Interpolation and embeddings of weighted tent spaces

Alex Amenta

Australian National University / Université Paris-Sud

December 1, 2015

(unweighted) Tent Spaces

For $f\colon \mathbb{R}^{n+1}_+ \to \mathbb{C}$, $0 < p, q < \infty$,

$$||f||_{T^{p,q}} := \left(\int_{\mathbb{R}^n} \left(\iint_{\Gamma(x)} |f(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p/q} dx \right)^{1/p} .$$

This (quasi)norm defines the *tent space* $T^{p,q} = T^{p,q}(\mathbb{R}^n)$.

R. R. Coifman, Y. Meyer, and E. M. Stein. "Some New Function Spaces and Their Applications to Harmonic Analysis". In: *J. Funct. Anal.* 62 (1985), pp. 304–335

Weighted Tent Spaces

For $f\colon \mathbb{R}^{n+1}_+ \to \mathbb{C}$, $0 < p, q < \infty$, $s \in \mathbb{R}$,

$$||f||_{T^{p,q}_s} := \left(\int_{\mathbb{R}^n} \left(\iint_{\Gamma(x)} |t^{-s} f(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p/q} dx \right)^{1/p} .$$

This (quasi)norm defines the *weighted tent space* $T^{p,q}_s = T^{p,q}_s(\mathbb{R}^n)$.

Complex interpolation of weighted tent spaces

Theorem

For $0 < p_0, p_1, q_0, q_1 < \infty$, $s_0, s_1 \in \mathbb{R}$, *and* $\theta \in (0,1)$,

$$[T^{p_0,q_0}_{s_0}, T^{p_1,q_1}_{s_1}]_\theta = T^{p_\theta,q_\theta}_{s_\theta}$$

where $p_\theta^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1}$, *likewise for* q_θ, *and* $s_\theta = (1-\theta)s_0 + \theta s_1$.

S. Hofmann, S. Mayboroda, and A. McIntosh. "Second Order Elliptic Operators with Complex Bounded Measurable Coefficients in L^p, Sobolev, and Hardy Spaces". In: *Ann. Sci. Ec. Norm. Supér. (4)* 44.5 (2011), pp. 723–800

Real interpolation of weighted tent spaces

Theorem (A., 2015)

For $0 < p_0, p_1, q < \infty$, $s_0 \neq s_1 \in \mathbb{R}$, *and* $\theta \in (0,1)$,

$$(T^{p_0,q}_{s_0}, T^{p_1,q}_{s_1})_{\theta,p_\theta} = Z^{p_\theta,q}_{s_\theta},$$

where $Z^{p,q}_s$ *is defined by the quasinorm*

$$||f||_{Z^{p,q}_s} := \left(\iint_{\mathbb{R}^{n+1}_+} \left(⨍_{\Omega(x,t)} |\tau^{-s} f(\xi,\tau)|^q \, d\xi \, d\tau \right)^{p/q} \frac{dx\,dt}{t} \right)^{1/p}$$

with $\Omega(x,t) = B(x,t) \times (t/2, 2t)$.

A. Amenta. “Interpolation and Embeddings of Weighted Tent Spaces”. arxiv:1509.05699. 2015

Idea of proof (a bit unrigorous):

Idea of proof (a bit unrigorous):

$$||f||_{T^{p,q}_s} = \left|\left| x \mapsto \mathbf{1}_{\Gamma(x)} f \right|\right|_{L^p(\mathbb{R}^n : L^q_s)}$$

where

$$L^q_s := L^q\left(\mathbb{R}^{n+1}_+, t^{-qs} \frac{dy\,dt}{t^{n+1}}\right).$$

Idea of proof (a bit unrigorous):

$$||f||_{T^{p,q}_s} = \left|\left|x \mapsto \mathbf{1}_{\Gamma(x)} f\right|\right|_{L^p(\mathbb{R}^n : L^q_s)}$$

where

$$L^q_s := L^q\left(\mathbb{R}^{n+1}_+, t^{-qs}\frac{dy\,dt}{t^{n+1}}\right).$$

So we will have

$$||f||_{(T^{p_\bullet,q}_{s_\bullet})_{\theta,p_\theta}} \simeq \left|\left|x \mapsto \mathbf{1}_{\Gamma(x)} f\right|\right|_{(L^{p_\bullet}(\mathbb{R}^n : L^q_{s_\bullet}))_{\theta,p_\theta}},$$

where

$$(L^{p_\bullet}(\mathbb{R}^n : L^q_{s_\bullet}))_{\theta,p_\theta} = L^{p_\theta}(\mathbb{R}^n : (L^q_{s_\bullet})_{\theta,p_\theta}).$$

The problem is reduced to identifying 'off-diagonal' real interpolation spaces $(L^q_{s_\bullet})_{\theta,p_\theta}$ between weighted L^q spaces, with q fixed.

Theorem (Gilbert, 1972)

Suppose (M, μ) is a σ-finite measure space and let w be a weight on (M, μ). Let $0 < \theta < 1$, $0 < p, q \leq \infty$. Then for each $r > 1$ the expression

$$\left|\left|\left(r^{-k\theta}\left|\left|\mathbf{1}_{x:w(x)\in(r^{-k},r^{-k+1}]}f\right|\right|_{L^q(M)}\right)_{k\in\mathbb{Z}}\right|\right|_{\ell^p(\mathbb{Z})},$$

gives an equivalent quasinorm on $(L^q(M), L^q(M, w^q))_{\theta,p}$.

Theorem (Gilbert, 1972)

Suppose (M,μ) is a σ-finite measure space and let w be a weight on (M,μ). Let $0<\theta<1$, $0<p,q\le\infty$. Then for each $r>1$ the expression

$$\left|\left|\left(r^{-k\theta}\,\left|\left|\mathbf{1}_{x:w(x)\in(r^{-k},r^{-k+1}]}f\right|\right|_{L^q(M)}\right)_{k\in\mathbb{Z}}\right|\right|_{\ell^p(\mathbb{Z})},$$

gives an equivalent quasinorm on $(L^q(M),L^q(M,w^q))_{\theta,p}$.

In our situation,

$$L^q\left(t^{-qs_1}\,\frac{dy\,dt}{t^{n+1}}\right)=L^q\left((t^{-(s_1-s_0)})^q t^{-qs_0}\,\frac{dy\,dt}{t^{n+1}}\right)$$

so we can apply Gilbert's theorem with $w(y,t)=t^{-(s_1-s_0)}$.

J. Gilbert. "Interpolation Between Weighted L^p-spaces". In: *Ark. Mat.* 10.1-2 (1972), pp. 235–249

Corollary

For $0 < p_0, p_1, q < \infty$, $s_0 \neq s_1 \in \mathbb{R}$, $\theta \in (0,1)$, *and* $r > 1$, *we have*

$$||f||_{(T^{p_0,q}_{s_0}, T^{p_1,q}_{s_1})_{\theta,p_\theta}} \simeq \left|\left| \left(r^{-k\theta(s_1-s_0)} \left|\left| \mathbf{1}_{(r^{-k}, r^{-k+1})}(t) f \right|\right|_{T^{p_\theta,q}_{s_0}} \right)_{k\in\mathbb{Z}} \right|\right|_{\ell^{p_\theta}(\mathbb{Z})}.$$

Corollary

For $0 < p_0, p_1, q < \infty$, $s_0 \neq s_1 \in \mathbb{R}$, $\theta \in (0,1)$, *and* $r > 1$, *we have*

$$||f||_{(T^{p_0,q}_{s_0}, T^{p_1,q}_{s_1})_{\theta,p_\theta}} \simeq \left|\left|\left(r^{-k\theta(s_1-s_0)}\, ||\mathbf{1}_{(r^{-k},r^{-k+1})}(t) f||_{T^{p_\theta,q}_{s_0}}\right)_{k\in\mathbb{Z}}\right|\right|_{\ell^{p_\theta}(\mathbb{Z})}.$$

This expression turns out to be equivalent to $||f||_{Z^{p_\theta,q}_{s_\theta}}$.

Embeddings of weighted tent spaces

Embeddings of weighted tent spaces

Theorem (A., 2015)

Let $0 < p_0 < p_1 \leq \infty$, $q \in (0, \infty)$, *and* $s_0 > s_1 \in \mathbb{R}$ *with*

$$s_1 - s_0 = n\left(\frac{1}{p_1} - \frac{1}{p_0}\right).$$

Then we have the continuous inclusion

$$T^{p_0,q}_{s_0}(\mathbb{R}^n) \hookrightarrow T^{p_1,q}_{s_1}(\mathbb{R}^n).$$

Theorem (A., 2015)

Let $0 < p_0 < p_1 \leq \infty$, $q \in (0, \infty)$, *and* $s_0 > s_1 \in \mathbb{R}$ *with*

$$s_1 - s_0 = n\left(\frac{1}{p_1} - \frac{1}{p_0}\right).$$

Then we have the continuous inclusion

$$T^{p_0,q}_{s_0}(\mathbb{R}^n) \hookrightarrow T^{p_1,q}_{s_1}(\mathbb{R}^n).$$

The proof relies on the atomic decomposition theorem.

A $T^{p,q}_s$-*atom* associated with a ball $B \subset \mathbb{R}^n$ is a function $a\colon \mathbb{R}^{n+1}_+ \to \mathbb{C}$ supported on the *tent* $T(B)$ (to be drawn on the board), such that

$$||a||_{T^{q,q}_s} \leq |B|^{\frac{1}{q}-\frac{1}{p}}.$$

A $T^{p,q}_s$-*atom* associated with a ball $B \subset \mathbb{R}^n$ is a function $a\colon \mathbb{R}^{n+1}_+ \to \mathbb{C}$ supported on the *tent* $T(B)$ (to be drawn on the board), such that

$$||a||_{T^{q,q}_s} \leq |B|^{\frac{1}{q}-\frac{1}{p}}.$$

Theorem (Coifman–Meyer–Stein 1985, with easy modifications)

Suppose that $p \in (0,1]$, $q \geq p$, and $s \in \mathbb{R}$. Then a function f is in $T^{p,q}_s$ if and only if there exists a sequence $(a_k)_{k\in\mathbb{N}}$ of $T^{p,q}_s$-atoms and a sequence of scalars $(\lambda_k)_{k\in\mathbb{N}} \in \ell^p(\mathbb{N})$ such that

$$f = \sum_{k\in\mathbb{N}} \lambda_k a_k$$

with convergence in $T^{p,q}_s$. Furthermore, we have

$$||f||_{T^{p,q}_s} \simeq \inf ||\lambda_k||_{\ell^p(\mathbb{N})}$$

where the infimum is taken over all such decompositions.

Sketch of (part of) the proof

Sketch of (part of) the proof

First show that $T^{1,q}_{s_0}$-atoms are uniformly contained in $T^{p_1,q}_{s_1}$ when $s_1 = n(p_1^{-1} - 1)$ (taking $p_0 = 1$)

Sketch of (part of) the proof

First show that $T^{1,q}_{s_0}$-atoms are uniformly contained in $T^{p_1,q}_{s_1}$ when $s_1 = n(p_1^{-1} - 1)$ (taking $p_0 = 1$)

$$||a||_{T^{p_1,q}_{s_1}} = \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_1} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1}$$

Sketch of (part of) the proof

First show that $T^{1,q}_{s_0}$-atoms are uniformly contained in $T^{p_1,q}_{s_1}$ when $s_1 = n(p_1^{-1} - 1)$ (taking $p_0 = 1$)

$$||a||_{T^{p_1,q}_{s_1}} = \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_1} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1}$$

$$\leq r_B^{-(s_1 - s_0)} \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_0} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1}$$

Sketch of (part of) the proof

First show that $T^{1,q}_{s_0}$-atoms are uniformly contained in $T^{p_1,q}_{s_1}$ when $s_1 = n(p_1^{-1} - 1)$ (taking $p_0 = 1$)

$$||a||_{T^{p_1,q}_{s_1}} = \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_1} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1}$$

$$\leq r_B^{-(s_1-s_0)} \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_0} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1}$$

$$\lesssim |B|^{-(s_1-s_0)/n} |B|^{\frac{1}{p_1 (q/p_1)'}} \, ||a||_{T^{q,q}_{s_0}}$$

Sketch of (part of) the proof

First show that $T^{1,q}_{s_0}$-atoms are uniformly contained in $T^{p_1,q}_{s_1}$ when $s_1 = n(p_1^{-1} - 1)$ (taking $p_0 = 1$)

$$||a||_{T^{p_1,q}_{s_1}} = \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_1} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1}$$

$$\leq r_B^{-(s_1-s_0)} \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_0} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1}$$

$$\lesssim |B|^{-(s_1-s_0)/n} |B|^{\frac{1}{p_1 (q/p_1)'}} \, ||a||_{T^{q,q}_{s_0}}$$

$$\leq |B|^{-(p_1^{-1}-1)+(p_1^{-1}-q^{-1})+(q^{-1}-1)}$$

Sketch of (part of) the proof

First show that $T^{1,q}_{s_0}$-atoms are uniformly contained in $T^{p_1,q}_{s_1}$ when $s_1 = n(p_1^{-1} - 1)$ (taking $p_0 = 1$)

$$
\begin{aligned}
||a||_{T^{p_1,q}_{s_1}} &= \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_1} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1} \\
&\leq r_B^{-(s_1-s_0)} \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_0} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1} \\
&\lesssim |B|^{-(s_1-s_0)/n} |B|^{\frac{1}{p_1 (q/p_1)'}} \, ||a||_{T^{q,q}_{s_0}} \\
&\leq |B|^{-(p_1^{-1}-1)+(p_1^{-1}-q^{-1})+(q^{-1}-1)} \\
&= 1.
\end{aligned}
$$

Sketch of (part of) the proof

First show that $T^{1,q}_{s_0}$-atoms are uniformly contained in $T^{p_1,q}_{s_1}$ when $s_1 = n(p_1^{-1} - 1)$ (taking $p_0 = 1$)

$$
\begin{aligned}
||a||_{T^{p_1,q}_{s_1}} &= \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_1} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1} \\
&\leq r_B^{-(s_1-s_0)} \left(\int_B \left(\iint_{\Gamma(x)\cap T(B)} |t^{-s_0} a(y,t)|^q \, \frac{dy\,dt}{t^{n+1}} \right)^{p_1/q} dx \right)^{1/p_1} \\
&\lesssim |B|^{-(s_1-s_0)/n} |B|^{\frac{1}{p_1(q/p_1)'}} \, ||a||_{T^{q,q}_{s_0}} \\
&\leq |B|^{-(p_1^{-1}-1)+(p_1^{-1}-q^{-1})+(q^{-1}-1)} \\
&= 1.
\end{aligned}
$$

Combining this with the atomic decomposition theorem proves the embedding $T^{1,q}_{s_0} \hookrightarrow T^{p_1,q}_{s_1}$.

We use a series of tricks to extend this result to the general theorem. For example, to show that $T^{p_0,q}_{s_0} \hookrightarrow T^{p_1,q}_{s_1}$ with $0 < p_0 < p_1 \leq q$ we argue by taking powers:

We use a series of tricks to extend this result to the general theorem. For example, to show that $T^{p_0,q}_{s_0} \hookrightarrow T^{p_1,q}_{s_1}$ with $0 < p_0 < p_1 \leq q$ we argue by taking powers:

$$||f||_{T^{p_1,q}_{s_1}} = ||f^{p_0}||^{1/p_0}_{T^{p_1/p_0,q/p_0}_{p_0 s_1}}$$

We use a series of tricks to extend this result to the general theorem. For example, to show that $T^{p_0,q}_{s_0} \hookrightarrow T^{p_1,q}_{s_1}$ with $0 < p_0 < p_1 \leq q$ we argue by taking powers:

$$\begin{aligned} ||f||_{T^{p_1,q}_{s_1}} &= ||f^{p_0}||^{1/p_0}_{T^{p_1/p_0,q/p_0}_{p_0 s_1}} \\ &\lesssim ||f^{p_0}||^{1/p_0}_{T^{1,q/p_0}_{p_0 s_0}} \end{aligned}$$

We use a series of tricks to extend this result to the general theorem. For example, to show that $T^{p_0,q}_{s_0} \hookrightarrow T^{p_1,q}_{s_1}$ with $0 < p_0 < p_1 \leq q$ we argue by taking powers:

$$\begin{aligned} ||f||_{T^{p_1,q}_{s_1}} &= ||f^{p_0}||^{1/p_0}_{T^{p_1/p_0,q/p_0}_{p_0 s_1}} \\ &\lesssim ||f^{p_0}||^{1/p_0}_{T^{1,q/p_0}_{p_0 s_0}} \\ &= ||f||_{T^{p_0,q}_{s_0}} . \end{aligned}$$

We use a series of tricks to extend this result to the general theorem. For example, to show that $T^{p_0,q}_{s_0} \hookrightarrow T^{p_1,q}_{s_1}$ with $0 < p_0 < p_1 \le q$ we argue by taking powers:

$$\begin{aligned} ||f||_{T^{p_1,q}_{s_1}} &= ||f^{p_0}||^{1/p_0}_{T^{p_1/p_0,q/p_0}_{p_0 s_1}} \\ &\lesssim ||f^{p_0}||^{1/p_0}_{T^{1,q/p_0}_{p_0 s_0}} \\ &= ||f||_{T^{p_0,q}_{s_0}} \,. \end{aligned}$$

Here we used that

$$\begin{aligned} p_0 s_0 - p_0 s_1 &= p_0 n \left(\frac{1}{p_1} - \frac{1}{p_0} \right) \\ &= n \left(\frac{p_0}{p_1} - 1 \right) . \end{aligned}$$

Results for general metric measure spaces (X, d, μ)

Here it is more natural to use the norm

$$||f||_{T_s^{p,q}(X)} := \left(\int_X \left(\iint_{\Gamma(x)} |\mu(B(y,t))^{-s} f(y,t)|^q \frac{dy}{\mu(B(y,t))} \frac{dt}{t} \right)^{p/q} dx \right)^{1/p}$$

weighting with powers of volumes instead of powers of t.

Results for general metric measure spaces (X, d, μ)
Here it is more natural to use the norm

$$||f||_{T_s^{p,q}(X)} := \left(\int_X \left(\iint_{\Gamma(x)} |\mu(B(y,t))^{-s} f(y,t)|^q \frac{dy}{\mu(B(y,t))} \frac{dt}{t} \right)^{p/q} dx \right)^{1/p}$$

weighting with powers of volumes instead of powers of t.

If X is doubling (i.e. a space of homogeneous type), then the embedding theorem is true with condition

$$s_1 - s_0 = \frac{1}{p_1} - \frac{1}{p_0}$$

(the dimension does not appear since we use volumes)

Results for general metric measure spaces (X, d, μ)
Here it is more natural to use the norm

$$||f||_{T^{p,q}_s(X)} := \left(\int_X \left(\iint_{\Gamma(x)} |\mu(B(y,t))^{-s} f(y,t)|^q \frac{dy}{\mu(B(y,t))} \frac{dt}{t} \right)^{p/q} dx \right)^{1/p}$$

weighting with powers of volumes instead of powers of t.

If X is doubling (i.e. a space of homogeneous type), then the embedding theorem is true with condition

$$s_1 - s_0 = \frac{1}{p_1} - \frac{1}{p_0}$$

(the dimension does not appear since we use volumes)

If X is unbounded and AD-regular, then the real interpolation theorem is true. (we need to use $\mu(B(y,t)) \simeq t^n$ for all $t > 0$ when identifying level sets of the weight $\mu(B(y,t))^{s_1 - s_0}$)