zarouf, 2014] so that $\|r_n\|_{H^\infty} \leq Cn$.
- $$|f(z) - r_n(z)| \leq C' n \Pi_{j=1}^n \left| \frac{z - z_j}{1 - z\bar{z}_j} \right|$$

Proof on the disk

- By outer continuity of the Green capacity, we may assume that f is bounded on $\mathbb{D}$, say $\|f\|_{H^\infty(\mathbb{D})} = 1$.
- For B_n a Blaschke product with zeros at $z_1, \cdots, z_n \in K$, projection of f onto $H^2 \ominus BH^2$ yields $r_n \in \mathcal{R}_n$ interpolating f at those points, $\|r_n\|_{H^2} \leq 1$. By a Bernstein-type estimate $\|r_n'\|_{H^\infty} \leq cn$ [Baranov-Zarouf, 2014] so that $\|r_n\|_{H^\infty} \leq Cn$.
- $$|f(z) - r_n(z)| \leq C' n \Pi_{j=1}^n \left| \frac{z - z_j}{1 - z\bar{z}_j} \right|$$
- Equivalently, with $\nu_n = \frac{1}{n}\Sigma_j \delta_{z_j}$,
$$|f(z) - B_n(z)| \leq C' n \exp\left\{ -n \int G_{\mathbb{D}}(z,t) d\nu_n(t) \right\}$$

Proof on the disk

- By outer continuity of the Green capacity, we may assume that f is bounded on $\mathbb{D}$, say $\|f\|_{H^\infty(\mathbb{D})} = 1$.
- For B_n a Blaschke product with zeros at $z_1, \cdots, z_n \in K$, projection of f onto $H^2 \ominus BH^2$ yields $r_n \in \mathcal{R}_n$ interpolating f at those points, $\|r_n\|_{H^2} \leq 1$. By a Bernstein-type estimate $\|r_n'\|_{H^\infty} \leq cn$ [Baranov-Zarouf, 2014] so that $\|r_n\|_{H^\infty} \leq Cn$.
- $$|f(z) - r_n(z)| \leq C' n \Pi_{j=1}^n \left| \frac{z - z_j}{1 - z\bar{z}_j} \right|$$
- Equivalently, with $\nu_n = \frac{1}{n}\Sigma_j \delta_{z_j}$,
$$|f(z) - B_n(z)| \leq C' n \exp\left\{ -n \int G_{\mathbb{D}}(z,t) d\nu_n(t) \right\}$$
- Taking n-th root while choosing the z_j so that ν_n converges weak* to $\omega^G_{K,\mathbb{D}}$ and letting $n \to \infty$ gives the desired bound up to $\varepsilon > 0$ for z close enough to K.

Proof on the disk

- By outer continuity of the Green capacity, we may assume that f is bounded on $\mathbb{D}$, say $\|f\|_{H^\infty(\mathbb{D})} = 1$.
- For B_n a Blaschke product with zeros at $z_1, \cdots, z_n \in K$, projection of f onto $H^2 \ominus BH^2$ yields $r_n \in \mathcal{R}_n$ interpolating f at those points, $\|r_n\|_{H^2} \leq 1$. By a Bernstein-type estimate $\|r_n'\|_{H^\infty} \leq cn$ [Baranov-Zarouf, 2014] so that $\|r_n\|_{H^\infty} \leq Cn$.
- $$|f(z) - r_n(z)| \leq C' n \Pi_{j=1}^{n} \left| \frac{z - z_j}{1 - z\bar{z}_j} \right|$$
- Equivalently, with $\nu_n = \frac{1}{n}\Sigma_j \delta_{z_j}$,
$$|f(z) - B_n(z)| \leq C' n \exp\left\{ -n \int G_{\mathbb{D}}(z,t) d\nu_n(t) \right\}$$
- Taking n-th root while choosing the z_j so that ν_n converges weak* to $\omega^G_{K,\mathbb{D}}$ and letting $n \to \infty$ gives the desired bound up to $\varepsilon > 0$ for z close enough to K.
- Using outer continuity of the Green capacity, we let $\varepsilon \to 0$.

The Gonchar conjecture

The Gonchar conjecture

- Motivated by certain constructions in multipoint Padé interpolation, A. A. Gonchar conjectured in 1978 that

$$\liminf_{n\to\infty} e_n^{1/n} \leq \exp\left(-\frac{2}{C(K,\Omega)}\right). \tag{1}$$

The Gonchar conjecture

- Motivated by certain constructions in multipoint Padé interpolation, A. A. Gonchar conjectured in 1978 that

$$\liminf_{n\to\infty} e_n^{1/n} \leq \exp\left(-\frac{2}{C(K,\Omega)}\right). \tag{1}$$

- In a sense, Gonchar's conjecture means that using rational approximants instead of linear ones improves the convergence like a Newton scheme does to in optimization to a steepest descent algorithm, by squaring the error, at least for a subsequence.

The Gonchar conjecture

- Motivated by certain constructions in multipoint Padé interpolation, A. A. Gonchar conjectured in 1978 that

$$\liminf_{n\to\infty} e_n^{1/n} \leq \exp\left(-\frac{2}{C(K,\Omega)}\right). \tag{1}$$

- In a sense, Gonchar's conjecture means that using rational approximants instead of linear ones improves the convergence like a Newton scheme does to in optimization to a steepest descent algorithm, by squaring the error, at least for a subsequence.
- Gonchar substantiated his conjecture by constructing classes of functions for which (1) is both an equality and a true limit, using (multipoint) Padé interpolants.

Padé interpolants and orthogonal polynomials

Padé interpolants and orthogonal polynomials

- Let $f(z) = \int \frac{d\mu(\xi)}{z-\xi}$ where μ is a complex measure supported on E compact. Here $\Omega = \overline{\mathbb{C}} \setminus E$.

Padé interpolants and orthogonal polynomials

- Let $f(z) = \int \frac{d\mu(\xi)}{z-\xi}$ where μ is a complex measure supported on E compact. Here $\Omega = \overline{\mathbb{C}} \setminus E$.
- If p_{n-1}/q_n interpolates f in $\{\xi_1^{(n)}, \cdots, \xi_{2n}^{(n)}, \infty\} \subset \Omega$ and if ω_{2n} is the (normalized) polynomial having zeros the $\xi_j^{(n)}$, then

$$\int \frac{q_{k_n}(\xi)}{\omega_{2n}(\xi)} \xi^k d\mu(\xi) = 0, \quad k \in \{0, 1, \ldots, k_n - 1\}. \qquad (2)$$

Padé interpolants and orthogonal polynomials

- Let $f(z) = \int \frac{d\mu(\xi)}{z-\xi}$ where μ is a complex measure supported on E compact. Here $\Omega = \overline{\mathbb{C}} \setminus E$.
- If p_{n-1}/q_n interpolates f in $\{\xi_1^{(n)}, \cdots, \xi_{2n}^{(n)}, \infty\} \subset \Omega$ and if ω_{2n} is the (normalized) polynomial having zeros the $\xi_j^{(n)}$, then

$$\int \frac{q_{k_n}(\xi)}{\omega_{2n}(\xi)} \xi^k d\mu(\xi) = 0, \quad k \in \{0, 1, \ldots, k_n - 1\}. \qquad (2)$$

- Note that orthogonality is non Hermitian.

Padé interpolants and orthogonal polynomials

- Let $f(z) = \int \frac{d\mu(\xi)}{z-\xi}$ where μ is a complex measure supported on E compact. Here $\Omega = \overline{\mathbb{C}} \setminus E$.
- If p_{n-1}/q_n interpolates f in $\{\xi_1^{(n)}, \cdots, \xi_{2n}^{(n)}, \infty\} \subset \Omega$ and if ω_{2n} is the (normalized) polynomial having zeros the $\xi_j^{(n)}$, then

$$\int \frac{q_{k_n}(\xi)}{\omega_{2n}(\xi)} \xi^k d\mu(\xi) = 0, \quad k \in \{0, 1, \ldots, k_n - 1\}. \qquad (2)$$

- Note that orthogonality is non Hermitian.
- To assess the asymptotic behavior of q_n, it was realized early that E should have special properties in connection with the asymptotic density of interpolation points, *i.e.* the weak* limit ν of the normalized counting measures of the $\xi_j^{(n)}$:

$$\frac{1}{2n} \sum_{\ell=1}^{2n} \delta_{\xi_\ell^{(n)}} \xrightarrow{w*} \nu.$$

Symmetric contours

Symmetric contours

- A weighted S-contour in the field ψ is a compact set $\mathcal{K}$ which is an analytic arc in the neighborhood of q.e. point, and such that at every such point

$$\partial\left(V^{\omega_{\mathcal{K},\psi}}+\psi\right)/\partial n^{+}=\partial\left(V^{\omega_{\mathcal{K},\psi}}+\psi\right)/\partial n^{-}$$

where $\partial^{\pm} n$ indicates normal derivatives from each side.

Symmetric contours

- A weighted S-contour in the field ψ is a compact set $\mathcal{K}$ which is an analytic arc in the neighborhood of q.e. point, and such that at every such point

$$\partial \left(V^{\omega_{\mathcal{K},\psi}} + \psi \right) / \partial n^{+} = \partial \left(V^{\omega_{\mathcal{K},\psi}} + \psi \right) / \partial n^{-}$$

  where $\partial^{\pm} n$ indicates normal derivatives from each side.
- The notion was introduced in nuce by [Nutall, 70's] and expounded by [Stahl, 1985] in the unweighted case, which is suitable to study classical Padé aproximants (*i.e.* high order inetrpolation at a single point).

The Gonchar-Rakhmanov theorem

The Gonchar-Rakhmanov theorem

Theorem [Gonchar-Rachmanov,87] If f is (essentially) a Cauchy integral on a weighted symmetric contour $\mathcal{K}_{f,\nu}$ in the field $-U^\nu$, with q.e. nonzero density on the arcs thereof, and if for each n the interpolation points $\xi_1^{(n)}, \cdots, \xi_{2n}^{(n)}$ are picked with asymptotic density ν:

$$\frac{1}{2n}\sum_{\ell=1}^{2n}\delta_{\xi_\ell^{(n)}} \overset{w*}{\longrightarrow} \nu.$$

The Gonchar-Rakhmanov theorem

Theorem [Gonchar-Rachmanov,87] If f is (essentially) a Cauchy integral on a weighted symmetric contour $\mathcal{K}_{f,\nu}$ in the field $-U^{\nu}$, with q.e. nonzero density on the arcs thereof, and if for each n the interpolation points $\xi_1^{(n)}, \cdots, \xi_{2n}^{(n)}$ are picked with asymptotic density ν:

$$\frac{1}{2n}\sum_{\ell=1}^{2n}\delta_{\xi_\ell^{(n)}} \overset{w*}{\longrightarrow} \nu.$$

then the Padé interpolants p_{n-1}/q_n in the points $\xi_\ell^{(n)}$ converge in capacity to f in the complement of $\mathcal{K}_{f,\nu}$:

$$\lim_{n\to\infty} \mathrm{cap}\{z \notin \mathcal{K}_{f,\nu} : \left||(f(z) - p_{n-1}(z)/q_n(z))| - e^{-2V_G^{\omega_{K,E^c}}}\right|^{1/n} > \varepsilon\} = 0.$$

The Gonchar-Rakhmanov theorem

Theorem [Gonchar-Rachmanov,87] If f is (essentially) a Cauchy integral on a weighted symmetric contour $\mathcal{K}_{f,\nu}$ in the field $-U^{\nu}$, with q.e. nonzero density on the arcs thereof, and if for each n the interpolation points $\xi_1^{(n)}, \cdots, \xi_{2n}^{(n)}$ are picked with asymptotic density ν:

$$\frac{1}{2n}\sum_{\ell=1}^{2n}\delta_{\xi_\ell^{(n)}} \stackrel{w*}{\longrightarrow} \nu.$$

then the Padé interpolants p_{n-1}/q_n in the points $\xi_\ell^{(n)}$ converge in capacity to f in the complement of $\mathcal{K}_{f,\nu}$:

$$\lim_{n\to\infty} \mathrm{cap}\{z \notin \mathcal{K}_{f,\nu} : \left| |(f(z) - p_{n-1}(z)/q_n(z))| - e^{-2V_G^{\omega_{K,E^c}}} \right|^{1/n} > \varepsilon\} = 0.$$

and the normalized counting measure of their poles converges towards $\omega_{\mathcal{K},-U^{\nu}}$.

Remarks

Remarks

- To substantiate the former's conjecture Gonchar and Rakhmanov used this theorem picking ν the equilibrium distribution on K of the plane condenser (K, E), and showing that the existence of $r_n \in \mathcal{R}_n$ converging in capacity to f as indicated implies existence of $R_n \in \mathcal{R}_n$ converging uniformly with the correct n-th root rate. This they could do if E consists of finitely many arcs.

Remarks

- To substantiate the former's conjecture Gonchar and Rakhmanov used this theorem picking ν the equilibrium distribution on K of the plane condenser (K, E), and showing that the existence of $r_n \in \mathcal{R}_n$ converging in capacity to f as indicated implies existence of $R_n \in \mathcal{R}_n$ converging uniformly with the correct n-th root rate. This they could do if E consists of finitely many arcs.
- It is of course required that a weighted symmetric contour E exists at all. For functions with polar singular set contained in K^c, an open set Ω exists to minimize $C(K, \Omega)$ with f analytic on Ω. Then $E = \Omega^c$ works [Stahl 1989]. Moreover, E has finitely many arcs if f has finitely many branch points.

Remarks

- To substantiate the former's conjecture Gonchar and Rakhmanov used this theorem picking ν the equilibrium distribution on K of the plane condenser (K, E), and showing that the existence of $r_n \in \mathcal{R}_n$ converging in capacity to f as indicated implies existence of $R_n \in \mathcal{R}_n$ converging uniformly with the correct n-th root rate. This they could do if E consists of finitely many arcs.
- It is of course required that a weighted symmetric contour E exists at all. For functions with polar singular set contained in K^c, an open set Ω exists to minimize $C(K, \Omega)$ with f analytic on Ω. Then $E = \Omega^c$ works [Stahl 1989]. Moreover, E has finitely many arcs if f has finitely many branch points. For general fields [Stahl-Yattselev-L.B., 2013][Buslaev-Suetin,2015].

Remarks

- To substantiate the former's conjecture Gonchar and Rakhmanov used this theorem picking ν the equilibrium distribution on K of the plane condenser (K, E), and showing that the existence of $r_n \in \mathcal{R}_n$ converging in capacity to f as indicated implies existence of $R_n \in \mathcal{R}_n$ converging uniformly with the correct n-th root rate. This they could do if E consists of finitely many arcs.
- It is of course required that a weighted symmetric contour E exists at all. For functions with polar singular set contained in K^c, an open set Ω exists to minimize $C(K, \Omega)$ with f analytic on Ω. Then $E = \Omega^c$ works [Stahl 1989]. Moreover, E has finitely many arcs if f has finitely many branch points. For general fields [Stahl-Yattselev-L.B., 2013][Buslaev-Suetin,2015].
- Altogether functions with finitely many branchpoints support Gonchar's conjecture in a strong sense (true limit).

The Parfenov-Prokhorov theorem

The Parfenov-Prokhorov theorem

- When the complement of K is connected, O.G. Parfenov proved Gonchar's conjecture in1986.

The Parfenov-Prokhorov theorem

- When the complement of K is connected, O.G. Parfenov proved Gonchar's conjecture in1986.
- In 1994 the result was extended to the finitely connected case by V. A. Prokhorov.

The Parfenov-Prokhorov theorem

- When the complement of K is connected, O.G. Parfenov proved Gonchar's conjecture in1986.
- In 1994 the result was extended to the finitely connected case by V. A. Prokhorov.
- Whereas Gonchar did approach his conjecture trying to construct approximants (interpolants), Parfenov's proof is non-constructive and relies on the Adamjan-Arov-Krein theory of best meromorphic approximation, along with the observation that n-th root asymptotics in rational and meromorphic approximation are equivalent.

The Parfenov-Prokhorov theorem

- When the complement of K is connected, O.G. Parfenov proved Gonchar's conjecture in1986.
- In 1994 the result was extended to the finitely connected case by V. A. Prokhorov.
- Whereas Gonchar did approach his conjecture trying to construct approximants (interpolants), Parfenov's proof is non-constructive and relies on the Adamjan-Arov-Krein theory of best meromorphic approximation, along with the observation that n-th root asymptotics in rational and meromorphic approximation are equivalent.

Meromorphic approximation

Meromorphic approximation

- We approximate f on ∂K by the sum of a rational function and (the trace of) a function in $H^\infty(K^c)$:

$$em_n := \|f - g_n - r_n\|_{L^\infty(\partial K)} = \inf_{g \in H^\infty(K^c),\ r_n \in \mathcal{R}_n} \|f - g - r_n\|_{L^\infty(\partial K)}.$$

Meromorphic approximation

- We approximate f on ∂K by the sum of a rational function and (the trace of) a function in $H^\infty(K^c)$:

$$em_n := \|f - g_n - r_n\|_{L^\infty(\partial K)} = \inf_{g \in H^\infty(K^c),\ r_n \in \mathcal{R}_n} \|f - g - r_n\|_{L^\infty(\partial K)}.$$

- In other words, we approximate f on ∂K by the trace of a meromorphic function with at most n poles in K^c. This makes conformal invariance obvious (if K regular).

Meromorphic approximation

- We approximate f on ∂K by the sum of a rational function and (the trace of) a function in $H^\infty(K^c)$:

$$em_n := \|f - g_n - r_n\|_{L^\infty(\partial K)} = \inf_{g \in H^\infty(K^c),\ r_n \in \mathcal{R}_n} \|f - g - r_n\|_{L^\infty(\partial K)}.$$

- In other words, we approximate f on ∂K by the trace of a meromorphic function with at most n poles in K^c. This makes conformal invariance obvious (if K regular).
- By the Cauchy formula

$$f(z) - r_n(z) = \frac{1}{2i\pi} \int_{\partial K} \frac{(f - r_n - g)(t)}{t - z}\, dt \quad \text{for } z \in \overset{\circ}{K},$$

which implies easily that

$$\limsup e_{n_k}^{1/n_k} = \limsup em_{n_k}^{1/n_k}, \quad \liminf e_{n_k}^{1/n_k} = \liminf em_{n_k}^{1/n_k}$$

along any subsequence n_k.

Parfenov's proof

Parfenov's proof

- By conformal mapping assume $K = \overline{\mathbb{C}} \setminus \mathbb{D}$ with $\mathbb{D}$ the unit disk, and $\Omega = \overline{\mathbb{C}} \setminus E$, with E compact lying interior to the unit circle $\mathbb{T}$.

Parfenov's proof

- By conformal mapping assume $K = \overline{\mathbb{C}} \setminus \mathbb{D}$ with $\mathbb{D}$ the unit disk, and $\Omega = \overline{\mathbb{C}} \setminus E$, with E compact lying interior to the unit circle $\mathbb{T}$.
- By outer regularity of capacity, one may further assume that ∂E is a smooth Jordan curve Γ.

Parfenov's proof

- By conformal mapping assume $K = \overline{\mathbb{C}} \setminus \mathbb{D}$ with $\mathbb{D}$ the unit disk, and $\Omega = \overline{\mathbb{C}} \setminus E$, with E compact lying interior to the unit circle $\mathbb{T}$.
- By outer regularity of capacity, one may further assume that ∂E is a smooth Jordan curve Γ.
- AAK theory tells that the best error in uniform approximation to f on $\mathbb{T}$ by meromorphic functions with n poles is the $n+1$ singular value of the Hankel operator:

$$\begin{array}{rcl} A_f : H^2(\mathbb{D}) & \rightarrow & H_0^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}) \\ u & \mapsto & \mathbf{P}_-(fu) \end{array}$$

Parfenov's proof

- By conformal mapping assume $K = \overline{\mathbb{C}} \setminus \mathbb{D}$ with $\mathbb{D}$ the unit disk, and $\Omega = \overline{\mathbb{C}} \setminus E$, with E compact lying interior to the unit circle $\mathbb{T}$.
- By outer regularity of capacity, one may further assume that ∂E is a smooth Jordan curve Γ.
- AAK theory tells that the best error in uniform approximation to f on $\mathbb{T}$ by meromorphic functions with n poles is the $n+1$ singular value of the Hankel operator:

$$\begin{array}{rcl} A_f : H^2(\mathbb{D}) & \to & H_0^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}) \\ u & \mapsto & \mathbf{P}_-(fu) \end{array}$$

where $\mathbf{P}_-$ is the projection $L^2(\mathbb{T}) \to H_0^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ in the orthogonal decomposition:

$$L^2(\mathbb{T}) = H^2(\mathbb{D}) \oplus H_0^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}).$$

Parfenov's proof cont'd

Parfenov's proof cont'd

- By Cauchy formula

$$f(z) = \frac{1}{2i\pi} \int_{\Gamma} \frac{f(\xi)}{z - \xi} \, d\xi, \qquad z \in \Omega.$$

Parfenov's proof cont'd

- By Cauchy formula

$$f(z) = \frac{1}{2i\pi} \int_\Gamma \frac{f(\xi)}{z - \xi}\, d\xi, \qquad z \in \Omega.$$

- Moreover by the residue theorem

$$\mathbf{P}_-(h)(z) = \frac{1}{2i\pi} \int_\mathbb{T} \frac{h(\xi)}{z - \xi}\, d\xi, \quad h \in L^2(\mathbb{T}), \qquad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

Parfenov's proof cont'd

- By Cauchy formula

$$f(z) = \frac{1}{2i\pi} \int_\Gamma \frac{f(\xi)}{z-\xi}\, d\xi, \qquad z \in \Omega.$$

- Moreover by the residue theorem

$$\mathbf{P}_-(h)(z) = \frac{1}{2i\pi} \int_\mathbb{T} \frac{h(\xi)}{z-\xi}\, d\xi, \quad h \in L^2(\mathbb{T}), \qquad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

- By the above, Fubini's theorem, and the residue formula, we get for $v \in H^2(\mathbb{D})$:

$$\begin{aligned} A_f(v)(z) &= \frac{1}{2i\pi} \int_\Gamma \left(\frac{1}{2i\pi} \int_\mathbb{T} \frac{v(\zeta)}{(z-\zeta)(\zeta-\xi)}\, d\zeta \right) f(\xi)\, d\xi \\ &= \frac{1}{2i\pi} \int_\Gamma \frac{v(\xi) f(\xi)}{(z-\xi)}\, d\zeta, \qquad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}. \end{aligned}$$

Parfenov's proof cont'd

Parfenov's proof cont'd

Therefore A_f is the composition of four elementary operators:

$$A_f = B_1\, B_2\, B_3\, B_4,$$

Parfenov's proof cont'd

Therefore A_f is the composition of four elementary operators:

$$A_f = B_1\, B_2\, B_3\, B_4,$$

- $B_4 : H^2(\mathbb{D}) \to L^2(\Gamma)$ is the embedding operator obtained by restricting functions to Γ,

Parfenov's proof cont'd

Therefore A_f is the composition of four elementary operators:

$$A_f = B_1\, B_2\, B_3\, B_4,$$

- $B_4 : H^2(\mathbb{D}) \to L^2(\Gamma)$ is the embedding operator obtained by restricting functions to Γ,
- $B_3 : L^2(\Gamma) \to L^2(\Gamma)$ is the multiplication by f,

Parfenov's proof cont'd

Therefore A_f is the composition of four elementary operators:

$$A_f = B_1\, B_2\, B_3\, B_4,$$

- $B_4 : H^2(\mathbb{D}) \to L^2(\Gamma)$ is the embedding operator obtained by restricting functions to Γ,
- $B_3 : L^2(\Gamma) \to L^2(\Gamma)$ is the multiplication by f,
- $B_2 : L^2(\Gamma) \to \mathcal{S}^2(\Omega)$ is the Cauchy projection onto the Smirnov class of Ω,

Parfenov's proof cont'd

Therefore A_f is the composition of four elementary operators:

$$A_f = B_1\, B_2\, B_3\, B_4,$$

- $B_4 : H^2(\mathbb{D}) \to L^2(\Gamma)$ is the embedding operator obtained by restricting functions to Γ,
- $B_3 : L^2(\Gamma) \to L^2(\Gamma)$ is the multiplication by f,
- $B_2 : L^2(\Gamma) \to \mathcal{S}^2(\Omega)$ is the Cauchy projection onto the Smirnov class of Ω,
- $B_1 : \mathcal{S}^2(\Omega) \to H^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ is the embedding operator arising by restriction.

Parfenov's proof cont'd

Therefore A_f is the composition of four elementary operators:

$$A_f = B_1\, B_2\, B_3\, B_4,$$

- $B_4 : H^2(\mathbb{D}) \to L^2(\Gamma)$ is the embedding operator obtained by restricting functions to Γ,
- $B_3 : L^2(\Gamma) \to L^2(\Gamma)$ is the multiplication by f,
- $B_2 : L^2(\Gamma) \to \mathcal{S}^2(\Omega)$ is the Cauchy projection onto the Smirnov class of Ω,
- $B_1 : \mathcal{S}^2(\Omega) \to H^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ is the embedding operator arising by restriction.
- B_3, B_2 are bounded, and for the singular values of B_1, B_4 we have [Zakharyuta-Skiba, 1976]

$$\lim_{k\to\infty} s_k^{1/k}(B_1) = \lim_{k\to\infty} s_k^{1/k}(B_4) = \exp\left(-\frac{1}{C(\overline{\mathbb{C}} \setminus \mathbb{D}, \Gamma)}\right).$$

Parfenov's proof cont'd

Therefore A_f is the composition of four elementary operators:

$$A_f = B_1\, B_2\, B_3\, B_4,$$

- $B_4 : H^2(\mathbb{D}) \to L^2(\Gamma)$ is the embedding operator obtained by restricting functions to Γ,
- $B_3 : L^2(\Gamma) \to L^2(\Gamma)$ is the multiplication by f,
- $B_2 : L^2(\Gamma) \to \mathcal{S}^2(\Omega)$ is the Cauchy projection onto the Smirnov class of Ω,
- $B_1 : \mathcal{S}^2(\Omega) \to H^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ is the embedding operator arising by restriction.
- B_3, B_2 are bounded, and for the singular values of B_1, B_4 we have [Zakharyuta-Skiba, 1976]

$$\lim_{k\to\infty} s_k^{1/k}(B_1) = \lim_{k\to\infty} s_k^{1/k}(B_4) = \exp\left(-\frac{1}{C(\overline{\mathbb{C}} \setminus \mathbb{D}, \Gamma)}\right).$$

- These estimates also follow from n-widths estimates by [Fischer-Micchelli, 1980].

Parfenov's proof cont'd

Parfenov's proof cont'd

- Applying now the Horn-Weyl inequalities:

$$\Pi_{k=0}^{n} s_k(AB) \leq \Pi_{k=0}^{n} s_k(A)\, \Pi_{k=0}^{n} s_k(B), \qquad n \in \mathbb{N}$$

valid for any pair of bounded operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $B : \mathcal{H}_2 \to \mathcal{H}_3$ between Hilbert spaces,

Parfenov's proof cont'd

- Applying now the Horn-Weyl inequalities:

$$\Pi_{k=0}^{n} s_k(AB) \leq \Pi_{k=0}^{n} s_k(A)\, \Pi_{k=0}^{n} s_k(B), \qquad n \in \mathbb{N}$$

  valid for any pair of bounded operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $B : \mathcal{H}_2 \to \mathcal{H}_3$ between Hilbert spaces,
- we obtain

$$\Pi_{k=0}^{n} s_k(A_f) \leq |||B_2|||^{n+1} |||B_3|||^{n+1} \Pi_{k=0}^{n} s_k(B_1))\, \Pi_{k=0}^{n} s_k(B_4),$$

Parfenov's proof cont'd

- Applying now the Horn-Weyl inequalities:

$$\Pi_{k=0}^{n} s_k(AB) \leq \Pi_{k=0}^{n} s_k(A)\, \Pi_{k=0}^{n} s_k(B), \qquad n \in \mathbb{N}$$

  valid for any pair of bounded operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $B : \mathcal{H}_2 \to \mathcal{H}_3$ between Hilbert spaces,
- we obtain

$$\Pi_{k=0}^{n} s_k(A_f) \leq |||B_2|||^{n+1} |||B_3|||^{n+1} \Pi_{k=0}^{n} s_k(B_1))\, \Pi_{k=0}^{n} s_k(B_4),$$

- from which Parfenov's theorem follows easily upon taking $1/n^2$-roots.

Parfenov's proof cont'd

- Applying now the Horn-Weyl inequalities:

$$\Pi_{k=0}^{n} s_k(AB) \leq \Pi_{k=0}^{n} s_k(A)\,\Pi_{k=0}^{n} s_k(B), \qquad n \in \mathbb{N}$$

  valid for any pair of bounded operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $B : \mathcal{H}_2 \to \mathcal{H}_3$ between Hilbert spaces,
- we obtain

$$\Pi_{k=0}^{n} s_k(A_f) \leq |||B_2|||^{n+1}|||B_3|||^{n+1}\Pi_{k=0}^{n} s_k(B_1))\,\Pi_{k=0}^{n} s_k(B_4),$$

- from which Parfenov's theorem follows easily upon taking $1/n^2$-roots.
- In short: quadratic estimates from spectral theory and AAK solve the problem.

Extremal domains of analyticity

Extremal domains of analyticity

- The Parfenov-Prokhorov theorem draws attention to the largest domain of analyticity for f, say Ω, containing a given compact set K, where "largest" means that $C(K, \Omega)$ is minimal.

Extremal domains of analyticity

- The Parfenov-Prokhorov theorem draws attention to the largest domain of analyticity for f, say Ω, containing a given compact set K, where "largest" means that $C(K, \Omega)$ is minimal. This is defined up to the complement of a closed polar set only, but we can make it unique by taking the union of all such domains.

Extremal domains of analyticity

- The Parfenov-Prokhorov theorem draws attention to the largest domain of analyticity for f, say Ω, containing a given compact set K, where "largest" means that $C(K, \Omega)$ is minimal. This is defined up to the complement of a closed polar set only, but we can make it unique by taking the union of all such domains.
- Existence of such an extremal domain was proved by H. Stahl in 1989. When the singular set of f is polar, it consists of countably many analytic arcs with branching plus a polar set.

Rational approximation to functions with polar singular set

Rational approximation to functions with polar singular set

Theorem (H. Stahl[†], M.Yattselev, L.B., 2015)

Let f be analytic in $\Omega \subset \mathbb{C}$ and continuable indefinitely except over a polar set.

Rational approximation to functions with polar singular set

Theorem (H. Stahl†, M.Yattselev, L.B., 2015)

Let f be analytic in $\Omega \subset \mathbb{C}$ and continuable indefinitely except over a polar set. Let $K \subset \Omega$ be compact with K^c connected.

Rational approximation to functions with polar singular set

Theorem (H. Stahl†, M.Yattselev, L.B., 2015)

Let f *be analytic in* $\Omega \subset \mathbb{C}$ *and continuable indefinitely except over a polar set. Let* $K \subset \Omega$ *be compact with* K^c *connected. Let further* Ω^* *maximize the Green capacity* $C(K, \Omega^*)$ *under the condition that* f *is analytic and single-valued in* Ω^*.

Rational approximation to functions with polar singular set

Theorem (H. Stahl[†], M.Yattselev, L.B., 2015)

Let f *be analytic in* $\Omega \subset \mathbb{C}$ *and continuable indefinitely except over a polar set. Let* $K \subset \Omega$ *be compact with* K^c *connected. Let further* Ω^* *maximize the Green capacity* $C(K, \Omega^*)$ *under the condition that* f *is analytic and single-valued in* Ω^**. Then*

Rational approximation to functions with polar singular set

Theorem (H. Stahl†, M.Yattselev, L.B., 2015)

Let f be analytic in $\Omega \subset \mathbb{C}$ and continuable indefinitely except over a polar set. Let $K \subset \Omega$ be compact with K^c connected. Let further Ω^* maximize the Green capacity $C(K, \Omega^*)$ under the condition that f is analytic and single-valued in Ω^*. Then

- $\lim_{n\to\infty} \; e_n^{1/n} = \exp\left\{\frac{-2}{C(K,\Omega^*)}\right\}$

Rational approximation to functions with polar singular set

Theorem (H. Stahl[†], M.Yattselev, L.B., 2015)

Let f be analytic in $\Omega \subset \mathbb{C}$ and continuable indefinitely except over a polar set. Let $K \subset \Omega$ be compact with K^c connected. Let further Ω^* maximize the Green capacity $C(K, \Omega^*)$ under the condition that f is analytic and single-valued in Ω^*. Then

- $\lim_{n\to\infty} \; e_n^{1/n} = \exp\left\{\frac{-2}{C(K,\Omega^*)}\right\}$
- *If there is a branchpoint and K is regular, then the asymptotic density of the poles $\xi_1^{(n)}, \cdots, \xi_n^{(n)}$ of an asymptotically optimal sequence r_n of rational approximants of degree n is ω^G_{K,Ω^*}:*

$$\frac{1}{n}\Sigma_{\ell=1}^{n}\delta_{\xi_\ell^{(n)}} \;\; {}^{w*}\!\longrightarrow \omega^G_{K,\Omega^*}.$$

Rational approximation to functions with polar singular set

Theorem (H. Stahl†, M.Yattselev, L.B., 2015)

Let f be analytic in $\Omega \subset \mathbb{C}$ and continuable indefinitely except over a polar set. Let $K \subset \Omega$ be compact with K^c connected. Let further Ω^* maximize the Green capacity $C(K, \Omega^*)$ under the condition that f is analytic and single-valued in Ω^*. Then

- $\lim_{n\to\infty} \; e_n^{1/n} = \exp\left\{\frac{-2}{C(K,\Omega^*)}\right\}$
- *If there is a branchpoint and K is regular, then the asymptotic density of the poles $\xi_1^{(n)}, \cdots, \xi_n^{(n)}$ of an asymptotically optimal sequence r_n of rational approximants of degree n is ω^G_{K,Ω^*}:*

$$\frac{1}{n}\Sigma_{\ell=1}^{n}\delta_{\xi_\ell^{(n)}} \; {}^{w*}\longrightarrow \omega^G_{K,\Omega^*}.$$

- *If there is no branchpoint convergence is faster than gometric, but asymptotic distribution of poles is unknown.*

About the proof

About the proof

- Assume $C(K, \Omega) > 0$. We know that

$$\liminf_{n \to \infty} e_n^{1/n} \leq \exp\left\{\frac{-2}{C(K\Omega)}\right\}.$$

About the proof

- Assume $C(K,\Omega) > 0$. We know that

$$\liminf_{n\to\infty} e_n^{1/n} \le \exp\left\{\frac{-2}{C(K\Omega)}\right\}.$$

$$\limsup_{n\to\infty} e_n^{1/n} \le \exp\left\{\frac{-1}{C(K,\Omega)}\right\}.$$

About the proof

- Assume $C(K,\Omega) > 0$. We know that

$$\liminf_{n\to\infty} e_n^{1/n} \leq \exp\left\{\frac{-2}{C(K\Omega)}\right\}.$$

$$\limsup_{n\to\infty} e_n^{1/n} \leq \exp\left\{\frac{-1}{C(K,\Omega)}\right\}.$$

- Dwelling on Horn-Weyl inequalities we prove

$$\limsup_{n\to\infty} e_n^{1/n} > \exp\left\{\frac{-2}{C(K,\Omega)}\right\} \Longrightarrow \liminf_{n\to\infty} e_n^{1/n} < \exp\left\{\frac{-2}{C(K,\Omega)}\right\}.$$

About the proof cont'd

About the proof cont'd

- In a second step, one shows that along any subsequence $\liminf e_n^{1/n} \geq \exp\left\{\frac{-2}{C(K,\Omega^*)}\right\}$ and that this speed of convergence is attained only if the asymptotic density of the poles is $\omega^G_{(K,\Omega^*)}$

About the proof cont'd

- In a second step, one shows that along any subsequence $\liminf e_n^{1/n} \geq \exp\left\{\frac{-2}{C(K,\Omega^*)}\right\}$ and that this speed of convergence is attained only if the asymptotic density of the poles is $\omega^G_{(K,\Omega^*)}$
- This is done by analyzing the limit L, along a subsequence, of $(\log e_n)/n$ on the Riemann surface of f. We divide it in three subsets E^+, E^-, E_0 where the limit is positive, negative or 0. The surface lies schlicht over G^- and saturated over G^+. Balayaging the mass of L (a δ-subharmonic function) out of G^+, G^-, we find thanks to schlichtness and Bagemihl-type arguments that the mass on G_0 is at most 2.

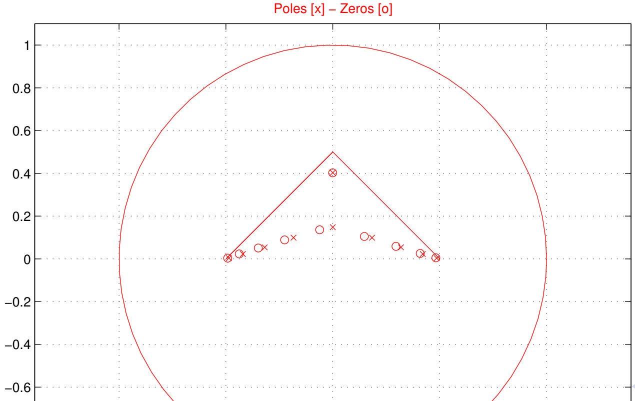
About the proof cont'd

- In a second step, one shows that along any subsequence $\liminf e_n^{1/n} \geq \exp\left\{\frac{-2}{C(K,\Omega^*)}\right\}$ and that this speed of convergence is attained only if the asymptotic density of the poles is $\omega^G_{(K,\Omega^*)}$
- This is done by analyzing the limit L, along a subsequence, of $(\log e_n)/n$ on the Riemann surface of f. We divide it in three subsets E^+, E^-, E_0 where the limit is positive, negative or 0. The surface lies schlicht over G^- and saturated over G^+. Balayaging the mass of L (a δ-subharmonic function) out of G^+, G^-, we find thanks to schlichtness and Bagemihl-type arguments that the mass on G_0 is at most 2.
- One dificulty is that L is only finely continuous, which leads us to work with fine tology, fine balayage, and fine Dirichlet problems.

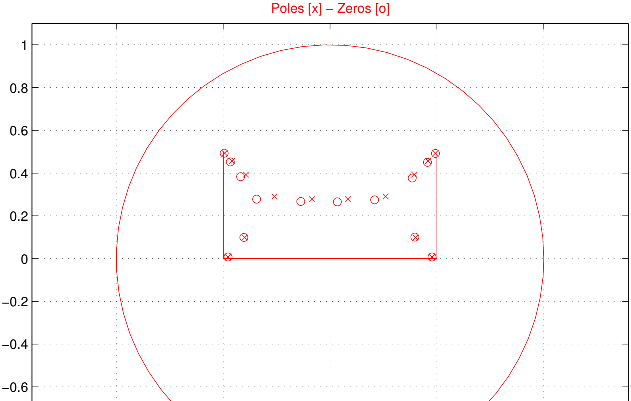
About the proof cont'd

- In a second step, one shows that along any subsequence $\liminf e_n^{1/n} \geq \exp\left\{\frac{-2}{C(K,\Omega^*)}\right\}$ and that this speed of convergence is attained only if the asymptotic density of the poles is $\omega^G_{(K,\Omega^*)}$
- This is done by analyzing the limit L, along a subsequence, of $(\log e_n)/n$ on the Riemann surface of f. We divide it in three subsets E^+, E^-, E_0 where the limit is positive, negative or 0. The surface lies schlicht over G^- and saturated over G^+. Balayaging the mass of L (a δ-subharmonic function) out of G^+, G^-, we find thanks to schlichtness and Bagemihl-type arguments that the mass on G_0 is at most 2.
- One dificulty is that L is only finely continuous, which leads us to work with fine tology, fine balayage, and fine Dirichlet problems.
- One has to connect poles in rational approximation with poles in meromorphic approximation.

Some experiments



Some experiments



A conjecture

A conjecture

- Using the identification $\mathbb{R}^2 \sim \mathbb{C}$, analytic functions may be viewed as (conjugates of) gradients of harmonic functions.

A conjecture

- Using the identification $\mathbb{R}^2 \sim \mathbb{C}$, analytic functions may be viewed as (conjugates of) gradients of harmonic functions.
- This way rational functions become gradients of discrete logarithmic potentials.

A conjecture

- Using the identification $\mathbb{R}^2 \sim \mathbb{C}$, analytic functions may be viewed as (conjugates of) gradients of harmonic functions.
- This way rational functions become gradients of discrete logarithmic potentials.
- This makes sense in higer dimension (Newtonian potentials).

A conjecture

- Using the identification $\mathbb{R}^2 \sim \mathbb{C}$, analytic functions may be viewed as (conjugates of) gradients of harmonic functions.
- This way rational functions become gradients of discrete logarithmic potentials.
- This makes sense in higer dimension (Newtonian potentials).
- **Is it true that:**
  if a potential whose mass lies inside a domain Ω gets optimally approximated in a Sobolev sense on $\partial\Omega$ by a discrete potential, then the discrete masses of best approximation asymptotically distribute, in the sense of limit points of normalized counting measure, on the set of minimal Green capacity outside of which the initial gradient is single valued?

A conjecture

- Using the identification $\mathbb{R}^2 \sim \mathbb{C}$, analytic functions may be viewed as (conjugates of) gradients of harmonic functions.
- This way rational functions become gradients of discrete logarithmic potentials.
- This makes sense in higer dimension (Newtonian potentials).
- **Is it true that:**
  if a potential whose mass lies inside a domain Ω gets optimally approximated in a Sobolev sense on $\partial\Omega$ by a discrete potential, then the discrete masses of best approximation asymptotically distribute, in the sense of limit points of normalized counting measure, on the set of minimal Green capacity outside of which the initial gradient is single valued?
- And if the initial field can be continued except over a set of capacity zero, is it true that these counting measures converge weak-* to the Green equilibrium distriution of the minimal set?

A conjecture

- Using the identification $\mathbb{R}^2 \sim \mathbb{C}$, analytic functions may be viewed as (conjugates of) gradients of harmonic functions.
- This way rational functions become gradients of discrete logarithmic potentials.
- This makes sense in higer dimension (Newtonian potentials).
- **Is it true that:**
  if a potential whose mass lies inside a domain Ω gets optimally approximated in a Sobolev sense on $\partial\Omega$ by a discrete potential, then the discrete masses of best approximation asymptotically distribute, in the sense of limit points of normalized counting measure, on the set of minimal Green capacity outside of which the initial gradient is single valued?
- And if the initial field can be continued except over a set of capacity zero, is it true that these counting measures converge weak-* to the Green equilibrium distriution of the minimal set?
- We proved the conjecture in dimension 2 when the singular set is polar.

A sad note

A sad note

In memoriam Herbert Stahl, August 3, 1942–April 22, 2013.

And most importantly

Thank you!