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Universality of the limiting spectral distribution for matrices with correlated entries

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joint work with F. Merlevède and M. Peligrad

Journées du Gdr Analyse Fonctionnelle, Harmonique et Probabilités

3 December 2015

- Let $\mathbf{X}_1, \ldots, \mathbf{X}_p \in \mathbb{R}^N$ be i.i.d centered random vectors with covariance matrix $\Sigma_N = \mathbb{E}(\mathbf{X}_1 \mathbf{X}_1^T) = \ldots = \mathbb{E}(\mathbf{X}_p \mathbf{X}_p^T)$.

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- For fixed N, the strong law of large numbers implies

$$\lim_{p \to +\infty} \mathbf{B}_{N,p} = \lim_{p \to +\infty} \frac{1}{p} \sum_{k=1}^{p} \mathbf{X}_k \mathbf{X}_k^T = \Sigma_N \quad \text{a.s.}$$

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- What happens once $N := N_p, p \to \infty$ s.t. $p \to \infty$?

- Let $\mathbf{B}_N := \mathbf{B}_{N,p} = \frac{1}{p}\mathcal{X}_{N,p}\mathcal{X}_{N,p}^T = \frac{1}{p}\sum_{k=1}^{p} \mathbf{X}_k\mathbf{X}_k^T$

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- The empirical spectral measure of $\mathbf{B}_N$ is defined by

$$\mu_{\mathbf{B}_N} = \frac{1}{N}\sum_{k=1}^{N} \delta_{\lambda_k}$$

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- We shall suppose that $c_N := \frac{N}{p} \xrightarrow[N\to+\infty]{} c \in (0, \infty)$.

Marčenko-Pastur Theorem

Theorem (Marčenko, Pastur '67)

Let $(X_{ij})_{i,j\geqslant 1}$ *be a family of i.i.d. random variables such that*

$$\mathbb{E}(X_{11}) = 0 \quad \textit{and} \quad \mathrm{Var}(X_{11}) = \sigma^2.$$

If $\lim_N N/p = c \in (0,\infty)$*, then*

$$\mu_{\mathbf{B}_N} \xrightarrow[N\to\infty]{w} \mu_{\mathrm{MP}} \quad \textit{a.s.}$$

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whose density is given by

$$\left(1 - \frac{1}{c}\right)_+ \delta_0 + \frac{1}{2\pi\, c\sigma^2 x}\sqrt{(b-x)(x-a)}\mathbf{1}_{[a,b]}(x)dx$$

with $.+ := \max(0,.)$, $a = \sigma^2(1-\sqrt{c})^2$ *and* $b = \sigma^2(1+\sqrt{c})^2$*.*

Marčenko-Pastur Theorem
Simulation

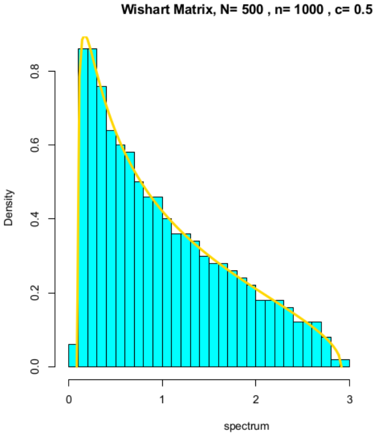

Figure : In blue, histogram of the eigenvalues. In yellow, the density of the Marčenko-Pastur distribution

Sample covariance matrix
Perturbations

- We consider

$$R_N^{1/2}\mathbf{B}_N R_N^{1/2} = \frac{1}{p} R_N^{1/2} \mathcal{X}_{N,p} \mathcal{X}_{N,p}^T R_N^{1/2}$$

where $R_N^{1/2}$ is a perturbation of the identity matrix:

$$R_N = \mathbf{I}_N + \theta \mathbf{u}\mathbf{u}^T, \quad \theta > 0$$

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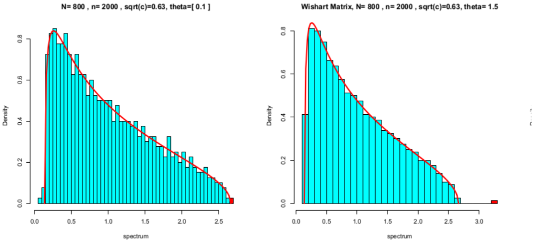

Figure : Histogram of the empirical eigenvalues and MP distribution

The Stieltjes transform $S_G : \mathbb{C}_+ \to \mathbb{C}$ of a measure ν on $\mathbb{R}$ is defined by

$$S_\nu(z) := \int \frac{1}{x - z} \, d\nu(x)$$

The Stieltjes Transform
Characterization of measures

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- The function S_ν is analytic over $\mathbb{C}_+$ and characterizes ν

$$\nu([a, b]) = \lim_{y \downarrow 0} \frac{1}{\pi} \int_a^b \mathfrak{Im} S_\nu(x + iy) \, dx$$

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- For a sequence of measures $(\nu_n)_n$ on $\mathbb{R}$, we have

$$\left(\nu_n \xrightarrow[n\to\infty]{\mathcal{L}} \nu\right) \Leftrightarrow \left(\forall z \in \mathbb{C}_+, \; S_{\nu_n}(z) \xrightarrow[n\to\infty]{} S_\nu(z)\right).$$

To prove the M-P Theorem, it is equivalent to prove

$$\forall z \in \mathbb{C}_+, \; S_{\mu_{\mathbf{B}_N}}(z) \to S_{\mu_{MP}}(z) \quad \text{a.s.}$$

where

$$S_{\mu_{MP}}(z) = \frac{\sigma^2(1-c) - z + \sqrt{(z-b)(z-a)}}{2cz\sigma^2}$$

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The Stieltjes transform
Characterization of the limiting distribution

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is the Stieltjes transform of the Marčenko-Pastur distribution satisfying $\forall z \in \mathbb{C}_+$ the equation:

$$S_{\mu_{MP}}(z) = \frac{1}{-z + 1 - c - czS_{\mu_{MP}}(z)}$$

Matrices with correlated entries

having independent columns

Theorem (Bai-Zhou '08, Yao '12)

$\mathbf{B}_N = \frac{1}{p}\sum_{k=1}^{p} \mathbf{X}_k \mathbf{X}_k^T$ *where the* $\mathbf{X}_k$*'s are independent copies of* $(X_1, \ldots, X_N)^T$ *where*

$$X_i = \sum_{j \geqslant 0} a_j \varepsilon_{i-j} \, .$$

1. *the* ε_i*'s are iid, centered and in* $\mathbf{L}^4$*.*
2. $\sum_{j \geq 1} |a_j| < \infty$

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then, a.s. $\mu_{\mathbf{B}_N} \xrightarrow[N\to\infty]{w} \mu$ *such that* $\underline{S} := S_{\underline{\mu}}(z)$ *verifies the equation*

$$z = -\frac{1}{\underline{S}} + \frac{c}{2\pi}\int_0^{2\pi} \frac{1}{\underline{S} + \big(2\pi f(\lambda)\big)^{-1}}\, d\lambda \,,$$

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$$f(x) = \frac{1}{2\pi}\sum_k Cov(X_0, X_k) e^{ixk} \, , \; x \in \mathbb{R}$$

Matrices with correlated entries
along both columns and rows

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- Boutet de Monvel, Khorunzhy and Vasilchuk '96

  Correlated Gaussian entries

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  Correlated Gaussian entries

- Hachem, Loubaton and Najim '05

$$X_{i,j} = \sum_{(k,\ell)\in\mathbb{Z}^2} a_{k,\ell} G_{i-k,j-\ell}$$

- Let $(\xi_{i,j})_{i,j\in\mathbb{Z}}$ be an array of iid random variables
- For all $(k,\ell)\in\mathbb{Z}^2$,

$$X_{k,\ell} := g(\xi_{k-i,\ell-j}\ ;\ (i,j)\in\mathbb{Z}^2)\,,$$

where $g:\mathbb{R}^{\mathbb{Z}^2}\to\mathbb{R}$ is a measurable function.

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Matrices whose entries are functions of iid random variables

- Let $(G_{i,j})_{i,j\in\mathbb{Z}}$ be an array of centered Gaussian random variables s.t. $\forall\,(i,j)\,,\,(k,\ell)\in\mathbb{Z}^2$,

$$\mathbb{E}(G_{k,\ell}G_{i,j}) = \mathbb{E}(X_{k,\ell}X_{i,j})$$

- Let $\mathbf{H}_N = \frac{1}{p}\mathcal{G}_{N,p}\mathcal{G}_{N,p}^T$

$$\mathcal{G}_{N,p} = \begin{array}{c} \begin{pmatrix} G_{1,1} & G_{1,2} & \dots & G_{1,p} \\ \vdots & \vdots & & \vdots \\ G_{N,1} & G_{N,2} & \dots & G_{N,p} \end{pmatrix} \\ \begin{array}{cccc} \mathbf{Z}_1 & \mathbf{Z}_2 & \dots & \mathbf{Z}_p \end{array} \end{array}$$

Theorem (B., Merlevède and Peligrad '14)

Provided that $N, p \to \infty$ s.t. $N/p \to c \in (o, \infty)$, we have $\forall\, z \in \mathbb{C}_+$,

$$\lim_{N \to \infty} \left| S_{\mathbf{B}_N}(z) - \mathbb{E}\big(S_{\mathbf{H}_N}(z)\big) \right| = 0 \ a.s.$$

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Corollary

If $N, p \to \infty$ *s.t.* $N/p \to c \in (0, \infty)$ *and if there exists* μ *such that for any continuous and bounded function* $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E} \int f d\mu_{\mathbf{H}_N} \xrightarrow[N\to+\infty]{} \int f d\mu,$$

then

$$\mu_{\mathbf{B}_N} \xrightarrow[N\to\infty]{w} \mu \quad a.s.$$

Theorem (B., Merlevède and Peligrad '14)

Let $N, p \to \infty$ t.q. $N/p \to c \in (0, \infty)$. Assume that

$$\sum_{k,\ell\in\mathbb{Z}} |\mathrm{Cov}(X_{0,0}, X_{k,\ell})| < \infty$$

Then,

$$\mu_{\mathbf{B}_N} \xrightarrow[N\to\infty]{w} \mu \quad a.s.$$

whose Stieltjes transform $S := S_\mu(z)$ verifies: $\forall z \in \mathbb{C}^+$

$$S(z) = \int_0^1 h(x,z)dx$$

where $h(x,z)$ is a solution of the equation

$$h(x,z) = \Big(-z + \int_0^1 \frac{f(x,s)}{1+c\int_0^1 f(u,s)h(u,z)du} ds \Big)^{-1},$$

avec

$$f(x,y) = \sum_{k,\ell\in\mathbb{Z}} \mathrm{Cov}(X_{0,0}, X_{k,\ell})\mathrm{e}^{-2\pi\mathrm{i}(kx+\ell y)}$$

Applications to linear processes

Corollary (B., Merlevède, Peligrad '14)

Let $(a_{i,j})_{(i,j)\in\mathbb{Z}^2}$ *be a double indexed sequence of numbers such that* $\sum_{i,j\in\mathbb{Z}} |a_{i,j}| < \infty$ *. Let*

$$X_{k,\ell} = \sum_{i,j\in\mathbb{Z}} a_{i,j}\xi_{k+i,\ell+j} \, .$$

The result follows with

$$f(x,y) = \mathbb{E}(\xi_{0,0}^2) \sum_{k,\ell\in\mathbb{Z}} \sum_{i,j\in\mathbb{Z}} a_{i,j} a_{k+i,\ell+j} \, e^{-2\pi(kx+\ell y)}$$

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We generalize the results of Hachem et al. '05, Yao '12 and Pan et al. '13.

Other Applications

Other possible applications:

- functions of linear processes,
- Volterra type processes,
- ARCH models, . . . etc

Thank you for your attention!