Spectral multipole theory for nano-antennas

(Directivity, Ideal absorption and unitarity)

Spectral Theory of Novel Materials : April 2016

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Maxwell's equations in inhomogeneous media

Time-harmonic Maxwell's equations : Inhomogeneous medium

$$\nabla \times \frac{1}{\mu(\vec{x})} \nabla \times \vec{\mathbf{E}}(\vec{x}) - \varepsilon(\vec{x}) \left(\frac{\omega}{c}\right)^2 \vec{\mathbf{E}}(\vec{x}) = i\mu_0 \omega \vec{j}_s(\vec{x})$$

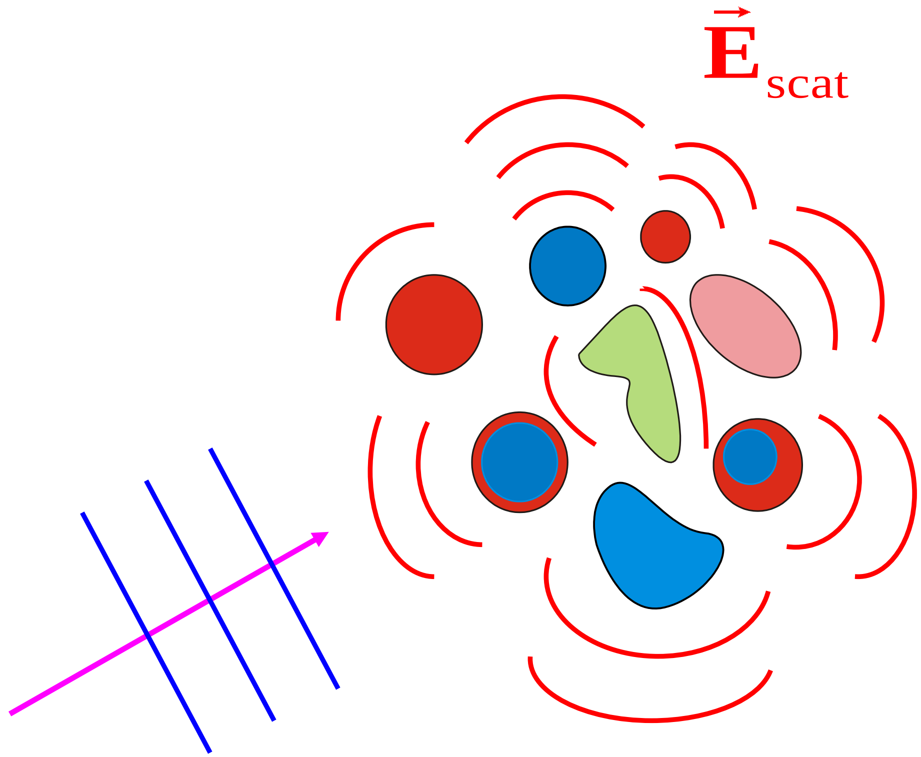
Green's function : Inhomogeneous medium

$$\nabla \times \frac{1}{\mu(\vec{x})} \nabla \times \overleftrightarrow{\mathbf{G}}(\vec{x}, \vec{x}') - \varepsilon(\vec{x}) \left(\frac{\omega}{c}\right)^2 \overleftrightarrow{\mathbf{G}}(\vec{x}, \vec{x}') = \overleftrightarrow{\mathbf{I}} \delta(\vec{x} - \vec{x}')$$

Total solution to the wave equation !

$$\vec{\mathbf{E}}(\vec{x}) = i\mu_0 \omega \int \overleftrightarrow{\mathbf{G}}(\vec{x}, \vec{x}') \cdot \vec{j}_s(\vec{x}')$$

Physical problems can often be formulated as a multiple-scattering problem



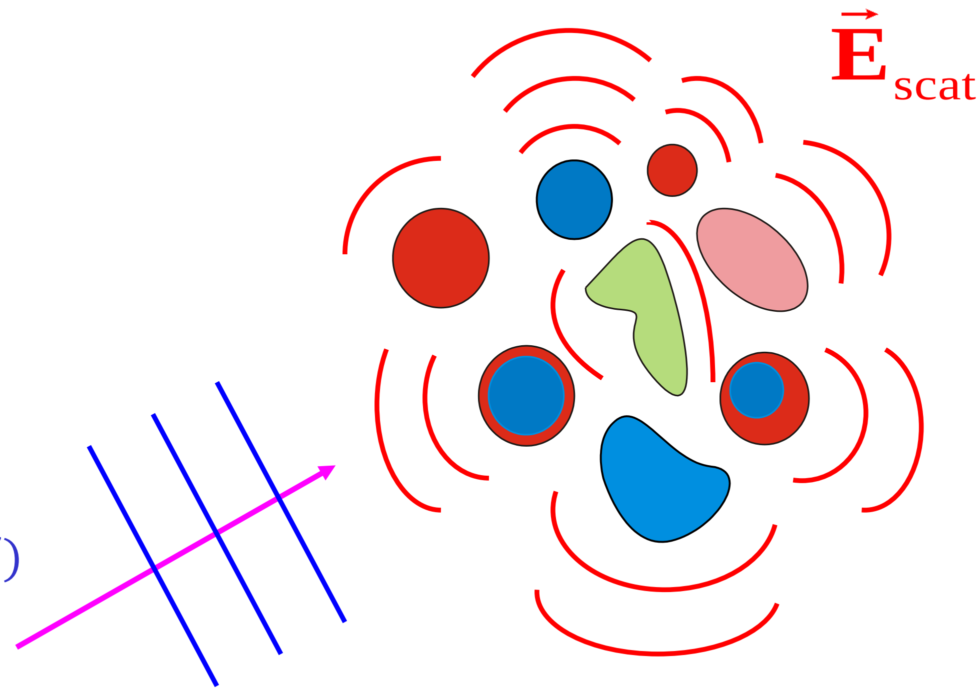
$$\vec{\mathbf{E}}_{\mathbf{inc}}(\vec{\boldsymbol{x}}) = i\mu_0\omega \int \overleftrightarrow{\mathbf{G}}_{\mathbf{0}}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}') \cdot \vec{\boldsymbol{j}}_s(\vec{\boldsymbol{x}}')$$

Meso-scopic approximation : permittivity, $\varepsilon(\vec{\boldsymbol{x}})$ is piecewise constant and $\mu(\vec{\boldsymbol{x}})$=1

$$\boldsymbol{\nabla} \times \frac{1}{\mu(\vec{\boldsymbol{x}})} \boldsymbol{\nabla} \times \overleftrightarrow{\mathbf{G}}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}') - \varepsilon(\vec{\boldsymbol{x}}) \left(\frac{\omega}{c}\right)^2 \overleftrightarrow{\mathbf{G}}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}') = \overleftrightarrow{\mathbf{I}}\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}')$$

T-matrix simplifies the formulation of the multiple-scattering problem

$$\overleftrightarrow{\mathbf{G}}(\vec{x},\vec{x}') = \overleftrightarrow{\mathbf{G}}_{\mathbf{0}}(\vec{x}-\vec{x}') + \overleftrightarrow{\mathbf{G}}_{\mathbf{0}}(\vec{x}-\vec{x}'') \cdot \overleftrightarrow{\mathbf{T}}(\vec{x}'',\vec{x}''') \cdot \overleftrightarrow{\mathbf{G}}_{\mathbf{0}}(\vec{x}'''-\vec{x}')$$



$$\vec{\mathbf{E}}_{\mathbf{inc}}(\vec{x}) = i\mu_0\omega \int \overleftrightarrow{\mathbf{G}}_{\mathbf{0}}(\vec{x},\vec{x}') \cdot \vec{j}_s(\vec{x}')$$

$$\vec{\mathbf{E}}_{\text{tot}}(\vec{x}) = \vec{\mathbf{E}}_{\mathbf{inc}}(\vec{x}) + \underbrace{\overleftrightarrow{\mathbf{G}}_{\mathbf{0}}(\vec{x}-\vec{x}'') \cdot \overleftrightarrow{\mathbf{T}}(\vec{x}'',\vec{x}''') \cdot \vec{\mathbf{E}}_{\mathbf{inc}}(\vec{x}''')}_{\vec{\mathbf{E}}_{\text{scat}}(\vec{x})}$$

Green's function of a homogeneous media

Scalar case

$$\Delta G_0(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') + k^2 G_0(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') = -\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}')$$

$$G_0(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') = \frac{e^{ik|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|}}{4\pi|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|} = \frac{i}{4\pi} h_0(|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|)$$

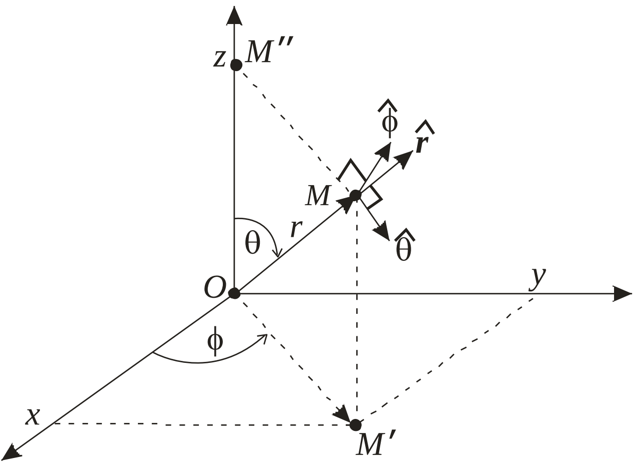
Vector case

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \overleftrightarrow{\mathbf{G}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') - k^2 \overleftrightarrow{\mathbf{G}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') = \overleftrightarrow{\mathbf{I}} \delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}')$$

$$\overleftrightarrow{\mathbf{G}}_0(\vec{\boldsymbol{r}}) = -\frac{e^{ikr}}{4\pi k^2 r^3} \mathrm{P.V.}\left\{(1 - ikr - k^2 r^2)\left(\overleftrightarrow{\mathbf{I}} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right) - 2(1 - ikr)\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right\} - \frac{\overleftrightarrow{\mathbf{I}}}{3k^2}\delta(\vec{\boldsymbol{r}})$$

$$\vec{\boldsymbol{r}} \equiv \vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'$$

Homogenous scalar Helmholtz equation with constant permittivity



$$\Delta\psi + k^2\psi = 0$$

$$\frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} - \frac{\bar{\bar{L}}^2}{r^2}\psi - k^2\psi = 0 \qquad k^2 = \varepsilon\left(\frac{\omega}{c}\right)^2$$

$$\psi(r,\theta,\phi) = \psi_r(r)Y_{n,m}(\theta,\phi) \qquad \bar{\bar{L}}^2 Y_{n,m}(\theta,\phi) = n(n+1)Y_{n,m}(\theta,\phi)$$

$$Y_{n,m}(\theta,\phi) = \bar{P}_n^m(\cos\theta)\exp(im\phi) \qquad \begin{cases} n = 0,1,2\ldots\infty \\ m = -n,\ldots,n \end{cases}$$

Change of variables:

$$j_n(kr) \equiv \psi_r(r) \equiv \frac{Z(r)}{(kr)^{1/2}}$$

Spherical Bessel function equation

$$r^2\frac{d^2Z}{dr^2} + r\frac{dZ}{dr} + \left[k^2r^2 - \left(n+\frac{1}{2}\right)^2\right]Z = 0$$

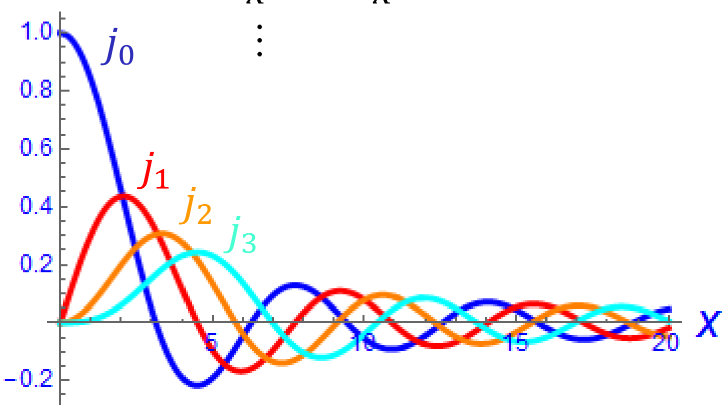
Linearly independent solutions $\Delta\psi + k^2\psi = 0$

Spherical Bessel functions (1) $j_n(x)$

$$j_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} x^{2s} = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$\vdots$$

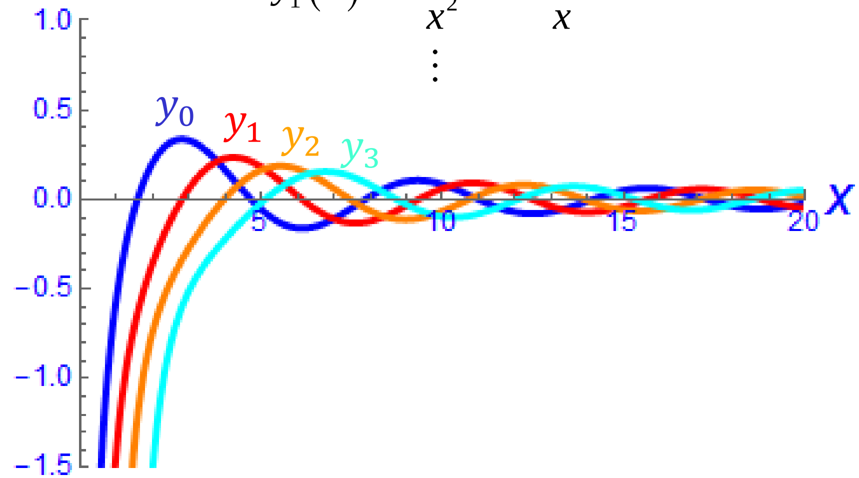


Spherical Neumann functions (2) $y_n(x)$

$$y_0(x) = -\frac{\cos x}{x}$$

$$y_1(x) = \frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$\vdots$$



Outgoing spherical Hankel functions (+)

$$h_n^{(+)}(x) = j_n(x) + i y_n(x)$$

$$h_0^{(+)}(x) = -\frac{i}{x} e^{ix}$$

$$h_1^{(+)}(x) = -e^{ix}\left(\frac{1}{x} + \frac{i}{x^2}\right)$$

$$\vdots$$

Incoming spherical Hankel functions (-)

$$h_n^{(-)}(x) = j_n(x) - i y_n(x)$$

$$h_0^{(-)}(x) = \frac{i}{x} e^{-ix}$$

$$h_1^{(-)}(x) = -e^{-ix}\left(\frac{1}{x} - \frac{i}{x^2}\right)$$

$$\vdots$$

Partial wave basis (scalar waves)

Plane wave expansion :

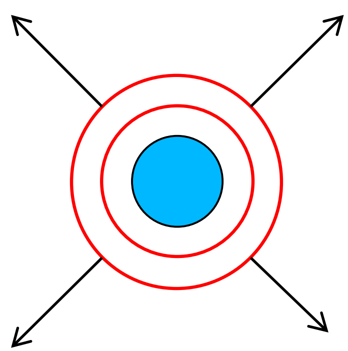
$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm}\varphi_{nm}(k\mathbf{r})$$

Plane wave Coefficients :

$$p_{n,m} = 4\pi i^n Y_{n,m}^*(\hat{\mathbf{k}})$$

Regular partial wave :

$$\varphi_{n,m}(k\boldsymbol{r}) = j_n(kr)Y_{n,m}(\theta,\phi)$$



Outgoing "partial" wave :

$$\varphi_{n,m}^{(+)}(k\boldsymbol{r}) = h_n^{(+)}(kr)Y_{n,m}(\theta,\phi)$$

Vector partial waves

$$\nabla\times\nabla\times\vec{\mathbf{E}}-k^2\,\vec{\mathbf{E}}=0$$

$$\vec{\mathbf{M}}^{(1)}_{n,m}(k\mathbf{r})=j_n(kr)\,\vec{\mathbf{X}}_{n,m}(\theta,\phi)$$

$$\vec{\mathbf{N}}^{(1)}_{n,m}(k\mathbf{r})=\frac{1}{kr}\Big[j_n(kr)\sqrt{n(n+1)}\,\vec{\mathbf{Y}}_{n,m}(\theta,\phi)$$

$$+\psi'_n(x)\,\vec{\mathbf{Z}}_{n,m}(\theta,\phi)\Big]$$

n=1,2,…, ∞ $\quad$ $-n < m < n$

Ricatti-Bessel functions

$$\psi_n(x)\equiv xj_n(x)$$

$$\psi'_n(x)=\left[j_n(x)+xj'_n(x)\right]$$

$$\nabla\cdot\vec{\mathbf{M}}_{n,m}=\nabla\cdot\vec{\mathbf{N}}_{n,m}=0 \qquad \nabla\times\vec{\mathbf{M}}_{n,m}=k\vec{\mathbf{N}}_{n,m} \quad \nabla\times\vec{\mathbf{N}}_{n,m}=k\vec{\mathbf{M}}_{n,m}$$

Vector spherical harmonics

$$\vec{\mathbf{Y}}_{n,m}(\theta,\phi) = \hat{\mathbf{r}} Y_{n,m} \qquad n = 0,\ldots,\infty \qquad m = -n,\ldots,n$$

$$\vec{\mathbf{X}}_{n,m}(\theta,\phi) = e^{im\phi}\left[\hat{\boldsymbol{\theta}} i\bar{u}_n^m - \hat{\boldsymbol{\varphi}}\bar{s}_n^m\right]$$

$$\vec{\mathbf{Z}}_{n,m}(\theta,\phi) = e^{im\phi}\left[\hat{\boldsymbol{\theta}}\bar{s}_n^m + \hat{\boldsymbol{\varphi}} i\bar{u}_n^m\right] \qquad n = 1,\ldots,\infty \qquad m = -n,\ldots,n$$

$$\bar{u}_{n,m}(\theta,\phi) = \frac{1}{\sqrt{n(n+1)}}\frac{m}{\sin\theta}\bar{P}_n^m(\cos\theta)$$

$$\bar{s}_{n,m}(\theta,\phi) = \frac{1}{\sqrt{n(n+1)}}\frac{d}{d\theta}\bar{P}_n^m(\cos\theta)$$

Partial wave expansions for Green's functions

$$G_0(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}')=\frac{e^{ik|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}}{4\pi|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}=ik\sum_{n=0}^{\infty}(-)^m\varphi_{n,m}^{(+)}(k\vec{\boldsymbol{r}}_>)\varphi_{n,-m}^{(1)}(k\vec{\boldsymbol{r}}_<)$$

$$\overleftrightarrow{\mathbf{G}}_0(\vec{\boldsymbol{r}})=-\frac{e^{ikr}}{4\pi k^2r^3}\mathrm{P.V.}\left\{(1-ikr-k^2r^2)\left(\overleftrightarrow{\mathbf{I}}-\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right)-2(1-ikr)\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right\}-\frac{\overleftrightarrow{\mathbf{I}}}{3k^2}\delta(\vec{\boldsymbol{r}})$$

$$=ik\sum_{n=0}^{\infty}(-)^m\left\{\overrightarrow{\boldsymbol{M}}_{n,m}^{(+)}(k\vec{\boldsymbol{r}}_>)\overrightarrow{\boldsymbol{M}}_{n,-m}^{(1)}(k\vec{\boldsymbol{r}}_<)+\overrightarrow{\boldsymbol{N}}_{n,m}^{(+)}(k\vec{\boldsymbol{r}}_>)\boldsymbol{N}_{n,-m}^{(1)}(k\vec{\boldsymbol{r}}_<)\right\}-\frac{\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{k^2}\delta(\vec{\boldsymbol{r}})$$

T-matrix of a scatterer (multipole representation)

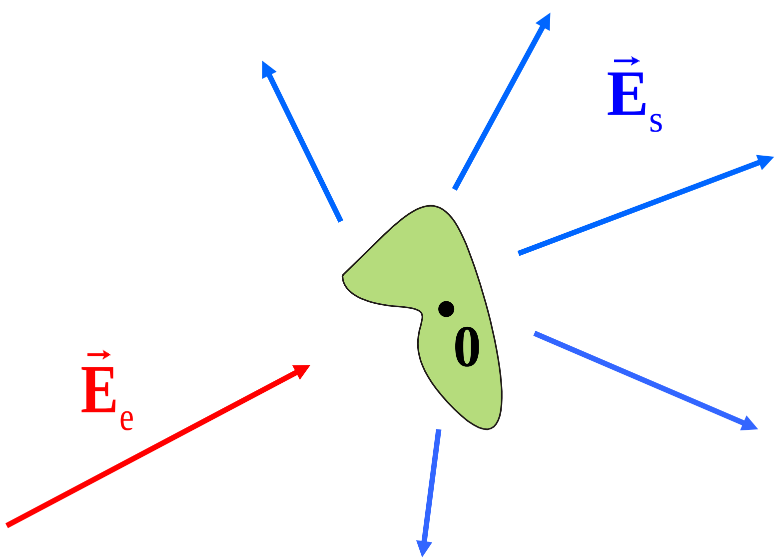
"Seeing is believing and all we see is scattered light" J.C. Stover

Excitation field

$$\vec{\boldsymbol{E}}_{\mathrm{e}}(k\vec{\boldsymbol{r}}) = \left[\vec{\boldsymbol{M}}^{(1)}(k\vec{\boldsymbol{r}}) \quad , \quad \vec{\boldsymbol{N}}^{(1)}(k\vec{\boldsymbol{r}})\right] . \, e$$

Scattering field

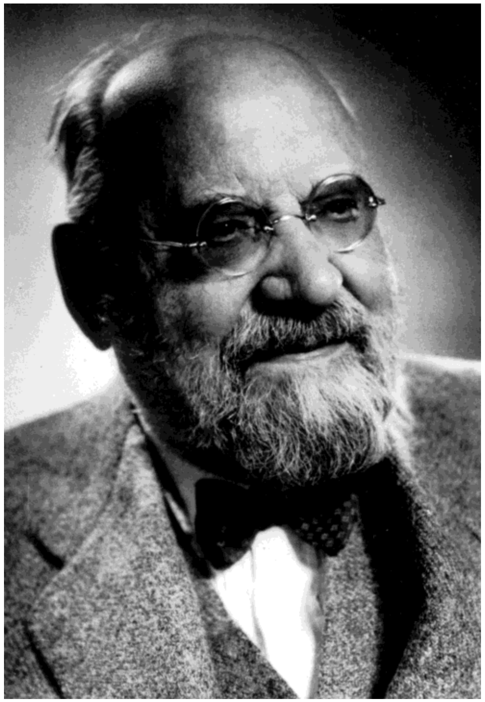
$$\vec{\boldsymbol{E}}_{s}(k\vec{\boldsymbol{r}}) = \left[\vec{\boldsymbol{M}}^{(+)}(k\vec{\boldsymbol{r}}) \quad , \quad \vec{\boldsymbol{N}}^{(+)}(k\vec{\boldsymbol{r}})\right] . \, f$$



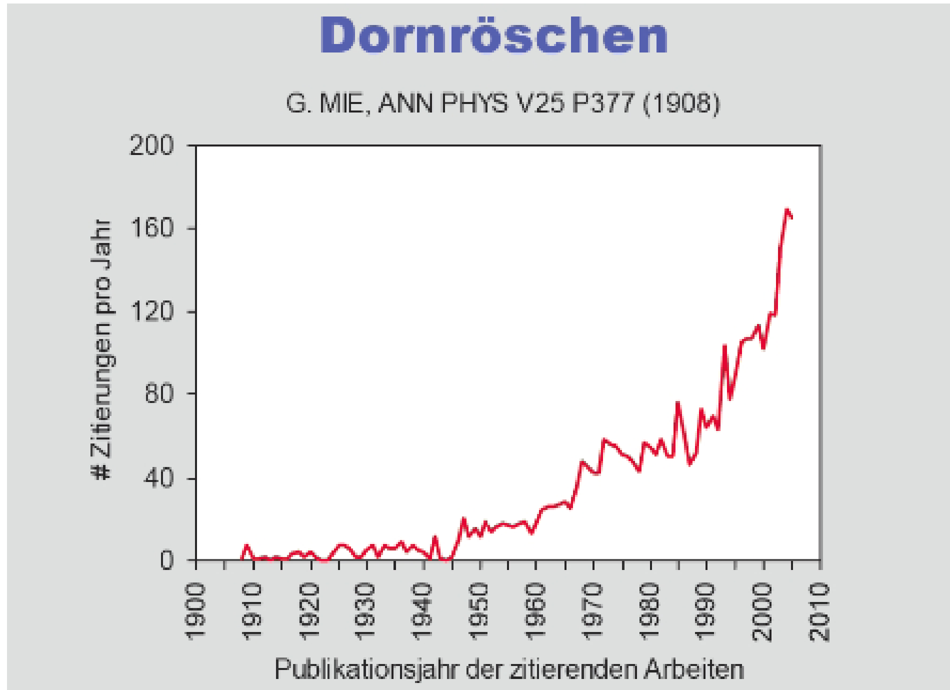
T-matrix :

$$f = \overline{\overline{T}} . e$$

Scattering by a sphere
Lorenz(1890)-Mie(1908)-Debye(1909) theory



Gustav Mie
(1868-1957)



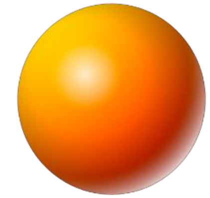
Ludvig Lorenz (1829–91)

“Light scattering and reflection by a transparent sphere (surface)”

in Oeuvres scientifiques de **L. Lorenz**. revues et annotées par H. Valentiner.

Tome Premier, Libraire Lehmann & Stage, Copenhague, 1898, p 403-529.

T-matrix of a sphere



$$f_{n,m}^{(e)} = T_n^{(e)} e_{n,m}^{(e)}$$

$$f_{n,m}^{(h)} = T_n^{(h)} e_{n,m}^{(h)}$$

$$\begin{bmatrix} f_p^{(h)} \\ f_p^{(e)} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} T_1^{(h)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_n^{(h)} \end{pmatrix} & \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} T_1^{(e)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_n^{(e)} \end{pmatrix} \end{bmatrix} \begin{bmatrix} e_p^{(h)} \\ e_p^{(e)} \end{bmatrix} = [\![T]\!] \begin{bmatrix} e_p^{(h)} \\ e_p^{(e)} \end{bmatrix}$$

Lorenz-Mie Coefficients : a_n , b_n $\quad T_n^{(e)} = -a_n \quad T_n^{(h)} = -b_n$

Traditional "Mie" form

$$a_n = \frac{\rho_s \psi_n(k_s R)\psi_n'(kR) - (\mu_s/\mu)\psi_n(kR)\psi_n'(k_s R)}{\rho_s \psi_n(k_s R)\xi_n'(kR) - (\mu_s/\mu)\xi_n(kR)\psi_n'(k_s R)}$$

$$b_n = \frac{\rho_s \psi_n(kR)\psi_n'(k_s R) - (\mu_s/\mu)\psi_n(k_s R)\psi_n'(kR)}{\rho_s \xi_n(kR)\psi_n'(k_s R) - (\mu_s/\mu)\psi_n(k_s R)\xi_n'(kR)}$$

$$\rho_s = \frac{k_s}{k} = \frac{N_s}{N}$$

The way we write things is important !

$$f_{n,m}^{(e)} = T_n^{(e)} e_{n,m}^{(e)}$$

$$f_{n,m}^{(h)} = T_n^{(h)} e_{n,m}^{(h)}$$

$$\Rightarrow \quad \begin{bmatrix} f_p^{(h)} \\ f_p^{(e)} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} T_1^{(h)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_n^{(h)} \end{pmatrix} & \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} T_1^{(e)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_n^{(e)} \end{pmatrix} \end{bmatrix} \begin{bmatrix} e_p^{(h)} \\ e_p^{(e)} \end{bmatrix} = [\![T]\!] \begin{bmatrix} e_p^{(h)} \\ e_p^{(e)} \end{bmatrix}$$

Logarithmic derivative form

Electric type $T_n^{(e)}$:

$$T_n^{(e)} = -\frac{j_n(kR)}{h_n^{(+)}(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

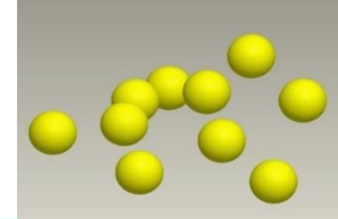
Magnetic type $T_n^{(h)}$:

$$T_n^{(h)} = -\frac{j_n(kR)}{h_n^{(+)}(kR)} \frac{\frac{\mu_s}{\mu}\varphi_n(kR) - \varphi_n(k_s R)}{\frac{\mu_s}{\mu}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

$$\varphi_n(z) = -\frac{[z j_n(z)]'}{j_n(kR)} \qquad \varphi_n^{(\pm)}(z) = -\frac{\left[z h_n^{(\pm)}(z)\right]'}{h_n(kR)}$$

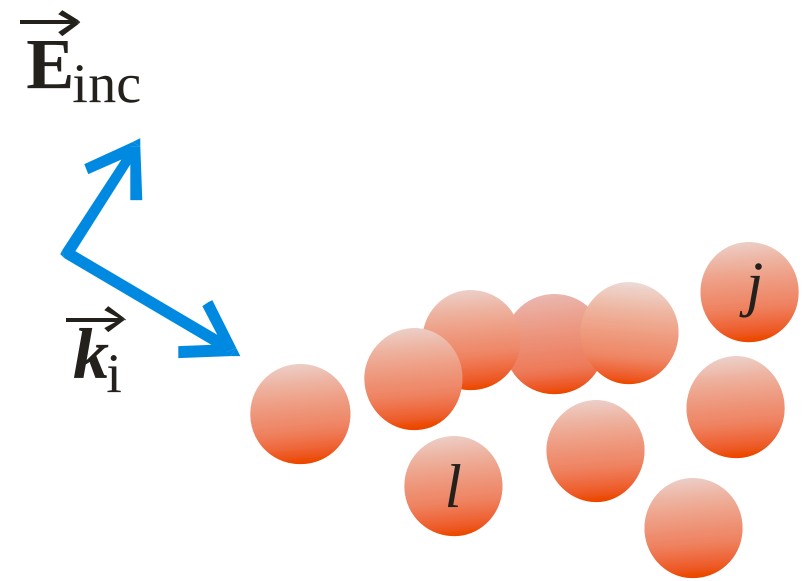
Multiple scattering

(Foldy-Lax excitation field formulation + T-matrix)



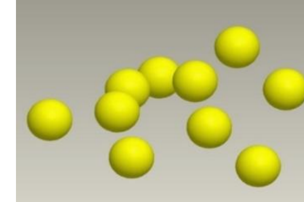
A**ggregate** of N objects subject to an incident field :

$$\begin{cases} e^{(j)} = a^{(j)} + \sum_{l=1, l\neq j}^{N} H^{(j,l)} . f^{(l)} \\ f^{(j)} = T^{(j)} . e^{(j)} \end{cases}$$



$$\begin{bmatrix} \left[T^{(1)}\right]^{-1} & -H^{(1,2)} & \dots & -H^{(1,N)} \\ -H^{(2,1)} & \left[T^{(2)}\right]^{-1} & \dots & -H^{(2,N)} \\ \vdots & & \ddots & \vdots \\ -H^{(N,1)} & -H^{(N,1)} & \dots & \left[T^{(N)}\right]^{-1} \end{bmatrix} \begin{bmatrix} f^{(1)} \\ f^{(2)} \\ \vdots \\ f^{(N)} \end{bmatrix} = \begin{bmatrix} a^{(1)} \\ a^{(2)} \\ \vdots \\ a^{(N)} \end{bmatrix}$$

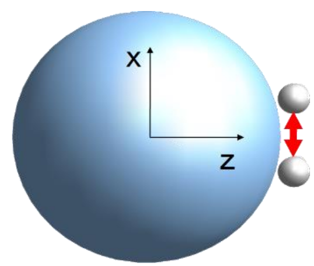
Matrix balancing

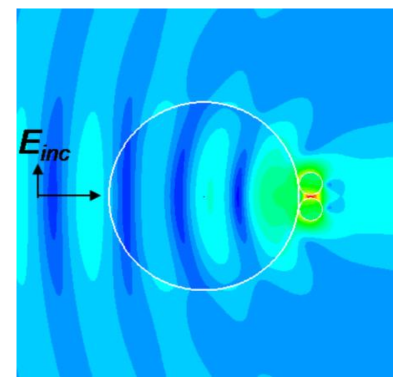


Matrix balanced equations are well conditioned for numerical inversion

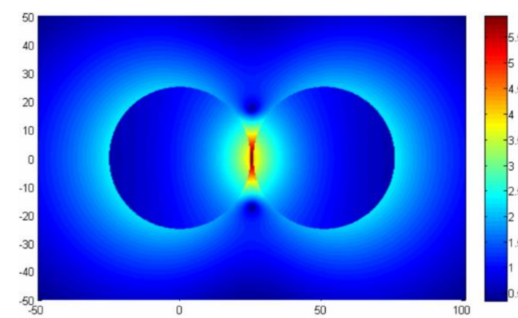
$$\begin{bmatrix} \left[\bar{T}^{(1)}\right]^{-1} & -\bar{H}^{(1,2)} & \cdots & -\bar{H}^{(1,N)} \\ -\bar{H}^{(2,1)} & \left[\bar{T}^{(2)}\right]^{-1} & \cdots & -\bar{H}^{(2,N)} \\ \vdots & & \ddots & \vdots \\ -\bar{H}^{(N,1)} & -\bar{H}^{(N,1)} & \cdots & \left[\bar{T}^{(N)}\right]^{-1} \end{bmatrix} \begin{bmatrix} f^{(1)} \\ f^{(2)} \\ \vdots \\ f^{(N)} \end{bmatrix} = \begin{bmatrix} a^{(1)} \\ a^{(2)} \\ \vdots \\ a^{(N)} \end{bmatrix}$$

$$T_n^{(e)} = -\frac{j_n(kR)}{h_n^{(+)}(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(kR) - \varphi_n(k_sR)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_sR)} \equiv \frac{j_n(kR)}{h_n^{(+)}(kR)} \bar{T}_n^{(e)}$$



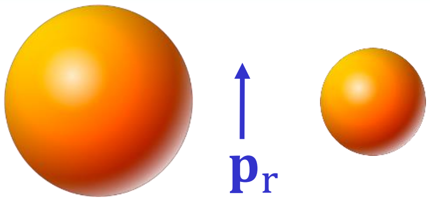


$$\varphi_n = \frac{[zj_n(z)]'}{j_n(z)} \qquad \varphi_n^{(+)}(x) = \frac{\left[xh_n^{(+)}(x)\right]'}{h_n^{(+)}(x)}$$



B. Stout et al., J. Opt. Soc. Am. **A**, 25 (2008)

Decay rates and directivity of quantum emitters near nano-antennas



$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \omega^2 \mu_0 \overline{\overline{\mathbf{G}}}(\vec{\boldsymbol{r}}, \vec{\mathbf{0}}) \cdot \vec{\boldsymbol{p}}$$

$$\langle P_{\boldsymbol{e}} \rangle = \frac{\omega^3}{2} \mu_0 \mathrm{Im}\left[\vec{\boldsymbol{p}}^* \cdot \overline{\overline{\mathbf{G}}}\left(\vec{\mathbf{0}}, \vec{\mathbf{0}}; \boldsymbol{\omega}\right) \cdot \vec{\boldsymbol{p}}\right] \propto \Gamma_e \qquad \tau \propto \frac{1}{\Gamma_e}$$

Quasi-analytic partial wave formulation :

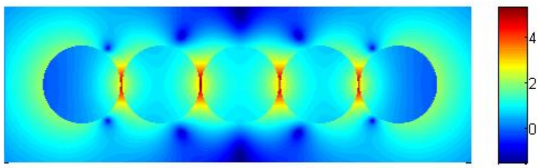
$$\frac{\Gamma_e}{\Gamma_0} = \frac{\langle P_{\boldsymbol{e}} \rangle}{\langle P_{\boldsymbol{e},\mathbf{0}} \rangle} = 1 + \frac{6\pi}{\mathrm{Re}\{k_b\}} \mathrm{Re}\left\{ k_b \sum_{j,l=1}^{N} f^{\dagger}.H^{(0,j)}.T^{(j,l)}.H^{(l,0)}.f \right\}$$

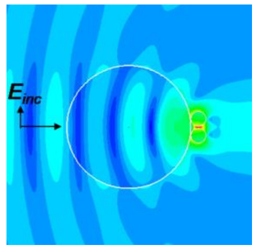
$$\frac{\Gamma_r}{\Gamma_0} = \frac{\langle P_{\boldsymbol{r}} \rangle}{\langle P_{\boldsymbol{r},\mathbf{0}} \rangle} = 1 + 6\pi \mathrm{Re}\left\{ k_b \sum_{i,j,k,l=1}^{N} \left[T^{(j,i)}.H^{(i,0)}.f\right]^{\dagger}.J^{(j,k)}.T^{(k,l)}.H^{(l,0)}.f \right\} + 12\,\pi\,\mathrm{Re}\left[\sum_{j,k=1}^{N} \left[J^{(k,0)}.f\right]^{\dagger}.T^{(k,j)}.H^{(j,0)}.f\right]$$

Method : B.Stout, et al. J. Opt. Soc. Am. B, 28, doi:10.1364/JOSAB.28.001213 (2011)

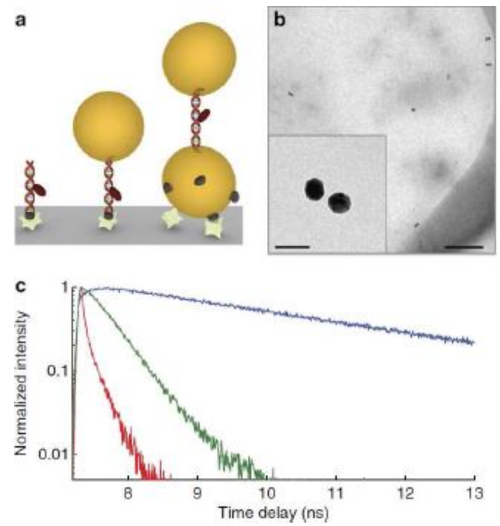
Multipole theory can simulate (Optical) Antenna properties

- Subwavelength field enhancements to efficiently excite quantum emitters



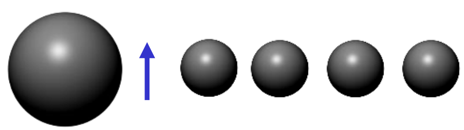


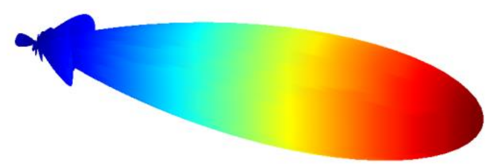
- Lifetime modification of quantum emitters



Nature Communications, 3 , 962 (2012)
Angewandte Chemie, 51 , (2012)
NANO Letters 11 , (2011)

- Directive emissions



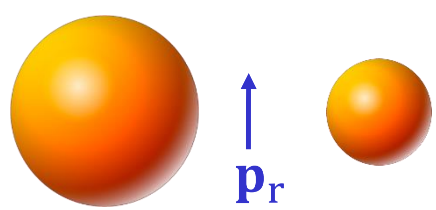


Optical receiving antenna ?

No equivalent in optics for co-axial to cables to extract energy from the antenna !

Need a sub-wavelength receiving element to convert light energy to another form !

(Receiving element can also be described via T-matrix)



T-matrix coefficients are constrained by the underlying physics !

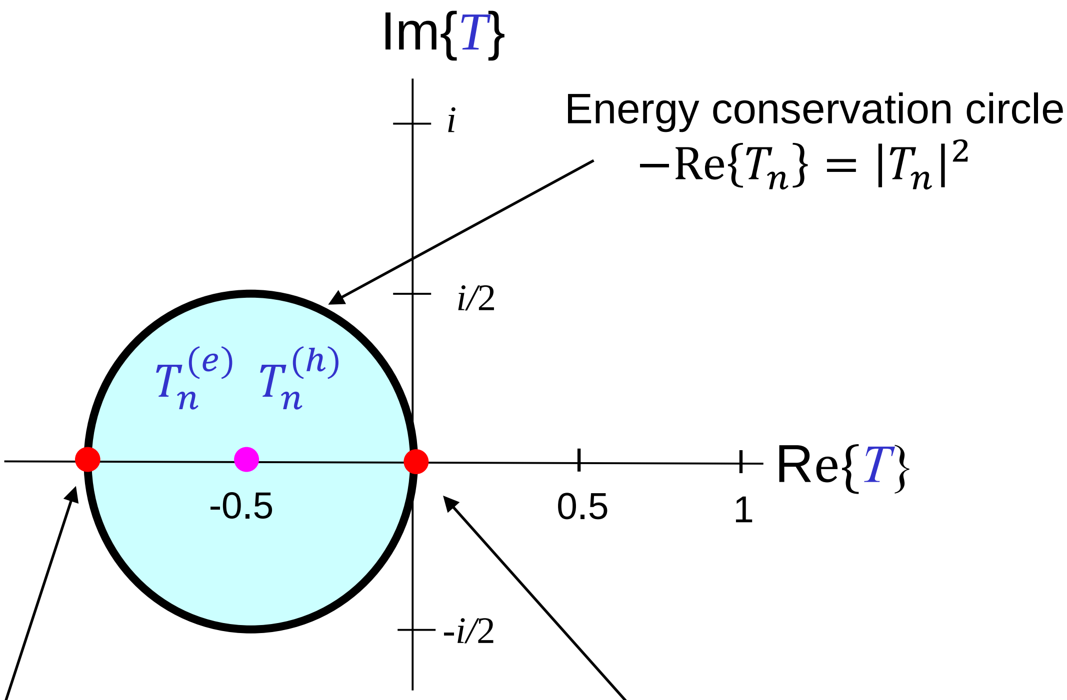
(spherically symmetric case) 

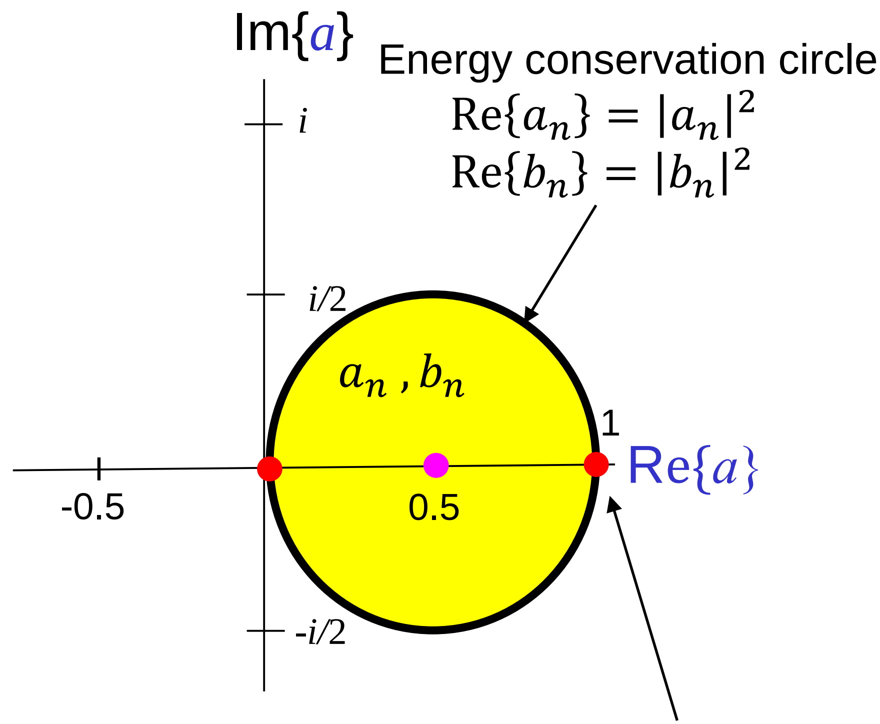
T-matrix coefficients

$T_n^{(e)}(\omega)$ and $T_n^{(h)}(\omega)$

Lorenz-Mie coefficients

$a_n(\omega)$ and $b_n(\omega)$





Unitary 'limit', $T_n^{(e,h)}$ = -1 'Invisible' $T_n^{(e,h)}$ = 0

Unitary 'limit', $a_n = 1$, b_n= 1

Cross sections in Mie theory

Valid for any scatterer

Extinction : $\sigma_e = -\frac{1}{k^2}\mathrm{Re}\left(f^\dagger . p\right) = -\frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\mathrm{Re}\left(t_n^{(e)} + t_n^{(h)}\right) = +\frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\mathrm{Re}\left(a_n + b_n\right)$

Scattering : $\sigma_s = \frac{1}{k^2} f^\dagger . f = \frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\left(\left|t_n^{(e)}\right|^2 + \left|t_n^{(h)}\right|^2\right) = \frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\left(\left|a_n\right|^2 + \left|b_n\right|^2\right)$

Absorption : $\sigma_a = \sigma_e - \sigma_s$

Exact electric dipole polarizability can be deduced from Mie theory !

$$\alpha(\omega) = 6\pi T_1^{(e)}/ik^3$$

Quasi-static limit – electric dipole

$$\lim_{x\to 0} j_n(x) \to \frac{x^n}{(2n+1)!!}$$

$$\lim_{x\to 0} \varphi_n^{(+)}(x) \to -n$$

$$x = kR$$

$$\lim_{x\to 0} h_n(x) \to \frac{(2n-1)!!}{i x^{n+1}}$$

$$\lim_{x\to 0} \varphi_n(x) \to n+1$$

$$\lim_{x\to 0} T_n^{(e)}(x) = \lim_{x\to 0} -\frac{j_n(x)}{h_n^{(+)}(x)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(x) - \varphi_n(\rho_s x)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(x) - \varphi_n(\rho_s x)} \to \frac{i x^{2n+1}(n+1)}{(2n-1)!!\,(2n+1)!!} \frac{\frac{\varepsilon_s}{\varepsilon} - 1}{\frac{\varepsilon_s}{\varepsilon} n + (n+1)}$$

Electric dipole term (n=1) dominates for small particle sizes :

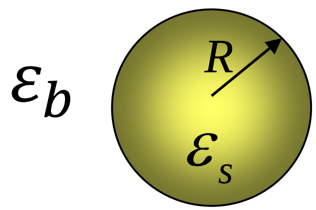
$$\lim_{kR\to 0} T_1^{(e)}(kR) \to ikR^3 \frac{2}{3} \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon}$$

Small particle → (plasmonics)

A particle much smaller than the wavelength generally interacts with incident light via its electric dipole moment

$$\vec{\mathbf{p}} = \epsilon_0 \varepsilon_b \alpha(\omega) \vec{\mathbf{E}}_{\mathrm{exc}}$$

$$\alpha(\omega) = 6\pi T_1^{(e)}/ik^3$$



$$\lim_{\omega\to 0} \alpha(\omega) = 4\pi R^3 \frac{\varepsilon_s - \varepsilon}{2\varepsilon + \varepsilon_s} \equiv \alpha_0$$

$$\sigma_{\mathrm{ext}} = k\,\mathrm{Im}\{\alpha(\omega)\}$$

$$\sigma_{\mathrm{scat}} = \frac{k^4 |\alpha(\omega)|^2}{6\pi}$$

$$\sigma_{\mathrm{abs}} = \sigma_{\mathrm{ext}} - \sigma_{\mathrm{scat}} \geq 0$$

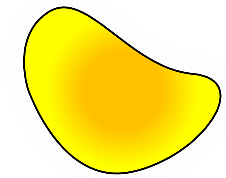
$$\mathrm{Im}\{\alpha\} \geq \frac{k^3 |\alpha|^2}{6\pi}$$

Unitary limit

$$|\alpha(\omega)| \leq \frac{6\pi}{k^3}$$

S-Matrix

(absorption/lasing and energy conservation/unitary limit)



The S–matrix relates the **outgoing part** of the **total field** to the **incoming part**

$$\vec{\mathbf{E}}_{\text{tot}}(k\vec{\boldsymbol{r}}) = \vec{\mathbf{E}}_{\text{exc}}(k\vec{\boldsymbol{r}}) + \vec{\mathbf{E}}_{\text{scat}}(k\vec{\boldsymbol{r}})$$

$$= \sum_{n,m}^{\infty} \left\{ \left[a_{n,m}^{(+,h)} \overrightarrow{\mathbf{M}}_{n,m}^{(+)}(k\vec{\boldsymbol{r}}) + a_{n,m}^{(+,e)} \overrightarrow{\mathbf{N}}_{n,m}^{(+)}(k\vec{\boldsymbol{r}}) \right] + \left[a_{n,m}^{(-,h)} \overrightarrow{\mathbf{M}}_{n,m}^{(-)}(k\vec{\boldsymbol{r}}) + a_{n,m}^{(-,e)} \overrightarrow{\mathbf{N}}_{n,m}^{(-)}(k\vec{\boldsymbol{r}}) \right] \right\}$$

$$a^{(+)} \equiv \overline{\overline{\mathrm{S}}}.\, a^{(-)} \quad \Rightarrow \quad \overline{\overline{\mathrm{S}}} = \overline{\overline{\mathrm{I}}} + 2\overline{\overline{\mathrm{T}}}$$

Energy conservation : $\Rightarrow$

S is unitary for lossless scatterers at real frequencies

$$\overline{\overline{\mathrm{S}}}^{\dagger}.\overline{\overline{\mathrm{S}}} = \overline{\overline{\mathrm{I}}} \quad \Rightarrow \quad \frac{1}{2}\left\{ \overline{\overline{\mathrm{T}}}^{\dagger} + \overline{\overline{\mathrm{T}}} \right\} = \overline{\overline{\mathrm{T}}}^{\dagger}.\overline{\overline{\mathrm{T}}}$$

Spherically symmetric particles :

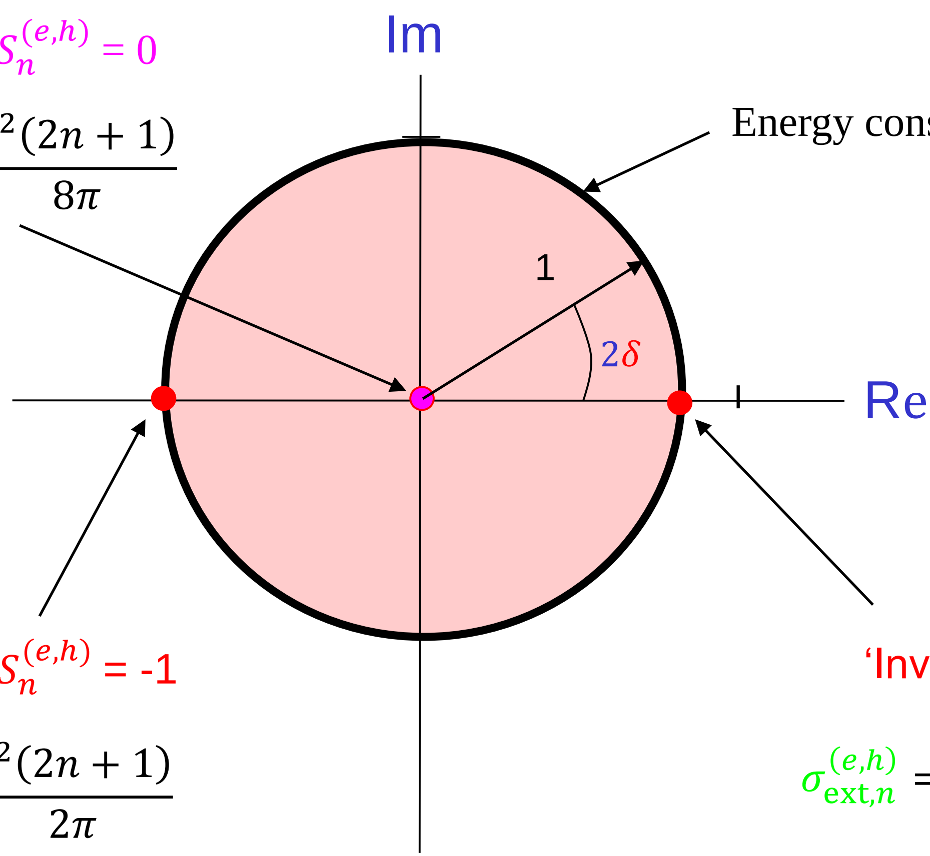
$$S_n^{(e)}(kR) = -\frac{h_n^{(-)}(kR)}{h_n^{(+)}(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(-)}(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

Limit behaviors correspond to cross section bounds

$$\sigma_{\text{scat},n}^{(e,h)} = \frac{\lambda^2(2n+1)}{8\pi}\left(\left|1 - S_n^{(e,h)}\right|^2\right) \qquad \sigma_{\text{ext},n}^{(e,h)} = \frac{\lambda^2(2n+1)}{4\pi}\text{Re}\left\{1 - S_n^{(e,h)}\right\} \qquad \sigma_{\text{abs},n}^{(e,h)} = \frac{\lambda^2(2n+1)}{8\pi}\left(1 - \left|S_n^{(e,h)}\right|^2\right)$$

Ideal absorption : $S_n^{(e,h)} = 0$

$$\sigma_{\text{scat},n}^{(e,h)} = \sigma_{\text{abs},n}^{(e,h)} = \frac{\lambda^2(2n+1)}{8\pi}$$



Energy conservation circle $|S_n^{(e,h)}| = 1$

$$S_n^{(e)} = e^{i2\delta_n^{(e)}} \;, \quad S_n^{(h)} = e^{i2\delta_n^{(h)}}$$

Unitary limit, $S_n^{(e,h)}$ = -1

$$\sigma_{\text{ext},n}^{(e,h)} = \sigma_{\text{scat},n}^{(e,h)} = \frac{\lambda^2(2n+1)}{2\pi}$$

'Invisible', $S_n^{(e,h)}$ = 1

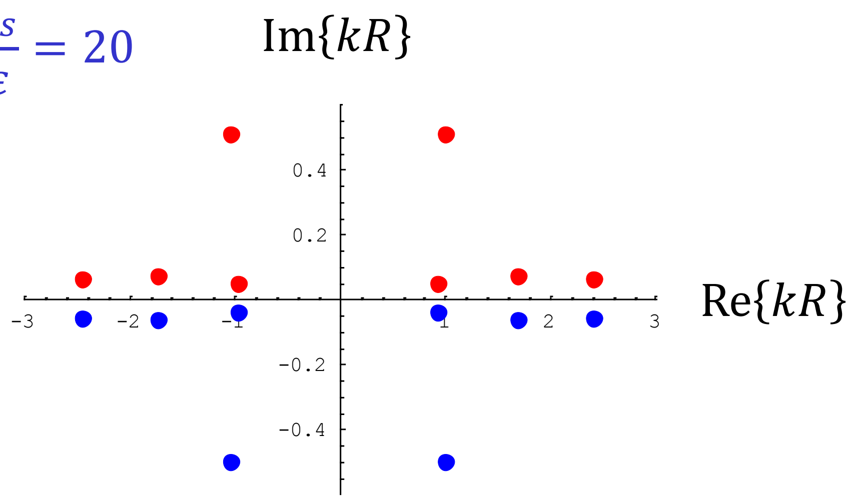
$$\sigma_{\text{ext},n}^{(e,h)} = \sigma_{\text{scat},n}^{(e,h)} = \sigma_{\text{abs},n}^{(e,h)} = 0$$

Simplicity of analytic structure

(Weierstrass factorization)

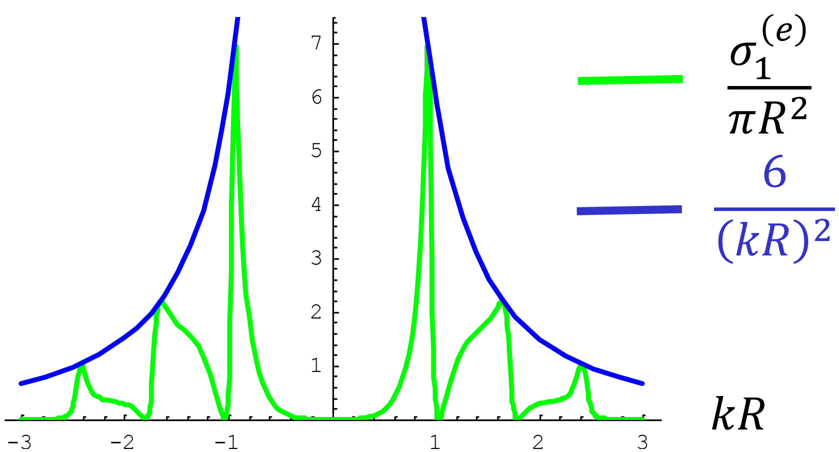
$$S_n(x) = A\exp(-2iBx)\prod_{\alpha}\frac{x - x_{z,\alpha}}{x - x_{p,\alpha}} = A\exp(-2iBx)\prod_{\alpha=1}^{\infty}\frac{(x - x_{z,\alpha})(x + x_{z,\alpha}^{*})}{(x - x_{p,\alpha})(x + x_{p,\alpha}^{*})}$$

$$\frac{\varepsilon_s}{\varepsilon} = 20$$



$$x \equiv kR = \frac{2\pi\omega R\sqrt{\varepsilon_b}}{c}$$

$$S_n^{(e)}(kR) = -\frac{h_n^{(-)}(kR)}{h_n^{(+)}(kR)}\,\frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(-)}(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$



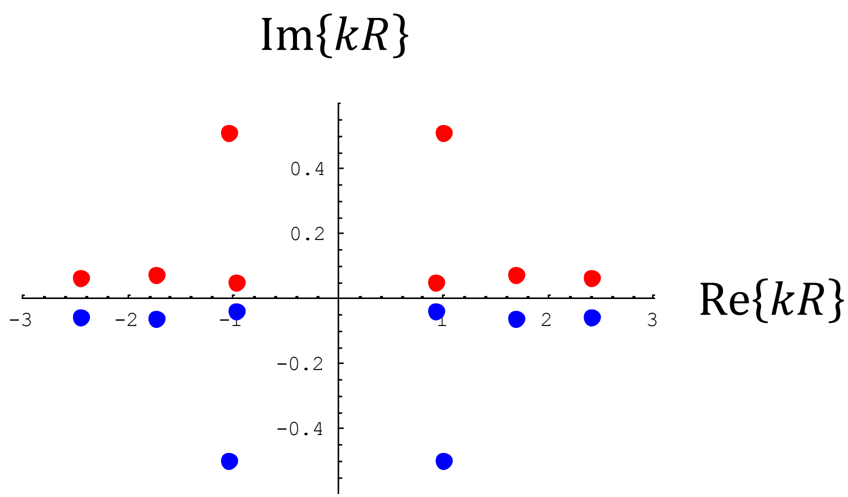
Unitary limit

Ideal absorption – 'Coherent perfect absorption'
Ultimate absorption ?

$$S_n^{(e,h)} = 0$$

$$s_n^{(e)}(kR) = -\frac{h_n^{(-)}(kR)}{h_n^{(+)}(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(-)}(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

$$\frac{\varepsilon_s}{\varepsilon} = 20$$



IA can be found by solving a complex transcendental equation

$$\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(-)}(kR) - \varphi_n(k_s R) = 0$$

V.Grigoriev, *et al* ACS Photonics 10.1021/ph500456w ACS Photonics (2015)

Ideal absorption is possible for both electric and magnetic modes in high index materials

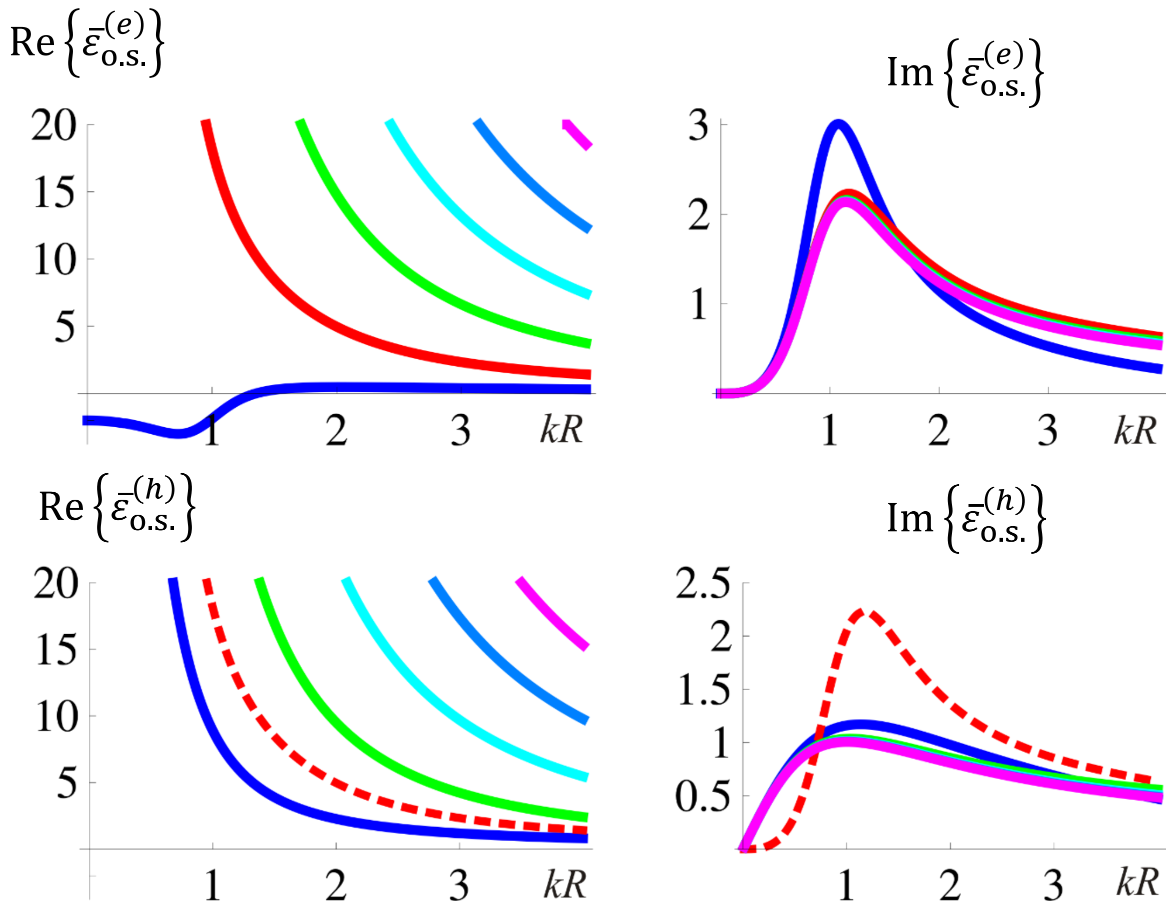
$$\bar{\varepsilon} \equiv \frac{\varepsilon_s}{\varepsilon}$$

Electric dipole IA modes

$$\frac{\varepsilon}{\varepsilon_s}\varphi_1(k_sR) = \varphi_1^{(-)}(kR)$$

Magnetic dipole IA modes

$$\frac{\mu}{\mu_s}\varphi_1(k_sR) = \varphi_1^{(-)}(kR)$$



$$S_n^{(e)}(kR) = -\frac{h_n^{(-)}(kR)}{h_n^{(+)}(kR)}\frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(-)}(kR) - \varphi_n(k_sR)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_sR)}$$

$$S_n^{(h)}(kR) = -\frac{h_n^{(-)}(kR)}{h_n^{(+)}(kR)}\frac{\frac{\mu_s}{\mu}\varphi_n^{(-)}(kR) - \varphi_n(k_sR)}{\frac{\mu_s}{\mu}\varphi_n^{(+)}(kR) - \varphi_n(k_sR)}$$

IA in homogeneous particles and realistic materials ?

Yes, but only for certain sizes and frequencies

IA electric dipole - exact calculation

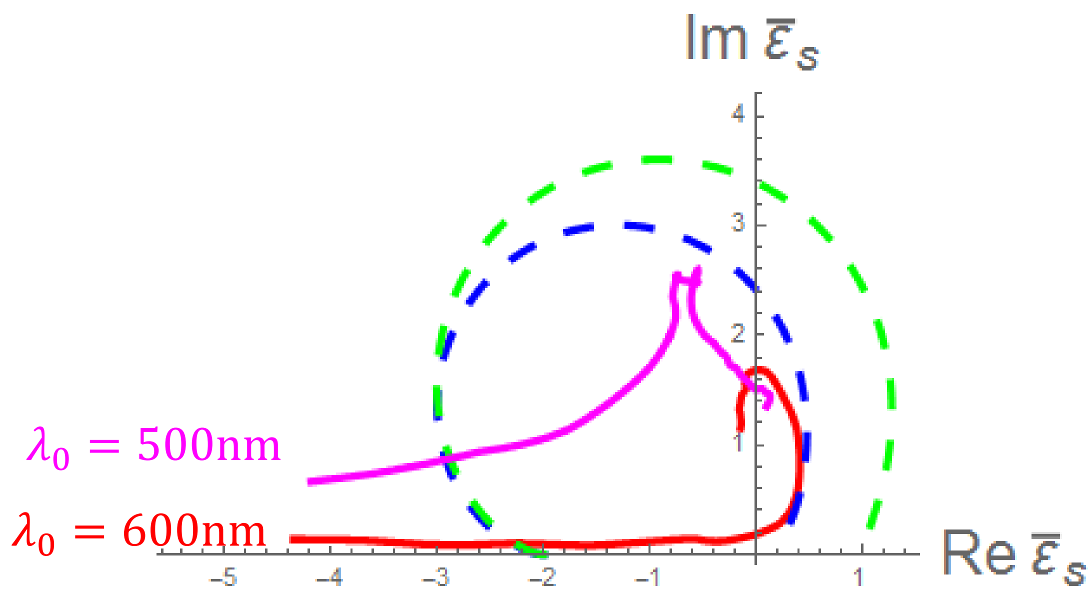
IA – electric dipole point-like approximation

Silver –

Gold –

Experimental dispersion curves : Johnson & Christy

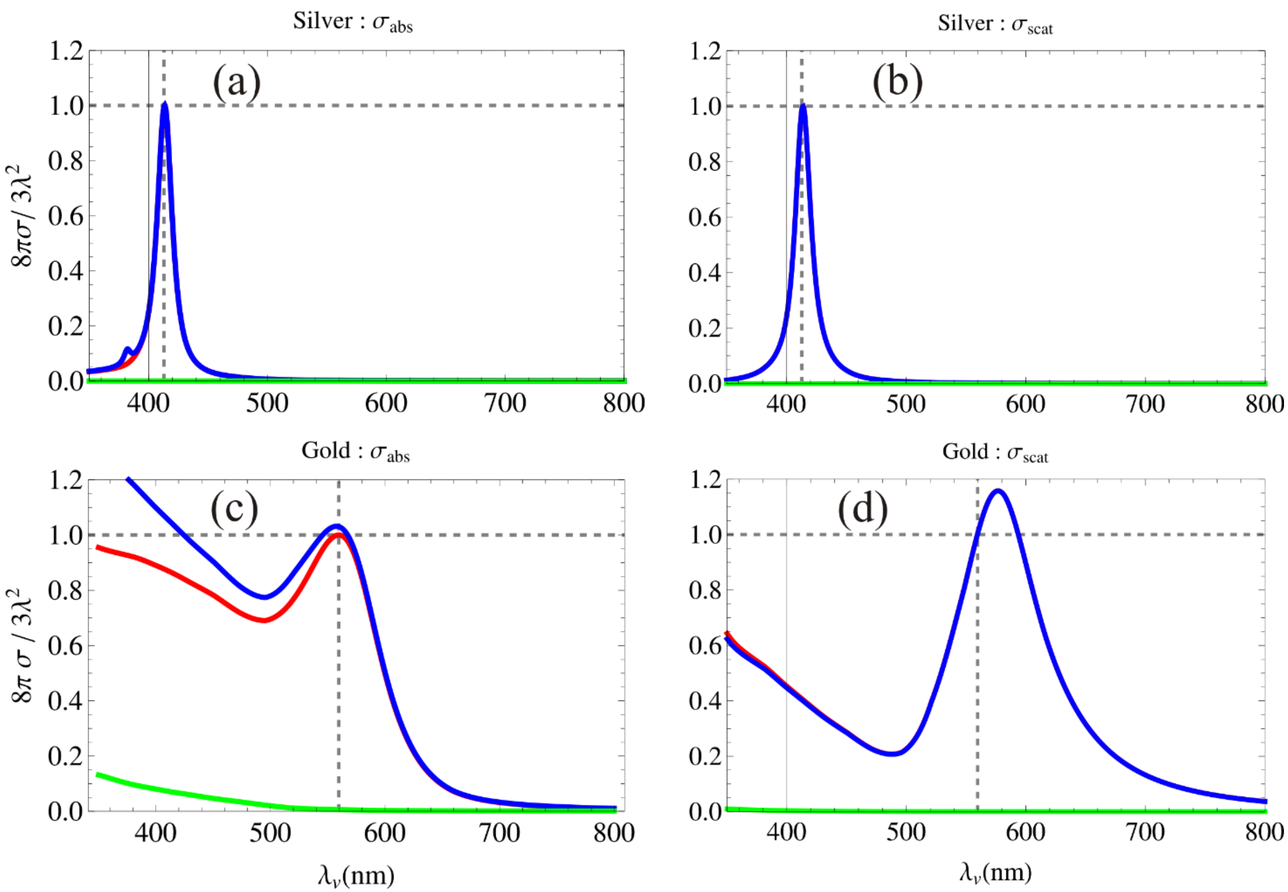
Polymer background : N_b=1.5



Ideal absorption can occur in subwavelength particles
(Using realistic materials like gold and silver)

in a N_b=1.5 background medium

Absorption cross section σ_{abs} Scattering cross section σ_{scat}



$D_{o.s.} = 30.6\text{nm}$

$kR_{o.s.} = 0.35$

$\lambda_{o.s.} = 413\text{nm}$

$\varepsilon_{o.s.} = -5.1 + i\,0.22$

$D_{o.s.} = 76.2\text{nm}$

$kR_{o.s.} = 0.64$

$\lambda_{o.s.} = 560\text{nm}$

$\varepsilon_{o.s.} = -6.6 + i\,1.9$

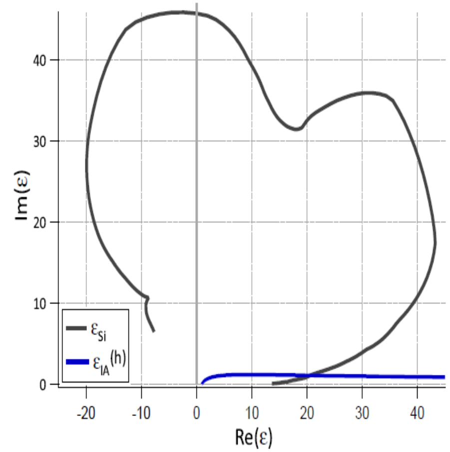
Electric dipole contribution to cross section

Cross section

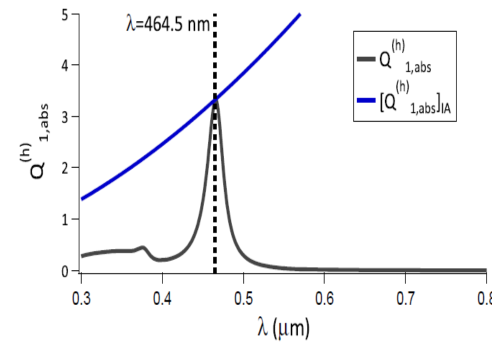
Silicon particles should also exhibit Ideal absorption

Magnetic dipole IA modes

$$S_n^{(h)}(kR) = -\frac{h_n^{(-)}(kR)}{h_n^{(+)}(kR)}\,\frac{\frac{\mu_s}{\mu}\varphi_n^{(-)}(kR) - \varphi_n(k_sR)}{\frac{\mu_s}{\mu}\varphi_n^{(+)}(kR) - \varphi_n(k_sR)}$$

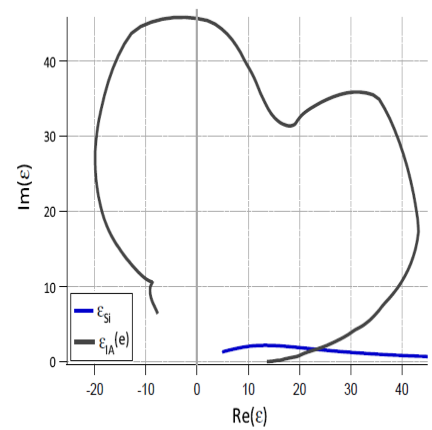


kR	λ (nm)	R (nm)	ε_{Si}
0.672	464.5	50	$20.57 + i1.06$

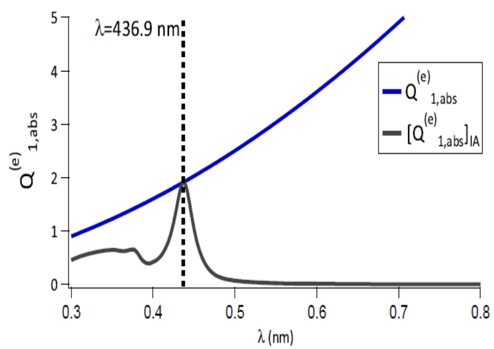


Electric dipole IA modes

$$S_n^{(e)}(kR) = -\frac{h_n^{(-)}(kR)}{h_n^{(+)}(kR)}\,\frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(-)}(kR) - \varphi_n(k_sR)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_sR)}$$



kR	$\lambda(nm)$	R (nm)	ε_{Si}
0.886	436.9	61.6	$23.3 + i1.7$



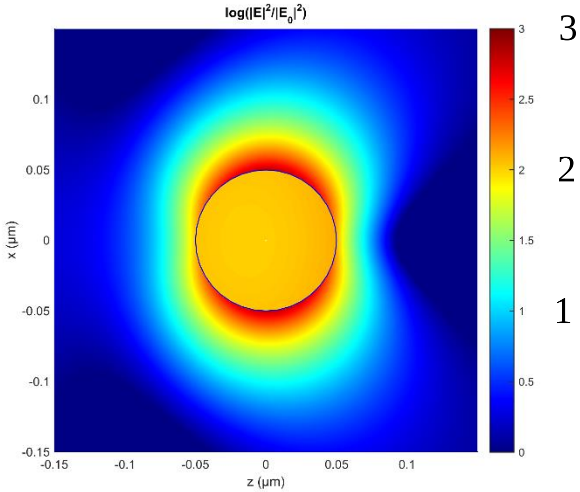
Electric near-field enhancements

$\frac{D}{\lambda} \cong \frac{1}{8}$ $kR = 0.4$

$\frac{\varepsilon_s}{\varepsilon} = -2.93 + i\,1.85$

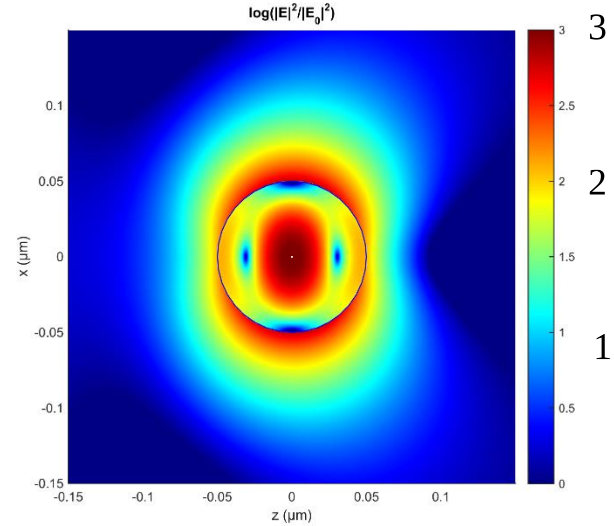
$\|\vec{E}\|^2 / \|\vec{E}_0\|^2$ Log scale

$\frac{\varepsilon_s}{\varepsilon} = 123.87 + i\,0.15$



Ideal Absorption limit

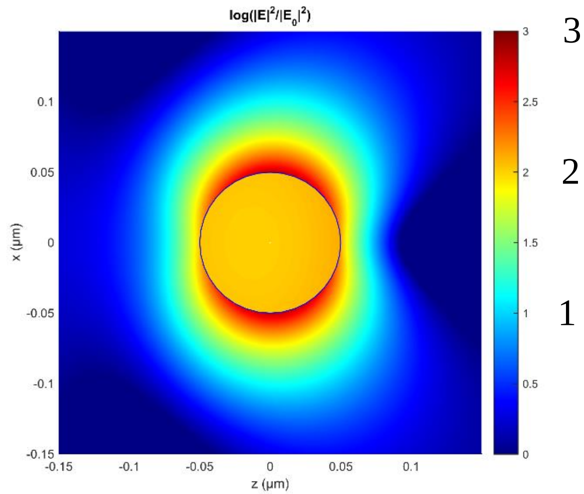
$\sigma_{\text{abs}} = \sigma_{\text{scat}} \sim \lambda^2/8$

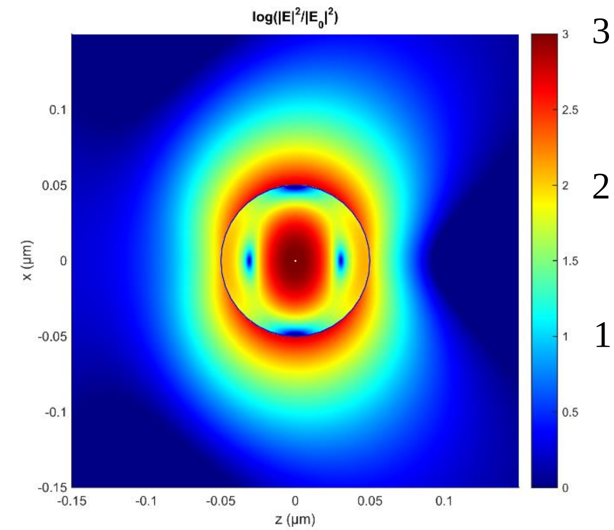


Electric near-field enhancements

$\frac{D}{\lambda} \cong \frac{1}{8}$ $kR = 0.4$

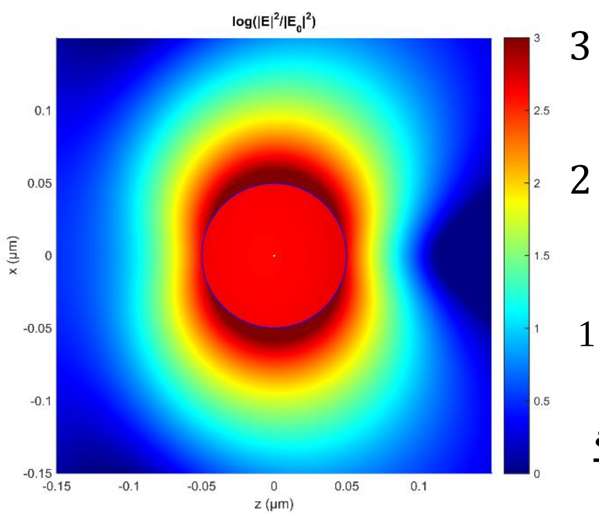
$\frac{\varepsilon_S}{\varepsilon} =$ - 2.93+i 1.85 $\|\vec{E}\|^2/\|\vec{E}_0\|^2$ Log scale $\frac{\varepsilon_S}{\varepsilon} =$123.87+ i 0.15





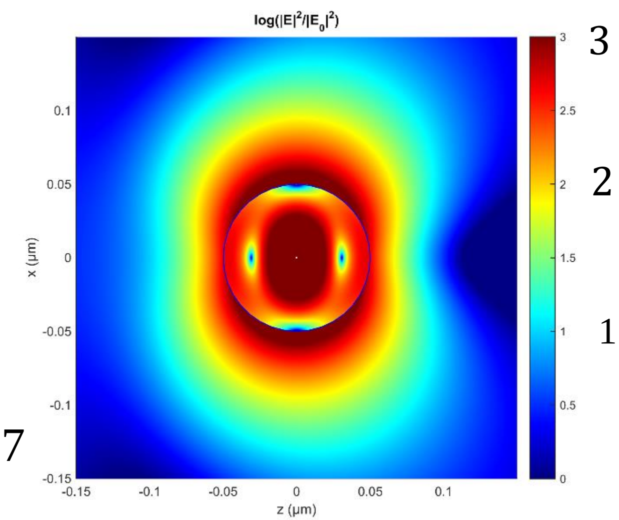
Ideal Absorption limit

$\sigma_{\text{abs}} = \sigma_{\text{scat}} \sim \lambda^2/8$



Unitary limit

$\sigma_{\text{scat}} \sim \lambda^2/2$

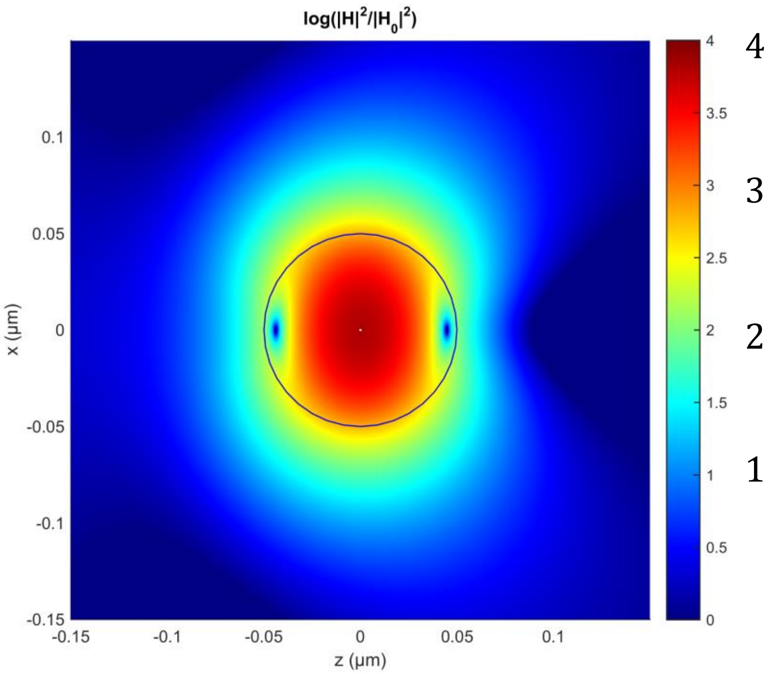


$\frac{\varepsilon_S}{\varepsilon} =$ - 2.4 $\frac{\varepsilon_S}{\varepsilon} =$123.87

Magnetic near-field enhancements

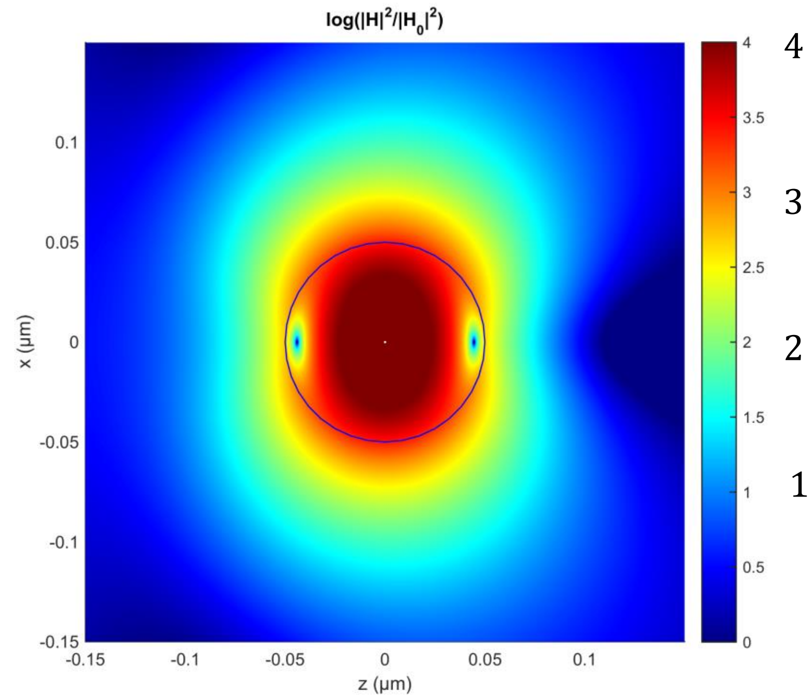
$\frac{D}{\lambda} \cong \frac{1}{8}$ $kR = 0.4$

$\frac{\varepsilon_s}{\varepsilon} = 59{,}930 + i\,0.72$ $\|\vec{H}\|^2/\|\vec{H}_0\|^2$ Log scale $\frac{\varepsilon_s}{\varepsilon} = 59{,}938$



Ideal Absorption limit

$\sigma_{\text{abs}} = \sigma_{\text{scat}} \sim \lambda^2/8$



Unitary limit

$\sigma_{\text{scat}} \sim \lambda^2/2$

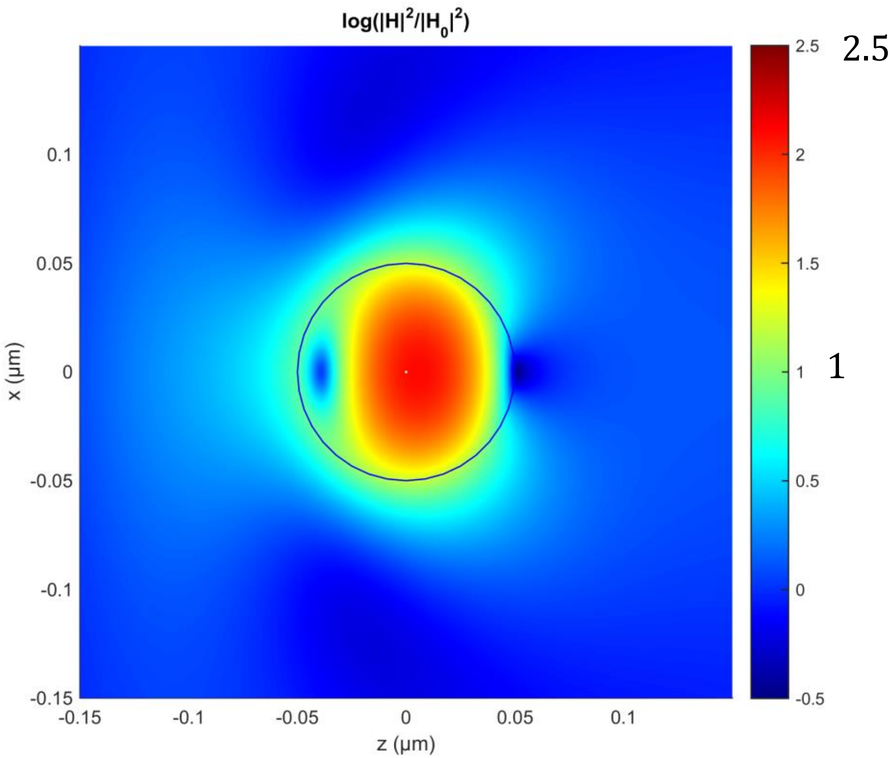
Magnetic near-field enhancements

$\frac{D}{\lambda} \cong \frac{1}{4}$

$\|\vec{H}\|^2 / \|\vec{H}_0\|^2$ Log scale

$kR = 0.8$

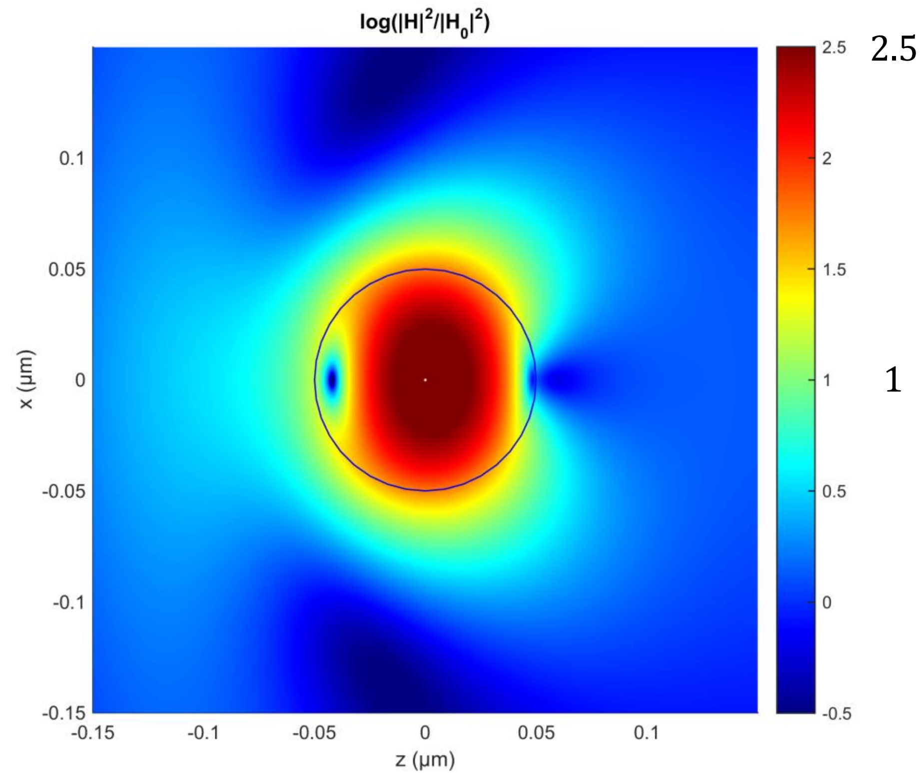
$\frac{\varepsilon_s}{\varepsilon} = 14.18\ i\ 1.09$



Ideal Absorption limit

$\sigma_{\text{abs}} = \sigma_{\text{scat}} \sim \lambda^2/8$

$\frac{\varepsilon_s}{\varepsilon} = 14.26$



Unitary limit

$\sigma_{\text{scat}} \sim \lambda^2/2$

Conclusions

- Multipole theory helps understand sub-wavelength dielectric antennas

- The S matrix is well adapted to studying fundamental limits of light-matter interactions like ideal absorption, unitarity, invisibility , and lasing.

- Limit behaviors can occur in either electric and magnetic modes of particles.

- Ideal Absorption with realistic materials at arbitrary frequencies will often require coated particle designs in order to introduce additional parameters.

http://www.fresnel.fr/perso/stout

K-Matrix (Reaction matrix)
adapted to studying energy conservation

The K–matrix relates the **regular part** of the **total field** to its **diverging part**

$$\vec{\mathbf{E}}_{\text{tot}}(k\vec{\boldsymbol{r}}) = \vec{\mathbf{E}}_{\text{exc}}(k\vec{\boldsymbol{r}}) + \vec{\mathbf{E}}_{\text{scat}}(k\vec{\boldsymbol{r}})$$

$$= \sum_{n,m}^{\infty} \left\{ \left[r_{n,m}^{(h)} \vec{\mathbf{M}}_{n,m}^{(1)}(k\vec{\boldsymbol{r}}) + r_{n,m}^{(e)} \vec{\mathbf{N}}_{n,m}^{(1)}(k\vec{\boldsymbol{r}}) \right] + \left[d_{n,m}^{(h)} \vec{\mathbf{M}}_{n,m}^{(2)}(k\vec{\boldsymbol{r}}) + d_{n,m}^{(e)} \vec{\mathbf{N}}_{n,m}^{(2)}(k\vec{\boldsymbol{r}}) \right] \right\}$$

$$d \equiv \overline{\overline{K}}.r$$

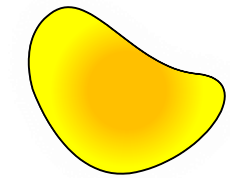
$$\overline{\overline{\mathrm{T}}} = -i\overline{\overline{K}}.\left(\overline{\overline{\mathrm{T}}} + \overline{\overline{\mathrm{I}}}\right) \qquad \overline{\overline{\mathrm{T}}}^{-1} = i\overline{\overline{K}}^{-1} - \overline{\overline{\mathrm{I}}}$$

$$\overline{\overline{K}} = i\left(\overline{\overline{\mathrm{S}}} - \overline{\overline{\mathrm{I}}}\right).\left(\overline{\overline{\mathrm{I}}} + \overline{\overline{\mathrm{S}}}\right)^{-1} \qquad \overline{\overline{\mathrm{S}}} = \left(\overline{\overline{\mathrm{I}}} - i\overline{\overline{K}}\right).\left(\overline{\overline{\mathrm{I}}} + i\overline{\overline{K}}\right)^{-1}$$

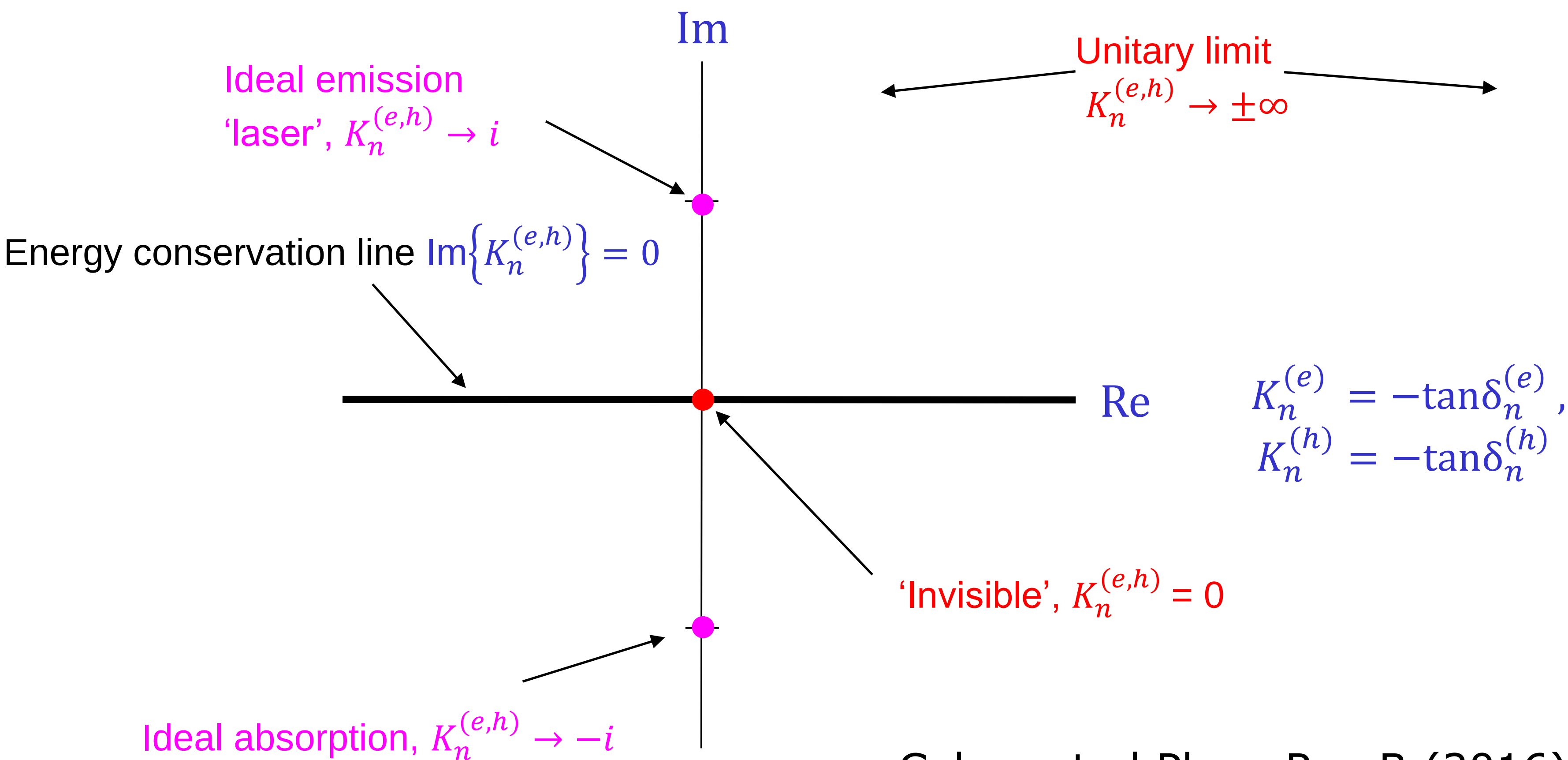
Caley Transform

$$K_n^{(e)} = -\tan\delta_n^{(e)},$$
$$K_n^{(h)} = -\tan\delta_n^{(h)}$$

K-Matrix is Hermitian for a lossless scatterer
(adapted to studying energy conservation and limit behaviors)



$$\overline{\overline{K}}^{\dagger} = \overline{\overline{K}} \quad : \text{Energy conservation}$$

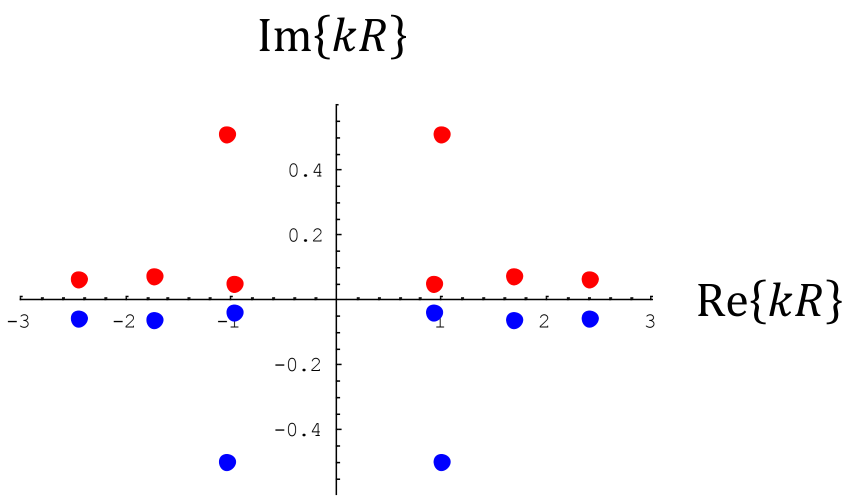


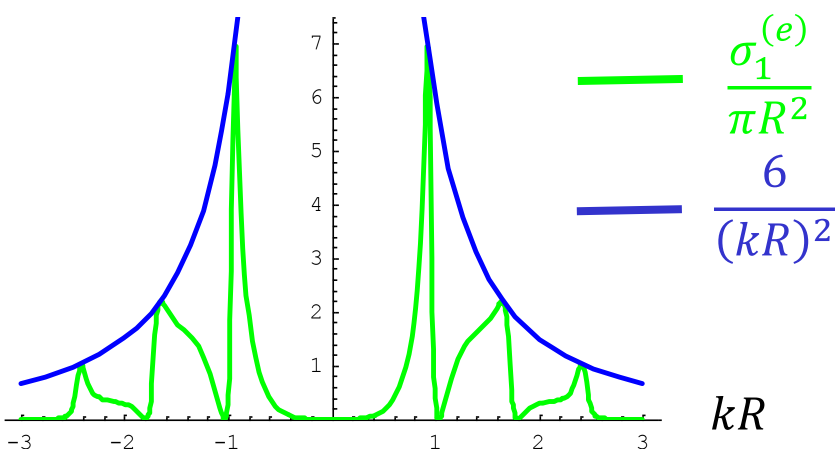
Colom et al Phys. Rev B (2016)

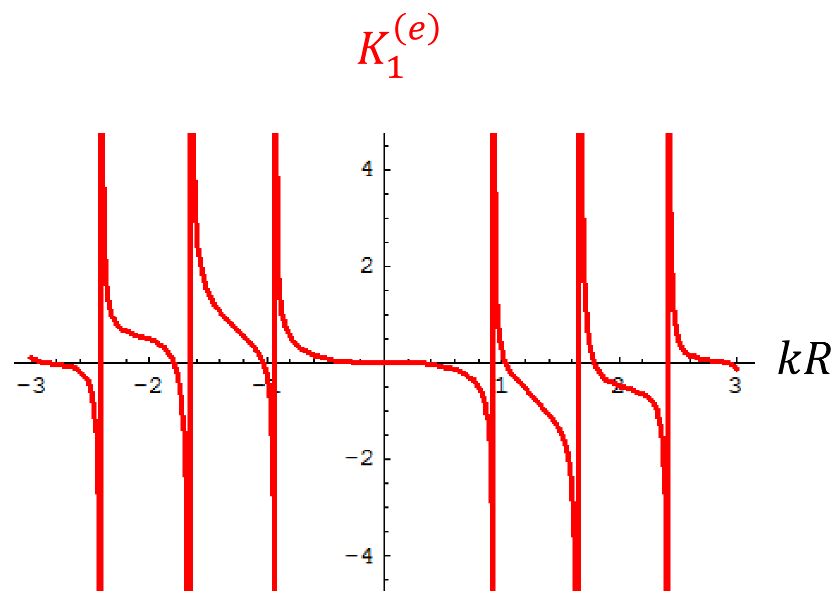
Simplicity of analytic structure

(Weierstrass factorization)

$$S_n(x) = A\exp(-2iBx)\prod_{\alpha}\frac{x - x_{z,\alpha}}{x - x_{p,\alpha}} = A\exp(-2iBx)\prod_{\alpha=1}^{\infty}\frac{(x - x_{z,\alpha})(x + x_{z,\alpha}^{*})}{(x - x_{p,\alpha})(x + x_{p,\alpha}^{*})} \qquad x \equiv kR$$







K-Matrix (Reaction matrix)

Gives access to all limit behaviors of light-matter interactions

Spherically symmetric particles :

$K_n^{(e,h)} \to \infty \quad \Rightarrow \quad S_n^{(e,h)} = T_n^{(e,h)} = -1$ ⟶ Unitary limit

$K_n^{(e,h)} = 0 \quad \Rightarrow \quad S_n^{(e,h)} \to 1 \;,\; T_n^{(e,h)} \to 0$ ⟶ Invisible

} lossless

$K_n^{(e,h)} = i \quad \Rightarrow \quad S_n^{(e,h)} \to \infty \;,\; T_n^{(e,h)} \to \infty$ ⟶ Emission - lasing } gain

$K_n^{(e,h)} = -i \quad \Rightarrow \quad S_n^{(e,h)} \to 0 \;,\; T_n^{(e,h)} \to -\frac{1}{2}$ ⟶ Ideal absorption } loss

Other formulations of physical limits

$$\sigma_{\text{ext}} = k\,\mathrm{Im}\{\alpha(\omega)\}$$

$$\sigma_{\text{scat}} = \frac{k^4|\alpha(\omega)|^2}{6\pi}$$

↑ $\vec{\mathbf{p}} = \epsilon_0 \varepsilon_b \alpha(\omega) \vec{\mathbf{E}}_{\text{exc}}$

$$\alpha(\omega) = 6\pi T_1^{(e)}/ik^3$$

Unitary limit : $\sigma_{\text{ext}} \leq \frac{3\lambda^2}{2\pi}$

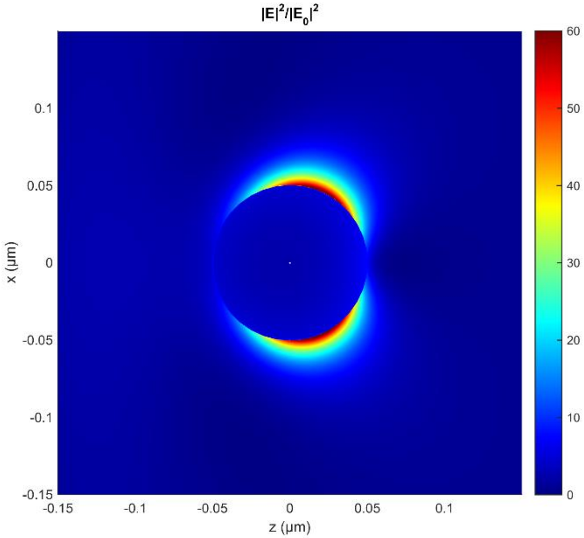
Unitarity : $\mathrm{Im}\{\alpha\} \geq \frac{k^3|\alpha|^2}{6\pi}$

$$\sigma_{\text{ext}} \leq \frac{3\lambda^2}{2\pi} = \frac{k|\alpha|^2}{\mathrm{Im}\{\alpha\}}$$

Electric near-field enhancements

$\frac{D}{\lambda} \cong \frac{1}{4}$ $\qquad$ $kR = 0.8$

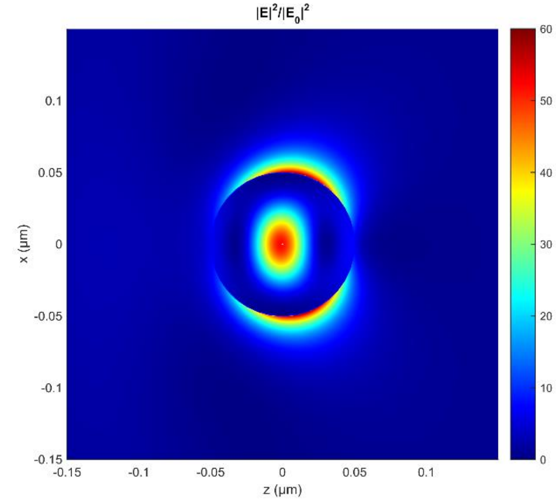
$\frac{\varepsilon_s}{\varepsilon} = -3.78$ $\qquad$ $\|\vec{E}\|^2/\|\vec{E}_0\|^2$ $\qquad$ $\frac{\varepsilon_s}{\varepsilon} = 28.43$



60

Unitary limit

$\sigma_{\text{scat}} \sim \lambda^2/2$



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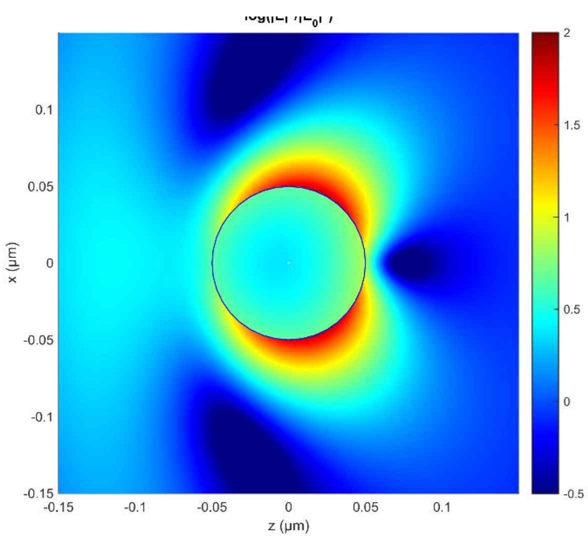
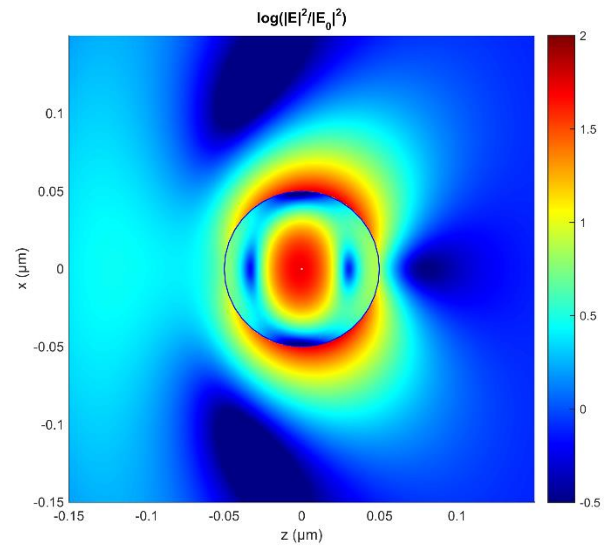


Log scale



Translation-addition theorem
(scalar wave case)

Scalar wave translation-addition matrices

$$\alpha_{\nu,\mu;n,m}\left(k\vec{\mathbf{r}}_0\right)=4\pi i^{\nu-m}\sum_{q=|n-\nu|}^{q=n+\nu} i^q\, 3Y\left(n,m,\nu,\mu;q\right)h_q\left(kr_0\right)Y_{q,m-\mu}\left(\theta_0,\phi_0\right)$$

$$\beta_{\nu,\mu;n,m}\left(k\vec{\mathbf{r}}_0\right)=Rg\left\{\alpha_{\nu,\mu;n,m}\left(k\vec{\mathbf{r}}_0\right)\right\}=4\pi i^{\nu-m}\sum_{q=|n-\nu|}^{q=n+\nu} i^q\, 3Y\left(n,m,\nu,\mu;q\right)j_q\left(kr_0\right)Y_{q,m-\mu}\left(\theta_0,\phi_0\right)$$

Looks horrible but relatively easy to derive in k-space !

$$3Y\left(\nu,\mu;n,m;q\right)\equiv\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi Y_{n,m}\left(\theta,\phi\right)Y^*_{\nu,\mu}\left(\theta,\phi\right)Y^*_{q,m-\mu}\left(\theta,\phi\right)$$

$$=\left(-\right)^m\left[\frac{\left(2n+1\right)\left(2\nu+1\right)\left[2q+1\right]}{4\pi}\right]^{1/2}\begin{pmatrix} n & \nu & q \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} n & \nu & q \\ -m & \mu & \mu-m\end{pmatrix}$$

Translation-addition theorem
(vector wave case)

$$\vec{\mathbf{\Psi}}^t(k\vec{\mathbf{r}}) = \vec{\mathbf{\Psi}}^t(k\vec{\mathbf{r}}').J(k\vec{\mathbf{r}}_0) \qquad r' > r_0$$

$$\vec{\mathbf{\Psi}}^t(k\vec{\mathbf{r}}) = Rg\left\{\vec{\mathbf{\Psi}}^t(k\vec{\mathbf{r}}')\right\}.H(k\vec{\mathbf{r}}_0) \qquad r' < r_0$$

$$Rg\left\{\vec{\mathbf{\Psi}}^t(k\vec{\mathbf{r}})\right\} = Rg\left\{\vec{\mathbf{\Psi}}^t(k\vec{\mathbf{r}}')\right\}.J(k\vec{\mathbf{r}}_0) \qquad \forall\left|\vec{\mathbf{r}}_0\right|$$

$$H(k\vec{\mathbf{r}}_0) = \begin{bmatrix} A_{\nu\mu,nm} & B_{\nu\mu,nm} \\ B_{\nu\mu,nm} & A_{\nu\mu,nm} \end{bmatrix}$$

$$J(k\vec{\mathbf{r}}_0) = Rg\left\{\begin{bmatrix} A_{\nu\mu,nm} & B_{\nu\mu,nm} \\ B_{\nu\mu,nm} & A_{\nu\mu,nm} \end{bmatrix}\right\}$$

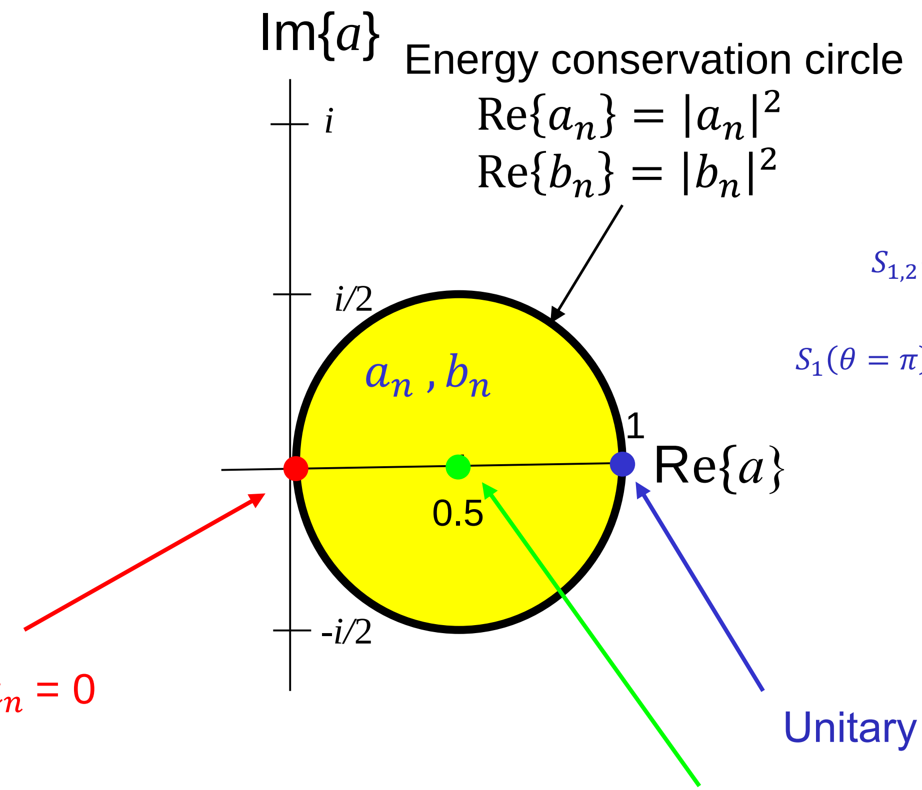
$$\begin{aligned} A_{\nu\mu,nm} = \frac{1}{2}\sqrt{\frac{1}{\nu(\nu+1)n(n+1)}}\Big[&2\mu m\alpha_{\nu,\mu;n,m} \\ &+\sqrt{(n-m)(n+m+1)}\sqrt{(\nu-\mu)(\nu+\mu+1)}\alpha_{\nu,\mu+1;n,m+1} \\ &+\sqrt{(n+m)(n-m+1)}\sqrt{(\nu+\mu)(\nu-\mu+1)}\alpha_{\nu,\mu-1;n,m-1}\Big] \end{aligned}$$

$$\begin{aligned} B_{\nu\mu,nm} = -i\frac{1}{2}\sqrt{\frac{2\nu+1}{2\nu-1}\frac{1}{\nu(\nu+1)n(n+1)}}\Big[&2m\sqrt{(\nu-\mu)(\nu+\mu)}\alpha_{\nu-1,\mu;n,m} \\ &+\sqrt{(n-m)(n+m+1)}\sqrt{(\nu-\mu)(\nu+\mu-1)}\alpha_{\nu-1,\mu+1;n,m+1} \\ &-\sqrt{(n+m)(n-m+1)}\sqrt{(\nu+\mu)(\nu-\mu-1)}\alpha_{\nu-1,\mu-1;n,m-1}\Big] \end{aligned}$$

Possible values of the Mie coefficients limited by the underlying physics !

Mie Coefficients

$a_n(\omega)$ and $b_n(\omega)$



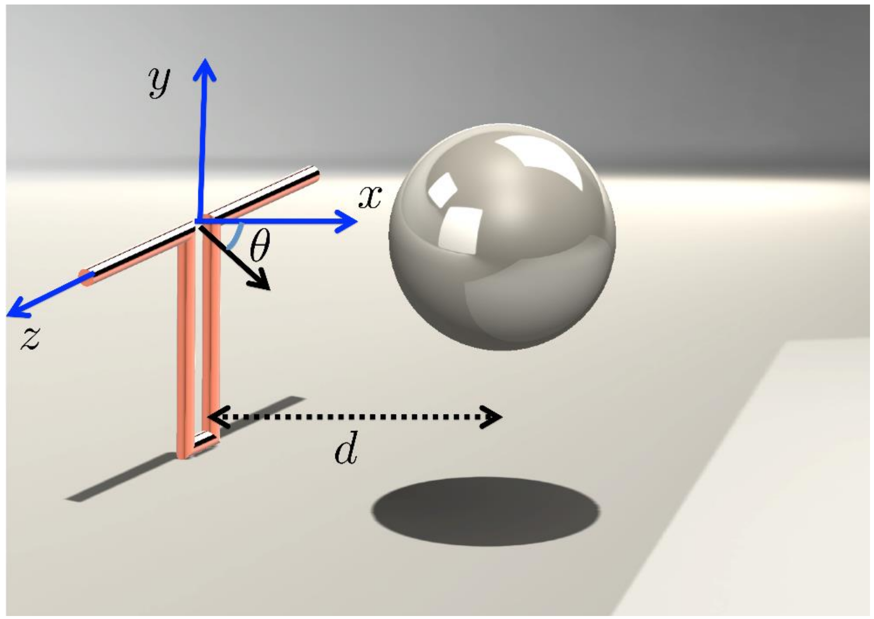
$$S_{1,2}(\theta = 0) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)[a_n + b_n]$$

$$S_1(\theta = \pi) = -S_2(\theta = \pi) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)[a_n - b_n]$$

Analogue Antennas in the micro-wave regime (low index : $N \cong 2$)

Nature - Scientific Reports, 3 3063 (2013)

Multimode interference in a dielectric sphere can yield directive emissions

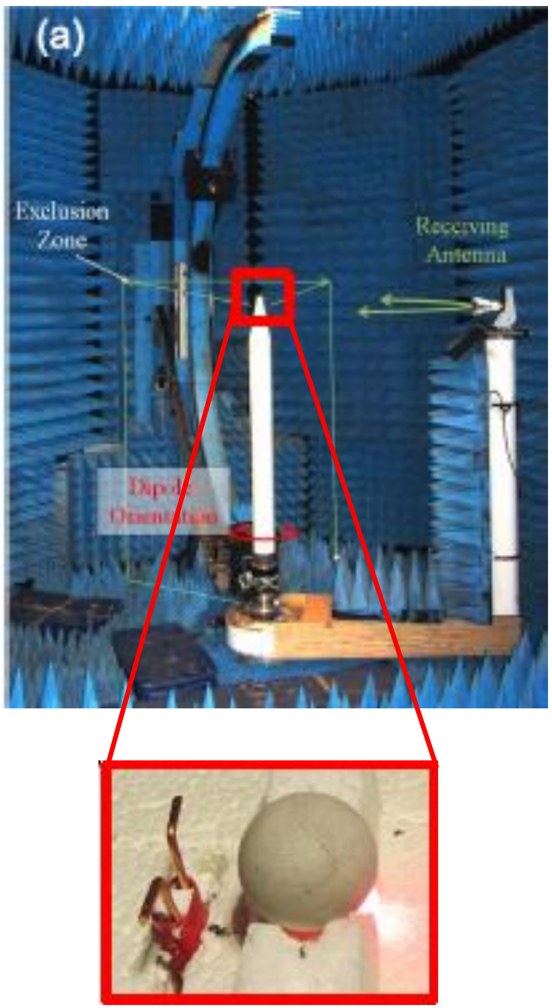


$$I_r(\theta,0) = I_0\left[\left(e^{i\varphi} + \gamma_1^{(a)}\alpha_1^{(e)} + \gamma_2^{(b)}\alpha_2^{(h)}\right)\cos\theta + 2\gamma_2^{(a)}\alpha_2^{(e)}\cos^2\theta + \gamma_1^{(b)}\alpha_1^{(h)} - \gamma_2^{[a]}\alpha_2^{(e)}\right]^2$$

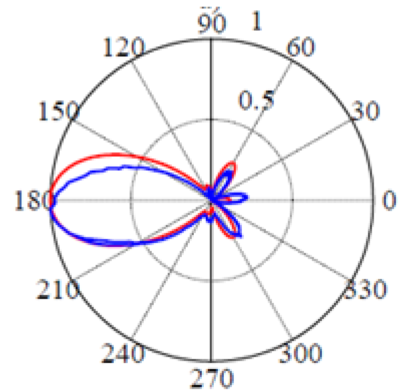
Analogue Antennas in the micro-wave regime (low index : $N \cong 2$)

Nature - Scientific Reports, 3 3063 (2013)

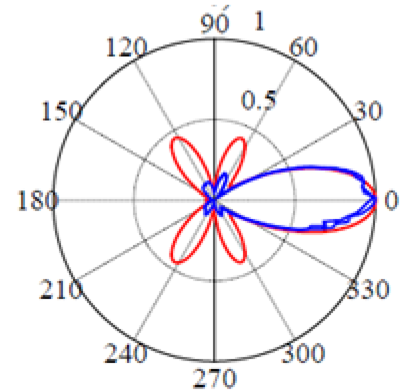
CCRM



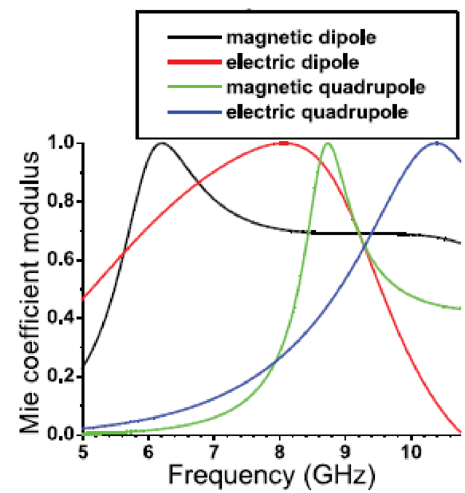
Measured emission and theory



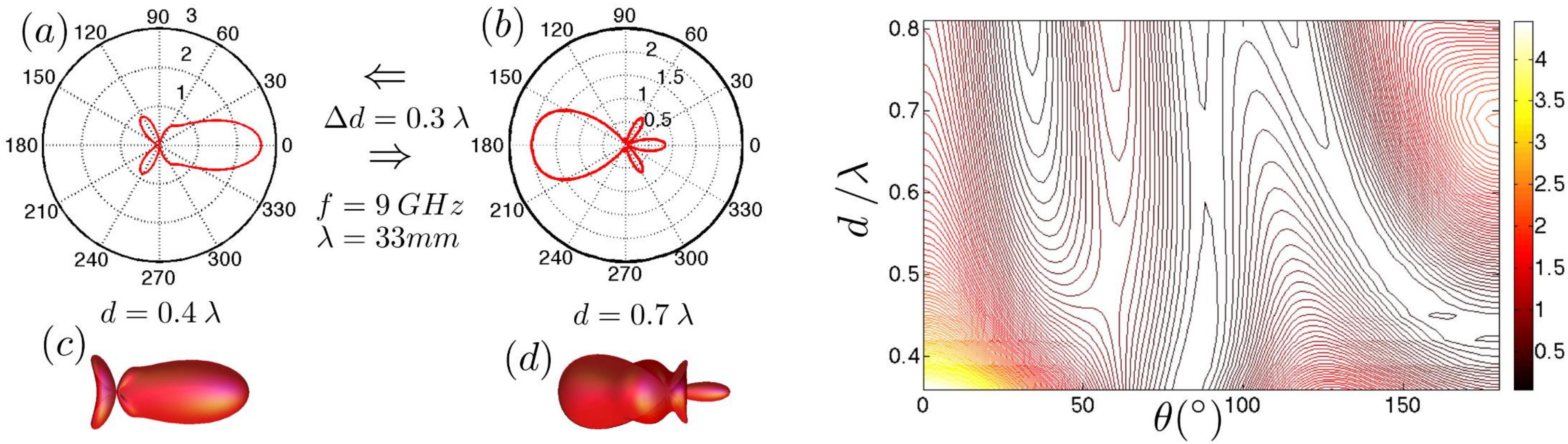
8,7 GHz



9,4 GHz



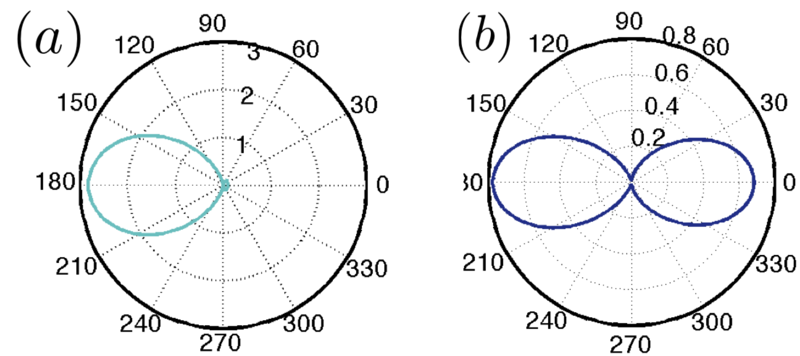
Changing antenna directivity with displacement



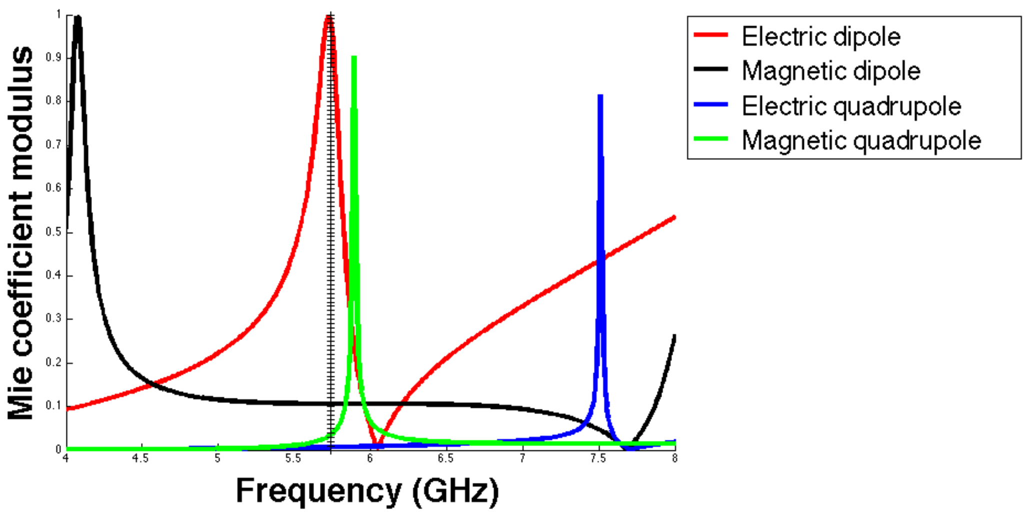
Benedicto; Bonod, Stout “Optimization of dielectric Antenna directivity” : in preparation

Suppressing forward scattering with high index dielectric antennas

- Isolated resonances
- Simpler interferences
- Fully suppressed forward emissions



(a) Actual emission
(b) Electric dipole suppressed



Modifying quantum decay rates with nano-antennas

Quasi-analytic multipole formalism :

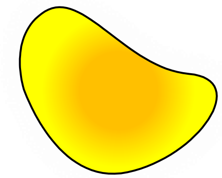
(radiative and non-radiative decay rates)

$$\frac{\Gamma_e}{\Gamma_0} = \frac{\langle P_e \rangle}{\langle P_{e,0} \rangle} = 1 + \frac{6\pi}{\mathrm{Re}\{k_b\}} \mathrm{Re}\left\{ k_b \sum_{j,l=1}^{N} f^{\dagger}.H^{(0,j)}.T^{(j,l)}.H^{(l,0)}.f \right\}$$

$$\frac{\Gamma_r}{\Gamma_0} = \frac{\langle P_r \rangle}{\langle P_{r,0} \rangle} = 1 + 6\pi \sum_{j,l,k,l=1}^{N} \left[T^{(j,i)}.H^{(i,0)}.f\right]^{\dagger}.J^{(j,k)}.T^{(k,l)}.H^{(i,0)}.f$$

$$+12\pi \mathrm{Re}\left\{ \sum_{j,l=1}^{N} \left[J^{(k,0)} f\right]^{\dagger}.T^{(k,j)}.H^{(j,0)}.f \right\}$$

K-Matrix (Reaction matrix)



The K–matrix relates the **regular part** of the **total field** to its **diverging part**

$$\vec{\mathbf{E}}_{\text{tot}}(k\vec{\boldsymbol{r}}) = \vec{\mathbf{E}}_{\text{exc}}(k\vec{\boldsymbol{r}}) + \vec{\mathbf{E}}_{\text{scat}}(k\vec{\boldsymbol{r}})$$

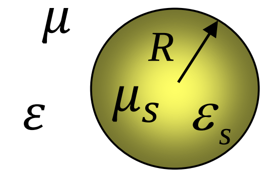
$$= \sum_{n,m}^{\infty}\left\{\left[r_{n,m}^{(h)}\overrightarrow{\mathbf{M}}_{n,m}^{(1)}(k\vec{\boldsymbol{r}}) + r_{n,m}^{(e)}\overrightarrow{\mathbf{N}}_{n,m}^{(1)}(k\vec{\boldsymbol{r}})\right] + \left[d_{n,m}^{(h)}\overrightarrow{\mathbf{M}}_{n,m}^{(2)}(k\vec{\boldsymbol{r}}) + d_{n,m}^{(e)}\overrightarrow{\mathbf{N}}_{n,m}^{(2)}(k\vec{\boldsymbol{r}})\right]\right\}$$

$$d \equiv \overline{\overline{\mathrm{K}}}.\,r \quad \Longrightarrow \quad \left\{ \quad \overline{\overline{\mathrm{T}}} = -i\overline{\overline{\mathrm{K}}}.\left(\overline{\overline{\mathrm{T}}} + \overline{\overline{\mathrm{I}}}\right) \qquad \overline{\overline{\mathrm{T}}}^{-1} = i\overline{\overline{\mathrm{K}}}^{-1} - \overline{\overline{\mathrm{I}}} \right.$$

$$K_n^{(e)} = -\frac{j_n(kR)}{y_n(kR)}\,\frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(kR) - \varphi_n(k_sR)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(2)}(kR) - \varphi_n(k_sR)}$$

$$K_n^{(e)} = -\tan\delta_n^{(e)}\,,$$
$$K_n^{(h)} = -\tan\delta_n^{(h)}$$

Reaction matrix is Hermitian for lossless scatterers



$$i\overline{\overline{\mathrm{T}}}^{-1} = -\overline{\overline{\mathrm{K}}}^{-1} - i\overline{\overline{\mathrm{I}}}$$

Unitary limit $\Rightarrow$ $K_n \to \infty$ $\Rightarrow$ $\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(2)}(kR) - \varphi_n(k_s R) = 0$

K-matrix is useful for deriving approximate non-transcendental approximations – to appear !

Rigorous formulation of “radiative corrections” – other “optical theorem”

$\alpha(\omega) = 6\pi T_1^{(e)}/ik^3$ $\alpha^{-1} = \alpha_{n.r.}^{-1} - i\frac{k^3}{6\pi}$ $\Rightarrow$ $\alpha_{\mathrm{n.r.}} = \frac{6\pi}{k^3}K_1^{(e)}$

Perspectives

- Use multipole methods to determine quasi-normal modes of systems
- Quasi-normal modes should help optimize the performance of photonic systems.
- Quasi-normal modes should help formulating theories of strong coupling between a quantum emitter and its nano-photonic environment.

Vector "partial" waves

Maxwell equation :

$$\nabla \times \nabla \times \vec{\mathbf{E}} - k^2 \vec{\mathbf{E}} = \vec{\mathbf{0}}$$

$$\nabla \cdot \vec{\mathbf{M}} = 0 \qquad \nabla \cdot \vec{\mathbf{N}} = 0$$

$$\psi_n(x) \equiv x j_n(x)$$

$$\psi_n'(x) = \left[j_n(x) + x j_n'(x) \right]$$

Electric and Magnetic types :

$$\vec{\mathbf{M}}^{(1)}{}_{n,m}(k\vec{\boldsymbol{r}}) = j_n(kr)\vec{\mathbf{X}}_{n,m}(\theta,\phi)$$

$$\vec{\mathbf{N}}^{(1)}{}_{n,m}(k\vec{\boldsymbol{r}}) = \frac{1}{kr}\Big[j_n(kr)\sqrt{n(n+1)}\,\vec{\mathbf{Y}}_{n,m}(\theta,\phi) + \psi_n'(kr)\vec{\mathbf{Z}}_{n,m}(\theta,\phi) \Big]$$

Solutions to Maxwell's propagation equations in spherical coordinates

$$\nabla \times \nabla \times \vec{\mathbf{E}} - k^2 \vec{\mathbf{E}} = 0$$

Transverse vector "partial" waves : $\nabla \cdot \vec{\mathbf{M}}_{n,m} = \nabla \cdot \vec{\mathbf{N}}_{n,m} = 0$

$$\vec{\boldsymbol{M}}_{n,m}(k\vec{\boldsymbol{r}}) = \frac{\nabla \times \left[\vec{\boldsymbol{r}}\varphi_{n,m}(k\vec{\boldsymbol{r}})\right]}{\sqrt{n(n+1)}}$$

magnetic modes

$$\vec{\boldsymbol{N}}_{n,m}(k\vec{\boldsymbol{r}}) = \frac{\nabla \times \vec{\boldsymbol{M}}_{n,m}(k\vec{\boldsymbol{r}})}{k}$$

electric modes

$$\Delta\varphi + k^2\varphi = 0$$

$$\varphi_{n,m}(k\vec{\boldsymbol{r}}) = j_n(kr)Y_{n,m}(\theta,\phi)$$

Linearly independent solutions

$\nabla \cdot \vec{\mathbf{M}}=0 \qquad \nabla \cdot \vec{\mathbf{N}}=0$

$$\vec{\mathbf{M}}^{(1)}{}_{n,m}(k\vec{\boldsymbol{r}}) = j_n(kr)\vec{\mathbf{X}}_{n,m}(\theta,\phi)$$

$$\vec{\mathbf{N}}^{(1)}{}_{n,m}(k\vec{\boldsymbol{r}}) = \frac{1}{kr}\left[j_n(kr)\sqrt{n(n+1)}\vec{\mathbf{Y}}_{n,m}(\theta,\phi) + \psi_n'(kr)\vec{\mathbf{Z}}_{n,m}(\theta,\phi)\right]$$

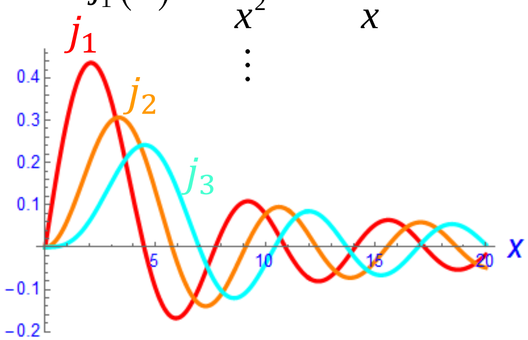
$$\psi_n'(kr) = [xj_n(x)]'$$

Spherical Bessel functions (1)

$$j_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} x^{2s} = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$\vdots$$



$$\vec{\mathbf{M}}^{(2)}{}_{n,m}(k\vec{\boldsymbol{r}}) = y_n(kr)\vec{\mathbf{X}}_{n,m}(\theta,\phi)$$

$$\vec{\mathbf{N}}^{(2)}{}_{n,m}(k\vec{\boldsymbol{r}}) = \frac{1}{kr}\left[y_n(kr)\sqrt{n(n+1)}\vec{\mathbf{Y}}_{n,m}(\theta,\phi) + \chi_n'(kr)\vec{\mathbf{Z}}_{n,m}(\theta,\phi)\right]$$

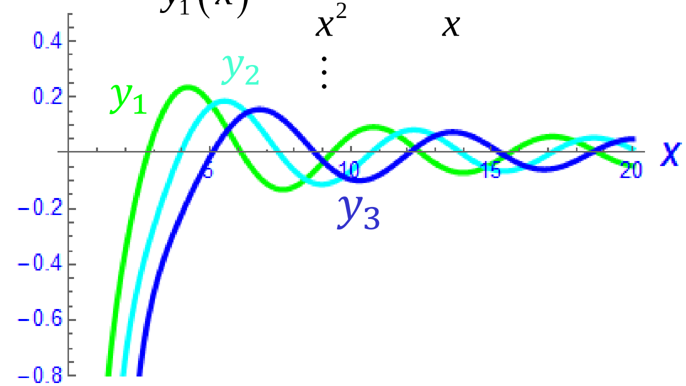
$$\chi_n'(x) = [xy_n(x)]'$$

Spherical Neumann functions (2)

$$y_0(x) = -\frac{\cos x}{x}$$

$$y_1(x) = \frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$\vdots$$



Outgoing and incoming boundary conditions

$$\vec{\mathbf{M}}^{(+)}{}_{n,m}(k\vec{\boldsymbol{r}}) = h_n^{(+)}(kr)\vec{\mathbf{X}}_{n,m}(\theta,\phi)$$

$$\vec{\mathbf{N}}^{(+)}{}_{n,m}(k\vec{\boldsymbol{r}}) = \frac{1}{kr}\left[h_n^{(+)}(kr)\sqrt{n(n+1)}\vec{\mathbf{Y}}_{n,m}(\theta,\phi) + \xi_n^{(+)\prime}(kr)\vec{\mathbf{Z}}_{n,m}(\theta,\phi)\right]$$

$$h_n^{(+)}(x) = j_n(x) + i y_n(x)$$

$$\xi_n^{(+)\prime}(x) = \left[h_n^{(+)}(x)\right]'$$

Outgoing spherical Hankel functions (+)

$$h_0^{(+)}(x) = -\frac{i}{x}e^{ix}$$

$$h_1^{(+)}(x) = -e^{ix}\left(\frac{1}{x} + \frac{i}{x^2}\right)$$

$$\vdots$$

$$\vec{\mathbf{M}}^{(-)}{}_{n,m}(k\vec{\boldsymbol{r}}) = h_n^{(-)}(kr)\vec{\mathbf{X}}_{n,m}(\theta,\phi)$$

$$\vec{\mathbf{N}}^{(-)}{}_{n,m}(k\vec{\boldsymbol{r}}) = \frac{1}{kr}\left[h_n^{(-)}(kr)\sqrt{n(n+1)}\vec{\mathbf{Y}}_{n,m}(\theta,\phi) + \xi_n^{(-)\prime}(kr)\vec{\mathbf{Z}}_{n,m}(\theta,\phi)\right]$$

$$h_n^{(-)}(x) = j_n(x) - i y_n(x)$$

$$\xi_n^{(-)\prime}(x) = \left[h_n^{(-)}(x)\right]'$$

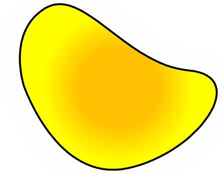
Incoming spherical Hankel functions (-)

$$h_0^{(-)}(x) = \frac{i}{x}e^{-ix}$$

$$h_1^{(-)}(x) = -e^{-ix}\left(\frac{1}{x} - \frac{i}{x^2}\right)$$

$$\vdots$$

K-Matrix (Reaction matrix)
adapted to studying energy conservation



The K–matrix relates the **regular part** of the **total field** to its **diverging part**

$$\vec{\mathbf{E}}_{\text{tot}}(k\vec{\boldsymbol{r}}) = \vec{\mathbf{E}}_{\text{exc}}(k\vec{\boldsymbol{r}}) + \vec{\mathbf{E}}_{\text{scat}}(k\vec{\boldsymbol{r}})$$

$$= \sum_{n,m}^{\infty} \left\{ \left[r_{n,m}^{(h)} \vec{\mathbf{M}}_{n,m}^{(1)}(k\vec{\boldsymbol{r}}) + r_{n,m}^{(e)} \vec{\mathbf{N}}_{n,m}^{(1)}(k\vec{\boldsymbol{r}}) \right] + \left[d_{n,m}^{(h)} \vec{\mathbf{M}}_{n,m}^{(2)}(k\vec{\boldsymbol{r}}) + d_{n,m}^{(e)} \vec{\mathbf{N}}_{n,m}^{(2)}(k\vec{\boldsymbol{r}}) \right] \right\}$$

$$d \equiv \overline{\overline{\mathrm{K}}}.r$$

$$\overline{\overline{\mathrm{T}}} = -i\overline{\overline{\mathrm{K}}}.\left(\overline{\overline{\mathrm{T}}} + \overline{\overline{\mathrm{I}}}\right) \qquad \overline{\overline{\mathrm{T}}}^{-1} = i\overline{\overline{\mathrm{K}}}^{-1} - \overline{\overline{\mathrm{I}}}$$

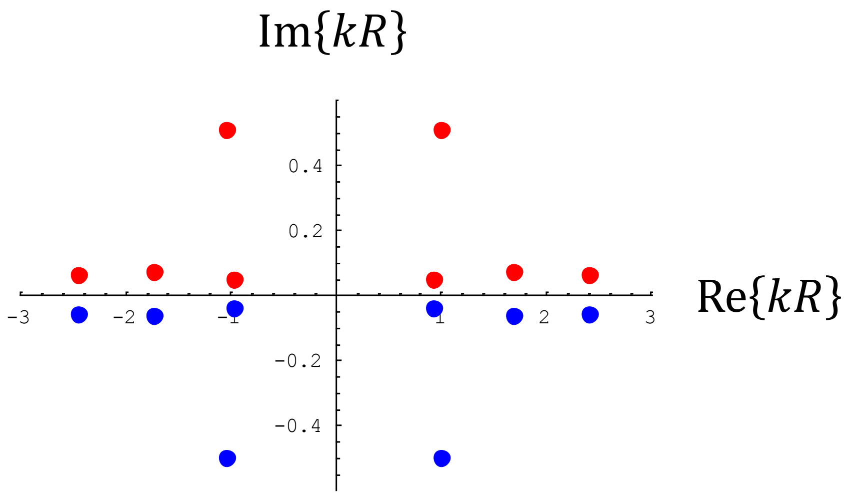
$$\overline{\overline{\mathrm{K}}} = i\left(\overline{\overline{\mathrm{S}}} - \overline{\overline{\mathrm{I}}}\right).\left(\overline{\overline{\mathrm{I}}} + \overline{\overline{\mathrm{S}}}\right)^{-1} \qquad \overline{\overline{\mathrm{S}}} = \left(\overline{\overline{\mathrm{I}}} - i\overline{\overline{\mathrm{K}}}\right).\left(\overline{\overline{\mathrm{I}}} + i\overline{\overline{\mathrm{K}}}\right)^{-1}$$

Caley Transform

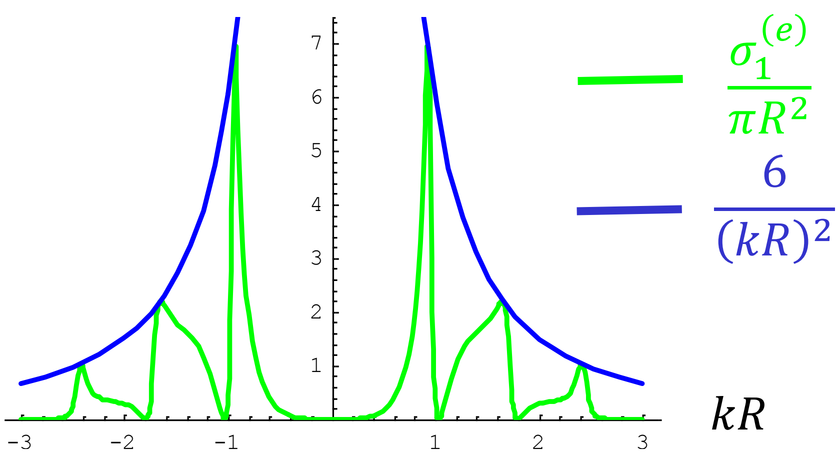
$$K_n^{(e)} = -\tan\delta_n^{(e)} ,$$
$$K_n^{(h)} = -\tan\delta_n^{(h)}$$

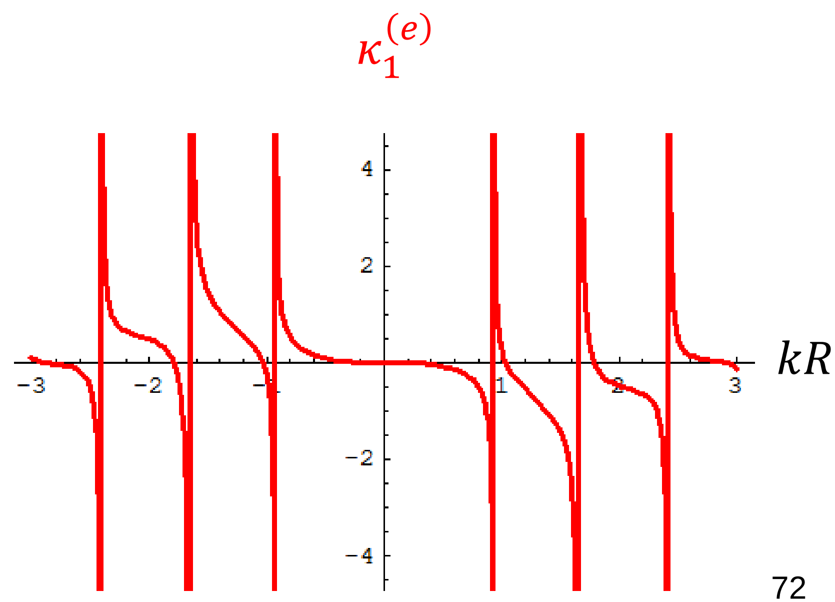
Simplicity of analytic structure
(Weierstrass factorization)

$$S_n(x) = A\exp(-2iBx)\prod_{\alpha}\frac{x - x_{z,\alpha}}{x - x_{p,\alpha}} = A\exp(-2iBx)\prod_{\alpha=1}^{\infty}\frac{(x - x_{z,\alpha})(x + x_{z,\alpha}^{*})}{(x - x_{p,\alpha})(x + x_{p,\alpha}^{*})} \qquad x \equiv kR$$



$$s_n^{(e)}(kR) = -\frac{h_n^{(-)}(kR)\,\frac{\varepsilon_s}{\varepsilon}\,\varphi_n^{(-)}(kR) - \varphi_n\,(k_sR)}{h_n^{(+)}(kR)\,\frac{\varepsilon_s}{\varepsilon}\,\varphi_n^{(+)}(kR) - \varphi_n\,(k_sR)}$$





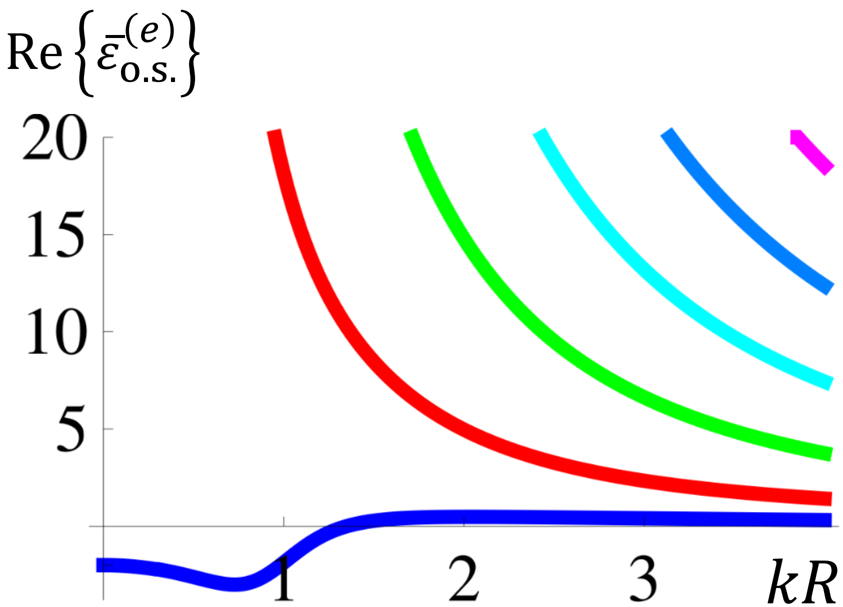
Ideal absorption given by the solution of a transcendental equation solving for of $\varepsilon_s/\varepsilon$ as a function of kR

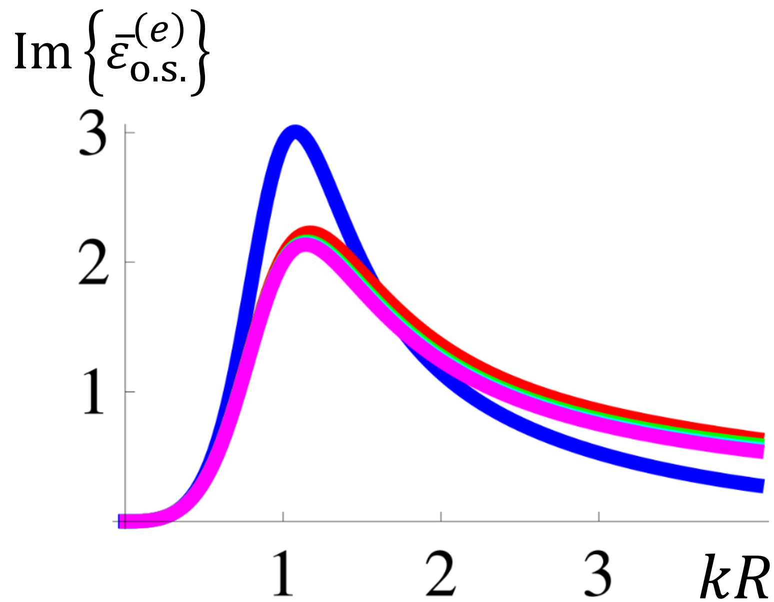
$$S_1^{(e)} = \frac{h_1^{(-)}(kR)}{h_1^{(+)}(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_1^{(-)}(kR) - \varphi_1(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_1^{(+)}(kR) - \varphi_1(k_s R)} = 0$$

Ideal absorption

$$\frac{\varepsilon}{\varepsilon_s}\varphi_1(k_s R) = \varphi_1^{(-)}(kR)$$

$$\varphi_1 = \frac{[zj_1(z)]'}{j_1(z)} \qquad \varphi_1^{(-)}(x) = \frac{\left[xh_1^{(-)}(x)\right]'}{h_1^{(-)}(x)}$$





Determining the Unitary limit condition

$$s_n^{(e,h)} = -1 \quad , \qquad t_n^{(e,h)} = -1 \quad , \qquad \kappa_n^{(e,h)} \to \infty$$

$$s_n^{(e)}(kR) = -\frac{h_n^{(-)}(kR)}{h_n^{(+)}(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(-)}(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

$$t_n^{(e)} = -\frac{j_n(kR)}{h_n^{(+)}(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

$$\kappa_n^{(e)} = -\frac{j_n(kR)}{y_n(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(2)}(kR) - \varphi_n(k_s R)}$$

The unitary limit can be found by solving a transcendental equation

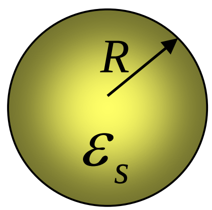
$$\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(2)}(kR) - \varphi_n(k_s R) = 0$$

Light interactions with a particle much smaller than the wavelength

➡ electric dipole description (polarizability)

$$\uparrow \qquad \vec{\mathbf{p}} = \epsilon_0 \varepsilon_b \alpha(\omega) \vec{\mathbf{E}}_{\mathrm{exc}}$$

ε_b



$$\lim_{\omega \to 0} \alpha(\omega) = 4\pi R^3 \frac{\varepsilon_s - \varepsilon}{2\varepsilon + \varepsilon_s} \equiv \alpha_0$$

$$\sigma_{\mathrm{ext}} = k \mathrm{Im}\{\alpha(\omega)\}$$

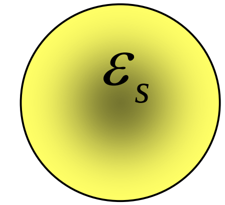
$$\sigma_{\mathrm{scat}} = \frac{k^4 |\alpha(\omega)|^2}{6\pi}$$

$$\sigma_{\mathrm{abs}} = \sigma_{\mathrm{ext}} - \sigma_{\mathrm{scat}} \geq 0$$

$$\mathrm{Im}\{\alpha\} \geq \frac{k^3 |\alpha|^2}{6\pi}$$

‘Point-like’ models – single resonant model

$$k = \frac{\omega}{c}\sqrt{\varepsilon}$$



$$\vec{\mathbf{p}} = \varepsilon_0 \varepsilon \alpha(\omega) \vec{\mathbf{E}}_{exc}$$

$$\alpha_0 = \lim_{kR \to 0} \alpha(\omega) = 4\pi R^3 \frac{\varepsilon_s - \varepsilon}{2\varepsilon + \varepsilon_s}$$

Radiative correction model :

$$\lim_{\omega \to 0} \alpha(\omega) = \frac{A(\omega)}{1 - i\frac{k^3}{6\pi}A(\omega)} \cong \frac{\alpha_0}{1 - i\frac{k^3}{6\pi}\alpha_0}$$

Green function point-like model :

$$\lim_{\omega \to 0} \alpha(\omega) \cong \frac{\alpha_0}{1 - \frac{1}{6\pi}\frac{k^2}{R}\alpha_0 - i\frac{k^3}{6\pi}\alpha_0}$$

Mie point-like model :

$$\lim_{\omega \to 0} \alpha(\omega) \cong \frac{\alpha_0}{1 - \frac{3}{20\pi}\left(\frac{\varepsilon_s - 2\varepsilon}{\varepsilon_s - \varepsilon}\right)\frac{k^2}{R}\alpha_0 - i\frac{k^3}{6\pi}\alpha_0}$$

$$\alpha^{-1} = \alpha_{q.s.}^{-1} - i\frac{k^3}{6\pi}$$

Fundamental limits (Dipole scattering)

$$\alpha(\omega) = t_1^{(e)} \frac{6\pi}{ik^3}$$

$$\sigma_{\text{ext}} = k\,\text{Im}\{\alpha(\omega)\}$$

$$\sigma_{\text{scat}} = \frac{k^4|\alpha(\omega)|^2}{6\pi}$$

Unitary limit (lossless scattering)

$$\alpha_{\text{u.l}} = 6\pi i/k^3 \quad \Rightarrow \quad \sigma_{\text{ext}} = \sigma_{\text{scat}} \sim \lambda^2/2$$

Ideal absorption – Optical sink

$$\alpha_{\text{I.A}} = 3\pi i/k^3 \quad \Rightarrow \quad \sigma_{\text{abs}} = \sigma_{\text{scat}} \sim \lambda^2/8$$

K-Matrix (Reaction matrix)

Gives access to all limit behaviors of light-matter interactions

Spherically symmetric particles :

$K_n^{(e,h)} \to \infty \quad \Rightarrow \quad S_n^{(e,h)} = T_n^{(e,h)} = -1$ ⟶ Unitary limit

$K_n^{(e,h)} = 0 \quad \Rightarrow \quad S_n^{(e,h)} \to 1 \;,\; T_n^{(e,h)} \to 0$ ⟶ Invisible

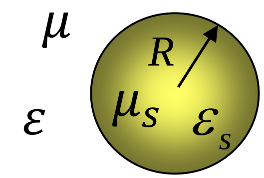
} lossless

$K_n^{(e,h)} = i \quad \Rightarrow \quad S_n^{(e,h)} \to \infty \;,\; T_n^{(e,h)} \to \infty$ ⟶ Emission - lasing → gain

$K_n^{(e,h)} = -i \quad \Rightarrow \quad S_n^{(e,h)} \to 0 \;,\; T_n^{(e,h)} \to -\frac{1}{2}$ ⟶ Ideal absorption → loss

Mie theory

Highlights the symmetry of the different matrices



$$T_n^{(e)} = -\frac{j_n(kR)}{h_n^{(+)}(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

$$T_n^{(h)} = -\frac{j_n(kR)}{h_n^{(+)}(kR)} \frac{\frac{\mu_s}{\mu}\varphi_n(kR) - \varphi_n(k_s R)}{\frac{\mu_s}{\mu}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

$$S_n^{(e)} = -\frac{h_n^{(-)}(kR)}{h_n^{(+)}(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(-)}(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

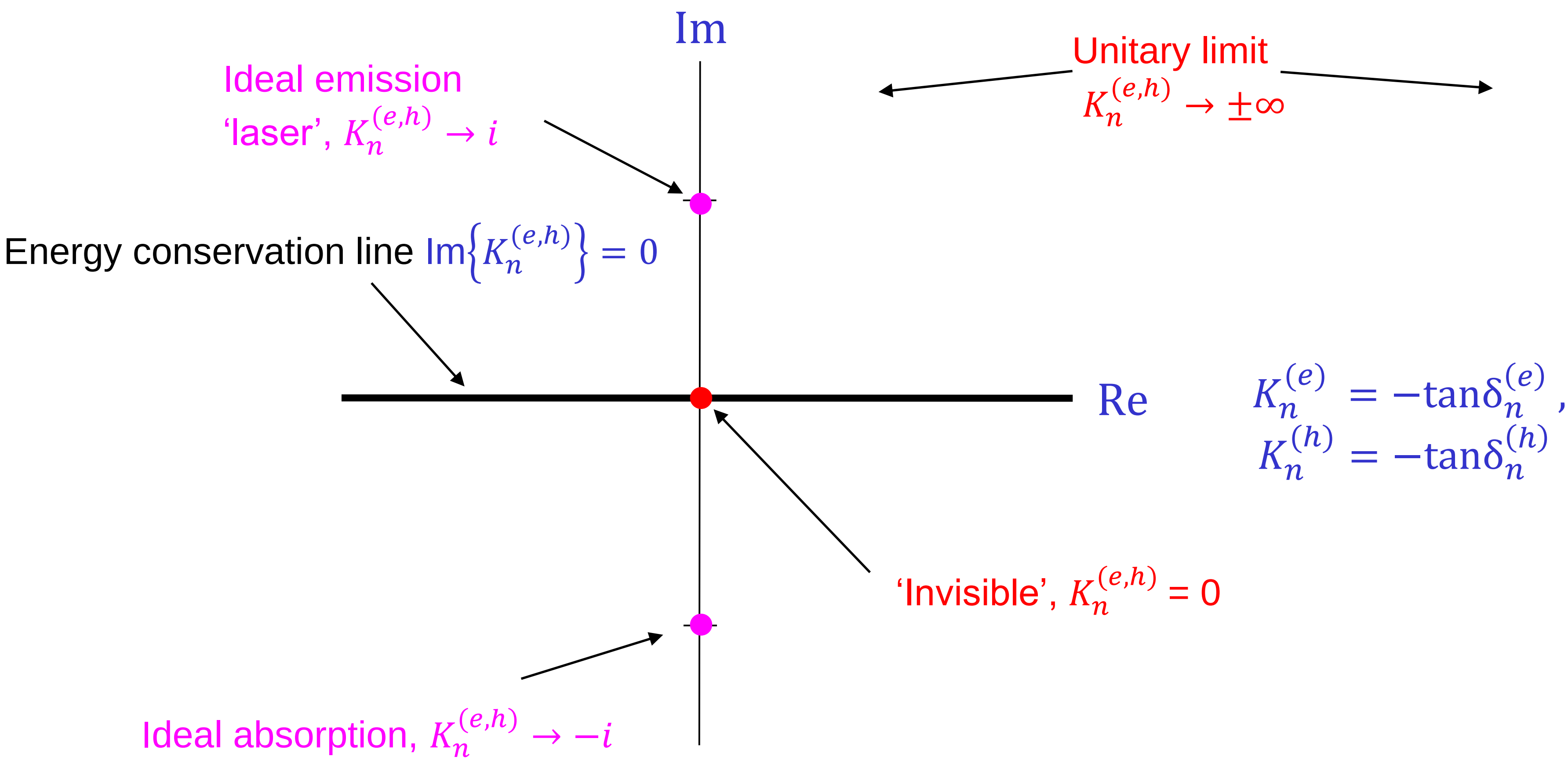
$$S_n^{(h)} = -\frac{h_n^{(-)}(kR)}{h_n^{(+)}(kR)} \frac{\frac{\mu_s}{\mu}\varphi_n^{(-)}(kR) - \varphi_n(k_s R)}{\frac{\mu_s}{\mu}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

$$K_n^{(e)} = -\frac{j_n(kR)}{y_n(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(2)}(kR) - \varphi_n(k_s R)}$$

$$K_n^{(h)} = -\frac{j_n(kR)}{y_n(kR)} \frac{\frac{\mu_s}{\mu}\varphi_n(kR) - \varphi_n(k_s R)}{\frac{\mu_s}{\mu}\varphi_n^{(2)}(kR) - \varphi_n(k_s R)}$$

K-Matrix is Hermitian for a lossless scatterer
(adapted to studying energy conservation and limit behaviors)

$$\overline{\overline{\mathrm{K}}}^{\dagger} = \overline{\overline{\mathrm{K}}}$$: Energy conservation



Weierstrass factorization
(useful for solving transcendental equations)

Electric Ideal absorption

$$\frac{\varepsilon}{\varepsilon_s}\varphi_n(k_s R) = \varphi_n^{(-)}(kR)$$

Magnetic Ideal absorption

$$\varphi_n(k_s R) = \varphi_n^{(-)}(kR)$$

Electric mode unitary limit

$$\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(2)}(kR) = \varphi_n(k_s R)$$

Magnetic mode unitary limit

$$\varphi_n^{(2)}(kR) = \varphi_n(k_s R)$$

$$\varphi_n(z) = \frac{[z j_n(z)]'}{j_n(z)} = 2 + \sum_{\alpha=1}^{\infty}\left(\frac{2z^2}{z^2 - a_{n,\alpha}^2}\right)$$

$$\varphi_1^{(-)}(x) = \frac{\left[x h_1^{(-)}(x)\right]'}{h_1^{(-)}(x)}$$

Vector spherical harmonics

Scalar spherical harmonics : $Y_{nm}(\theta,\phi) \equiv \left[\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\right] e^{im\phi} P_n^m(\cos\theta)$

3 types of **vector** spherical harmonics :

$$\vec{\boldsymbol{W}}^{(1)}_{n,m}(\theta,\phi) \equiv \vec{\boldsymbol{Y}}_{n,m}(\theta,\phi) = \hat{\boldsymbol{r}} Y_{n,m}(\theta,\phi) \qquad n = 0,\ldots\infty \qquad m = -n,\ldots n$$

$$\vec{\boldsymbol{W}}^{(2)}_{n,m}(\theta,\phi) \equiv \vec{\boldsymbol{X}}_{n,m}(\theta,\phi) = \vec{\boldsymbol{Z}}_{n,m}(\theta,\phi) \times \hat{\boldsymbol{r}}$$

$$n = 1,\ldots,\infty \qquad m = -n,\ldots,n$$

$$\vec{\boldsymbol{W}}^{(3)}_{n,m}(\theta,\phi) \equiv \vec{\boldsymbol{Z}}_{n,m}(\theta,\phi) = \frac{r\nabla Y_{n,m}}{\sqrt{n(n+1)}} = \hat{\boldsymbol{r}} \times \vec{\boldsymbol{X}}_{n,m}(\theta,\phi)$$

$$\int_0^{4\pi} d\Omega\, \vec{\boldsymbol{W}}^{(j),*}_{n,m}(\theta,\phi) \cdot \vec{\boldsymbol{W}}^{(k)}_{n,m}(\theta,\phi) = \delta_{j,k}\delta_{n,\nu}\delta_{m,\mu}$$

IA in homogeneous particles and realistic materials ?

Yes, but only for certain sizes and frequencies

IA electric dipole - exact calculation

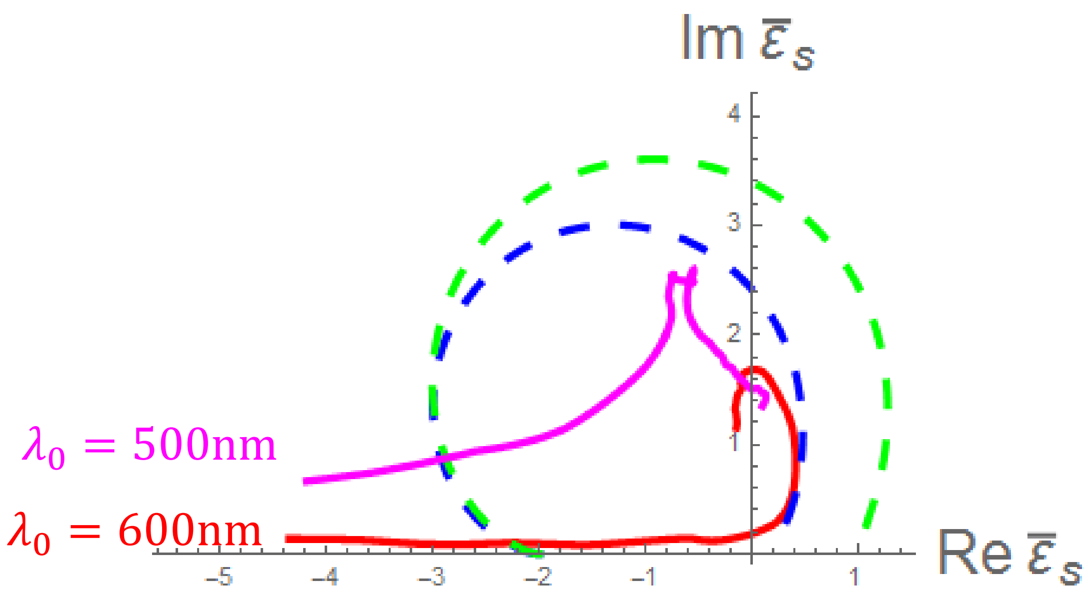
IA – electric dipole point-like approximation

Silver –

Gold –

Experimental dispersion curves : Johnson & Christy

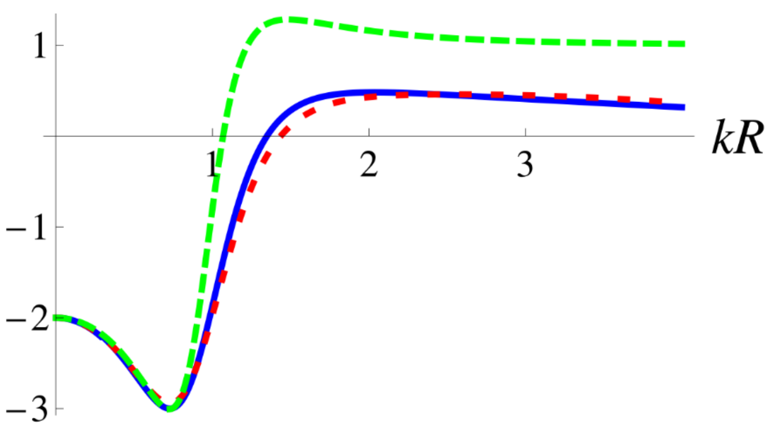
Polymer background : N_b=1.5

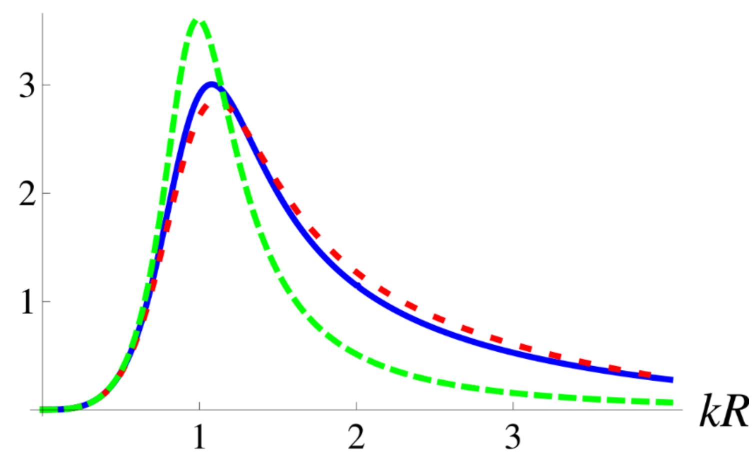


Point-like models and their extensions can yield analytic formulas for approximating the IA conditions

Point-like model : $\alpha^{-1} = \alpha_0^{-1} - \frac{k^2}{6\pi R} - i\,\frac{k^3}{6\pi}$ ✚ IA condition : $\alpha_{o.s} = 3\pi i / k^3$

$$\bar{\varepsilon}^{(e)}_{o.s.}(kR) = -\frac{2 + \frac{2}{3}(kR)^2(1 - ikR)}{1 - \frac{2}{3}(kR)^2(1 - ikR)}$$

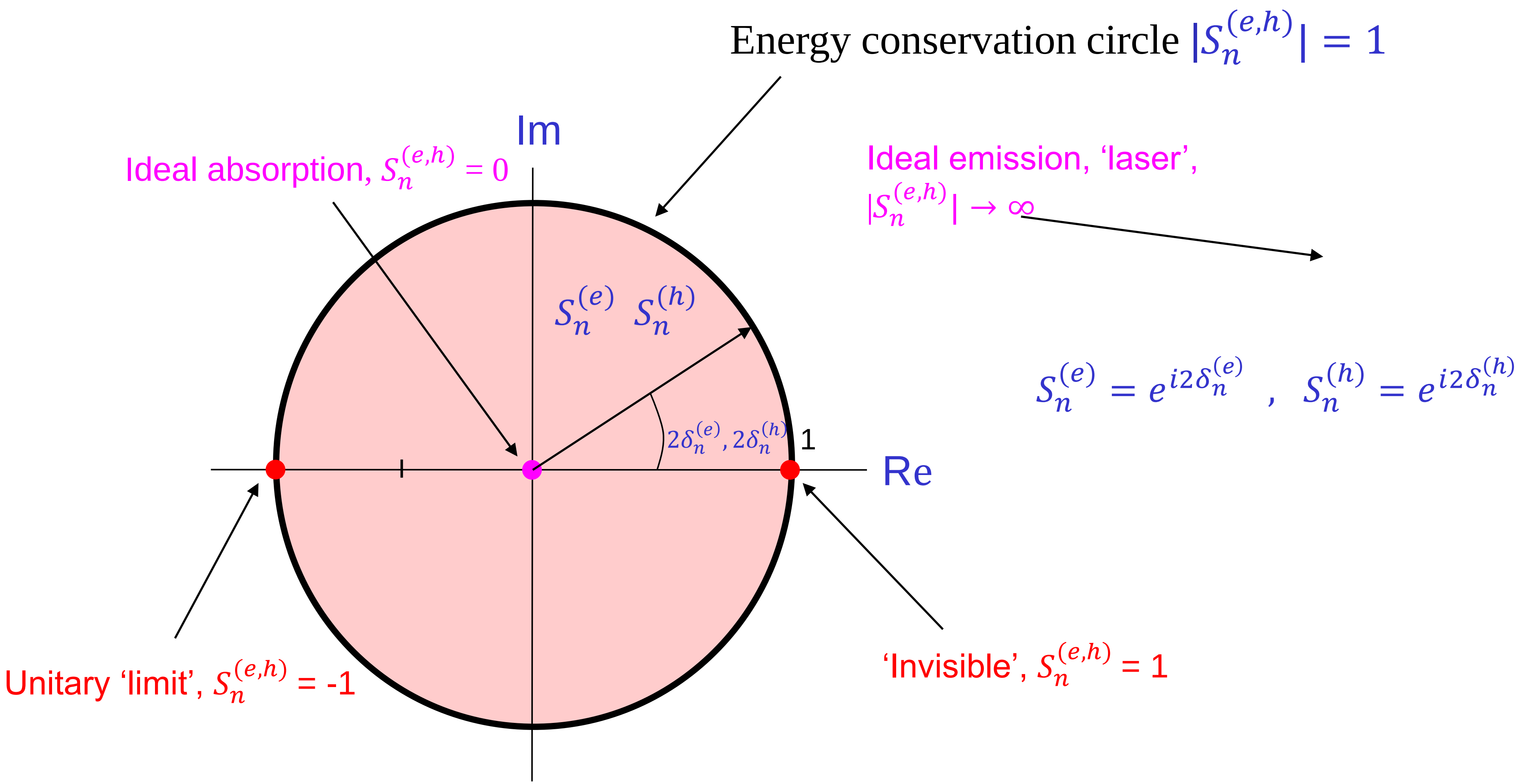




Improved analytic model

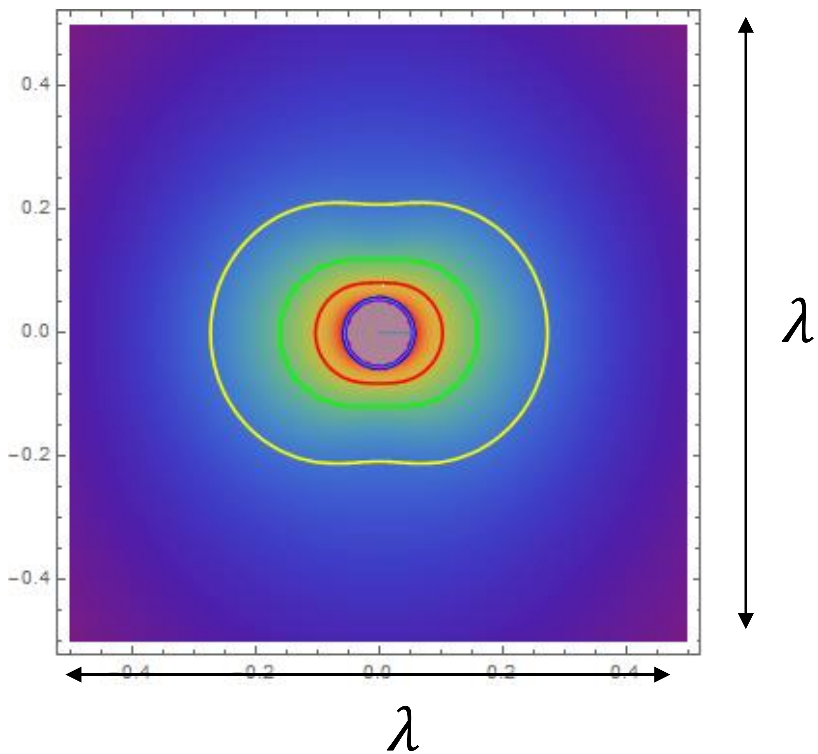
Exact calculations

S-matrix symmetrizes the limit behaviors



Ideal Absorption (optimizes ?) near field enhancements

$\|\vec{E}\|^2$ Log scale



$C_{abs} = C_{scat} \sim \lambda^2/8$

Sub-wavelength IA particles
produce large field enhancements