# A short proof of the Gap Theorem for separated sequences

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Joint work with Yurii Belov (Saint Petersburg)
and Alexander Ulanovskii (Stavanger)

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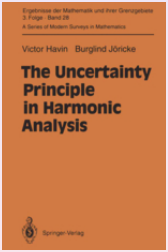
Special session dedicated to the memory of Victor Havin


Victor Petrovich Havin
07.03.1933–21.09.2015

# Uncertainty Principle in Harmonic Analysis

A nonzero function and its Fourier transform can not be too "small" simultaneously.

The word "small" can be understood in many different ways: smallness of support, fast decay at some point, etc.

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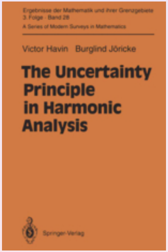


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# Two problems for functions with bounded spectra

Let $a > 0$ and let $f \in L^2(\mathbb{R})$, $\operatorname{supp} f \subset [-a, a]$.

Problem 1. For which functions $w \geq 0$, the estimate $|\widehat{f}| \leq w$ implies that $f = 0$ a.e.?

If there exists a nonzero function $f$ with $|\widehat{f}| \leq w$, we say that $w$ is an admissible majorant.

Problem 2. For which discrete sets $\Lambda \subset \mathbb{R}$ the condition $\widehat{f}|_\Lambda = 0$ implies that $f = 0$ a.e.?

This condition is equivalent to the completeness of the system of exponentials $\mathcal{E}_\Lambda := \{e^{i\lambda t}\}_{\lambda\in\Lambda}$ in $L^2(-a, a)$ or of reproducing kernels (cardinal sine functions) $\{k_\lambda\}_{\lambda\in\Lambda}$ in $PW_a = \widehat{L^2(-a, a)}$.

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# Beurling–Malliavin Theory

A necessary condition for $w$ to be an admissible majorant is the convergence of the logarithmic integral

$$\int_{\mathbb{R}} \frac{\log w(x)}{1+x^2}\,dx > -\infty.$$

This is a criterion for being an admissible majorant in the Hardy space (functions with semi-bounded spectra).

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If the logarithmic integral converges and the function $\Omega = -\log w$ is Lipschitz on $\mathbb{R}$, then $w$ is an admissible majorant for $PW_\sigma$ for any $\sigma > 0$.

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$$R(\Lambda) = \pi D^{BM}(\Lambda).$$

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This definition belongs to J.-P. Kahane. The original definition given by Beurling and Malliavin used the notion of short system of intervals.

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# Havin's approach to BM1

All previous proofs (Beurling–Malliavin, Koosis, de Branges) used essentially complex analysis. Havin (joint with J. Mashreghi and F. Nazarov) found a real proof of BM1. Another advantage of this approach is that it applies to more general spaces of analytic functions (model spaces $K_\Theta$, de Branges spaces).


- V.P. Havin, J. Mashreghi, Admissible majorants for model subspaces of $H^2$. Part I: slow winding of the generating inner function, Can. J. Math. (2003).
- V.P. Havin, J. Mashreghi, Admissible majorants for model subspaces of $H^2$. Part II: fast winding of the generating inner function, Can. J. Math. (2003).
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1. Parametrization of admissible majorants.

Let $w \geq 0$ and with $\Omega = -\log w \in L^1(\Pi)$, $d\Pi(t) = \frac{dt}{t^2+1}$.

$$\widetilde{\Omega}(x) = v.p.\,\frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{x-t} + \frac{t}{t^2+1}\right)\Omega(t)dt.$$

Then $w$ is an admissible majorant if and only if there exists a bounded function $m \geq 0$ with $mw \in L^2(\mathbb{R})$ and $\log m \in L^1(\Pi)$ such that

$$at + \widetilde{\Omega}(t) = \widetilde{\log m}(t) + \pi k \qquad \text{a.e. on } \mathbb{R},$$

where $k$ is a measurable function with integer values.

The same holds for spaces $K_\Theta$ with $at$ replaced by $\frac{1}{2}\arg\Theta(t)$.

This representation is based on an observation due to K. Dyakonov: $h = |f|$ for $f \in K_\Theta$ iff $h^2\Theta \in H^1$.

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2. Conditions in terms of $\widetilde{\Omega}$.

**Theorem**

If $\widetilde{\Omega}$ is Lipschitz, then $w$ is an admissible majorant for $PW_a$ for any $a > \|(\widetilde{\Omega})'\|_\infty$.

This result is a special case of a much more general sufficient admissibility condition applicable to general de Branges spaces (= spaces $K_\Theta$ with meromorphic $\Theta$).

But $\Omega \in Lip$ does not imply that $\widetilde{\Omega} \in Lip$.

3. Nazarov's correction theorem.

**Theorem**

If $\Omega \in Lip$, then for any $\varepsilon > 0$ there exists $\Omega_1 \geq \Omega$ such that $\Omega_1 \in L^1(\Pi)$, $\widetilde{\Omega}_1$ is Lipschitz and $\|(\widetilde{\Omega}_1)'\|_\infty < \varepsilon$.

Now $w_1 = \exp(-\Omega_1)$ is admissible and $w \geq w_1$.

The proof is based on an ingenious combinatorial construction.

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Combining the last two results one obtains a real variable and, probably, the shortest proof of the Multiplier Theorem.

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# Further developments

The papers by Havin and Mashreghi cited above contain also a number of results about admissible majorants in de Branges spaces of entire functions. In particular, it is shown that the theory bifurcates in two essentially different situations:

- Fast (linear or super-linear) growth of $\arg\Theta$ on $\mathbb{R}$
- Slow (sub-linear) growth of $\arg\Theta$ on $\mathbb{R}$

Some further developments:

- A. Baranov, V. Havin (2006), A. Baranov, A. Borichev, V. Havin (2007) – slow growth case (e.g., zeros of $\Theta$ of the form $z_n = n^\alpha + i$ or $z_n = \pm n^\alpha + i$, $n \in \mathbb{N}$).
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Let $X$ be a closed subset of $\mathbb{R}$. Denote by $M(X)$ the set of finite complex measures supported by $X$. The gap characteristic $G(X)$ is defined by

$$G(X) = \sup\{a : \exists \mu \in M(X) \text{ such that } \widehat{\mu}(x) = 0, x \in (-a, a)\}.$$

Solution:

- M. Mitkovski, A. Poltoratski (2010) – for the case when $X = \Lambda$ is a separated sequence, i.e.,

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**Theorem (Mitkovski, Poltoratski, 2010)**

Let $\Lambda$ be a separated sequence. Then $G(\Lambda) = \pi D_{BM}(\Lambda)$.

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# Short proof of the Gap Theorem

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**Proposition 1.**

Assume $\Lambda \subset \alpha\mathbb{Z}, \alpha > 0$. Then $D_{BM}(\Lambda) + D^{BM}(\alpha\mathbb{Z} \setminus \Lambda) = 1/\alpha$.

**Proposition 2.**

Assume $\Lambda \subset \alpha\mathbb{Z}, \alpha > 0$. Then $G(\Lambda) + R(\alpha\mathbb{Z} \setminus \Lambda) = \pi/\alpha$.

Proof for the case $\Lambda \subset \alpha\mathbb{Z}$:

$$G(\Lambda) = \pi/\alpha - R(\alpha\mathbb{Z} \setminus \Lambda) = \pi/\alpha - \pi D^{BM}(\alpha\mathbb{Z} \setminus \Lambda)$$

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# Short proof of the Gap Theorem

The general case – a perturbation argument.

Given a separated set $\Lambda$, consider its perturbations:

$$\tilde{\Lambda} = \{\lambda + \varepsilon_\lambda : \lambda \in \Lambda\}.$$

**Proposition 3.**

Assume $\Lambda$ is a separated set. For every positive number $\delta < d(\Lambda)/4$, and every number $\varepsilon_\lambda$ satisfying $\delta/2 < \varepsilon_\lambda < \delta$ the set $\tilde{\Lambda}$ satisfies

$$G(\tilde{\Lambda}) = G(\Lambda), \qquad D_{BM}(\tilde{\Lambda}) = D_{BM}(\Lambda).$$

Equality $D_{BM}(\tilde{\Lambda}) = D_{BM}(\Lambda)$ is obvious, since $n_\Lambda - n_{\tilde{\Lambda}} \in L^\infty$.
To prove that $G(\tilde{\Lambda}) = G(\Lambda)$, we use the fact that $\hat{\mu}$ vanishes on $[-a, a]$ iff the Cauchy transform of $\mu$ decays faster than $e^{-b|z|}$ along $i\mathbb{R}$ for any $b < a$.

End of the proof: Fix any separated set $\Lambda$ and $0 < \delta < d(\Lambda)/4$.
Clearly, there is a set $\tilde{\Lambda}$ such that $\tilde{\Lambda} \subset \alpha\mathbb{Z}$, for some $\alpha > 0$.

**Proposition 2.**

Assume $\Lambda \subset \alpha\mathbb{Z}, \alpha > 0$. Then $G(\Lambda) + R(\alpha\mathbb{Z} \setminus \Lambda) = \pi/\alpha$.

Let $\alpha = 1$, $\Lambda \subset \mathbb{Z}$, $\Gamma := \mathbb{Z} \setminus \Lambda$.
If $\mathcal{E}_\Gamma$ is not complete in $L^2(0, 2a)$, $0 < a < \pi$, then $\exists f \in L^2(\mathbb{R})$ which vanishes outside $(0, 2a)$ and $f \perp \mathcal{E}_\Gamma$. For $\epsilon > 0$ put $g = f * h$, where $h$ is a smooth function supported by $[0, \varepsilon]$. Then $g$ is smooth, vanishes outside $(0, 2a + \varepsilon)$ and is orthogonal to $\mathcal{E}_\Gamma$.

$$g(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx} = \sum_{n\in\Lambda} a_n e^{inx}, \qquad \{a_n\} \in \ell^1.$$

So, $\mu := \sum_{n\in\Lambda} a_n \delta_n \in M(\Lambda)$ and has a spectral gap of length at least $2\pi - 2a - \varepsilon$. So $R(\Gamma) + G(\Lambda) \geq \pi$.

Now, suppose that there exists a nontrivial measure $\mu$ such that $\hat{\mu} \equiv 0$ on $(0, 2a)$. Put $g(x) = \hat{\mu}\big|_{(2a,2\pi)}$. Then $g \in L^2(0, 2\pi)$ and $g \perp \mathcal{E}_\Gamma$. Hence, $R(\Gamma) \leq \pi - a$. So, $R(\Gamma) + G(\Lambda) \leq \pi$.

**Proposition 2.**

Assume $\Lambda \subset \alpha\mathbb{Z}, \alpha > 0$. Then $G(\Lambda) + R(\alpha\mathbb{Z} \setminus \Lambda) = \pi/\alpha$.

Let $\alpha = 1$, $\Lambda \subset \mathbb{Z}$, $\Gamma := \mathbb{Z} \setminus \Lambda$.
If $\mathcal{E}_\Gamma$ is not complete in $L^2(0, 2a)$, $0 < a < \pi$, then $\exists f \in L^2(\mathbb{R})$ which vanishes outside $(0, 2a)$ and $f \perp \mathcal{E}_\Gamma$. For $\epsilon > 0$ put $g = f * h$, where $h$ is a smooth function supported by $[0, \varepsilon]$. Then $g$ is smooth, vanishes outside $(0, 2a + \varepsilon)$ and is orthogonal to $\mathcal{E}_\Gamma$.

$$g(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx} = \sum_{n\in\Lambda} a_n e^{inx}, \qquad \{a_n\} \in \ell^1.$$

So, $\mu := \sum_{n\in\Lambda} a_n \delta_n \in M(\Lambda)$ and has a spectral gap of length at least $2\pi - 2a - \varepsilon$. So $R(\Gamma) + G(\Lambda) \geq \pi$.

Now, suppose that there exists a nontrivial measure $\mu$ such that $\hat{\mu} \equiv 0$ on $(0, 2a)$. Put $g(x) = \hat{\mu}\big|_{(2a,2\pi)}$. Then $g \in L^2(0, 2\pi)$ and $g \perp \mathcal{E}_\Gamma$. Hence, $R(\Gamma) \leq \pi - a$. So, $R(\Gamma) + G(\Lambda) \leq \pi$.