Spectral resolution of the Neumann-Poincaré operator and plasmon resonance

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Neumann-Poincaré operator

Let D be a bounded Lipschitz domain in $\mathbb{R}^d$. The single-layer potential $\mathcal{S}_{\partial D}$ and the Neumann-poincaré operator $\mathcal{K}_{\partial D}^*$ are defined as

$$\mathcal{S}_{\partial D}[\phi](x) = \int_{\partial D} \Gamma(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^d,$$
$$\mathcal{K}_{\partial D}^*[\phi](x) = p.v. \int_{\partial D} \frac{\partial}{\partial \nu_x}\Gamma(x-y)\phi(y)d\sigma(y), \quad x \in \partial D,$$

where ν is the outward unit normal vector to ∂D and Γ is the fundamental solution

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi}\ln|x|, & \text{for } d=2, \\ -\frac{1}{4\pi}|x|^{-1}, & \text{for } d=3. \end{cases}$$

Solution to transmission problems

Let D and $\mathbb{R}^d \setminus \overline{D}$ have the constant conductivities $\sigma_\star$ and 1, respectively. For a given entire harmonic function h consider the solution v to

$$\begin{cases} \nabla \cdot \sigma \nabla v = 0 \quad \text{in } \mathbb{R}^d, \\ v(x) - h(x) = O(|x|^{1-d}) \quad \text{as } |x| \to \infty. \end{cases}$$

The solution v admits the expression in terms of the single layer potential and the Neumann-poincaré operator:

$$v(x) = h(x) + \mathcal{S}_{\partial D}[\phi](x), \quad x \in \mathbb{R}^d$$

with

$$\phi(y) = (\lambda I - \mathcal{K}^*_{\partial D})^{-1}[\nu(x) \cdot \nabla h(x)](y), \quad y \in \partial D$$

and

$$\lambda = \frac{\sigma_\star + 1}{2(\sigma_\star - 1)}.$$

Symmetrization of $\mathcal{K}^*_{\partial D}$

- $\mathcal{K}^*_{\partial\Omega}$ is self-adjoint on the usual L^2-space only for a disc or a ball (L., 01).
- For general Lipschitz domain, it can be symmetrized using Plemelj's symmetrization principle (Khavinson-Putinar-Shapiro, 2007):

$$\mathcal{S}_{\partial\Omega}\mathcal{K}^*_{\partial\Omega} = \mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}$$

If we define, for $\varphi, \psi \in H_0^{-1/2}(\partial\Omega)$,

$$(\varphi,\psi)_* := -\langle \varphi, \mathcal{S}_{\partial\Omega}[\psi]\rangle = -\frac{1}{2\pi}\int_{\partial\Omega}\int_{\partial\Omega} \ln|x-y|\varphi(x)\overline{\psi(y)}\,ds(x)ds(y),$$

then the induced norm $\|\cdot\|_*$ is equivalent to the $H^{-1/2}(\partial\Omega)$ norm and $\mathcal{K}^*_{\partial\Omega}$ is self-adjoint on $\mathcal{H}^*_0 = (H_0^{-1/2}(\partial\Omega), \|\cdot\|_*)$.

$$(\varphi, \mathcal{K}^*\psi) = -\langle\varphi, S\mathcal{K}^*\psi\rangle = -\langle\varphi, \mathcal{K}S\psi\rangle = (\mathcal{K}^*\varphi, \psi).$$

Spectrum of $\mathcal{K}^*_{\partial\Omega}$ on $\mathcal{H}^*_0$

- Let $\sigma(\mathcal{K}^*_{\partial\Omega})$ be the spectrum of $\mathcal{K}^*_{\partial\Omega}$ on $\mathcal{H}^*_0$. It is known that

$$\sigma(\mathcal{K}^*_{\partial\Omega}) \subset (-1/2, 1/2).$$

- Since $\mathcal{K}^*_{\partial\Omega}$ is self-adjoint on $\mathcal{H}^*$, $\sigma(\mathcal{K}^*_{\partial\Omega})$ is real and consists of pure point spectrum (eigenvalues),absolutely continuous spectrum, and singularly continuous spectrum, namely,

$$\sigma(\mathcal{K}^*_{\partial\Omega}) = \sigma_{\rm ac}(\mathcal{K}^*_{\partial\Omega}) \cup \sigma_{\rm sc}(\mathcal{K}^*_{\partial\Omega}) \cup \sigma_{\rm pp}(\mathcal{K}^*_{\partial\Omega}).$$

- By the spectral resolution theorem there is a family of projection operators $\mathcal{E}_t$ on $\mathcal{H}^*$ (called a resolution of the identity) such that

$$\mathcal{K}^*_{\partial\Omega} = \int_{-1/2}^{1/2} t \, d\mathcal{E}_t. \tag{1}$$

Eigenvalue distribution of smooth domains

For $\mathcal{C}^{1,\alpha}$-domain D, $\mathcal{K}^*_{\partial D}$ is compact on $\mathcal{H}^*_0$ and admits the decomposition in terms of the eigenvalue (counted with the multiplicity) and eigenfunctions:

$$\mathcal{K}^*_{\partial D} = \sum_{n=1}^{\infty} \lambda_j \varphi_j \otimes \varphi_j$$

$\left(\frac{1}{2} > |\lambda_1| \geq |\lambda_2| \geq \cdots \to 0\right)$.

- Disk: $\mathcal{K}^*_{\partial D}$ is the averaging operator.
- Ellipse with the eccentricity r: $\lambda_n(r) = \pm\frac{1}{2}\left(\frac{1-r}{1+r}\right)^n, \ n = 1, 2, \ldots.$
- Ball: $\lambda_n^m = \frac{1}{2(2n+1)}, \ m = 1, \ldots, 2n+1$. Note that the eigenvalues are positive.
- Ellipsoid: Eigenvalues have the closed form of in terms of integral of Lamé functions. And for any number $\lambda \in (1/2, 1/2)$ there is an ellipsoid including 0 (Feng-Kang).

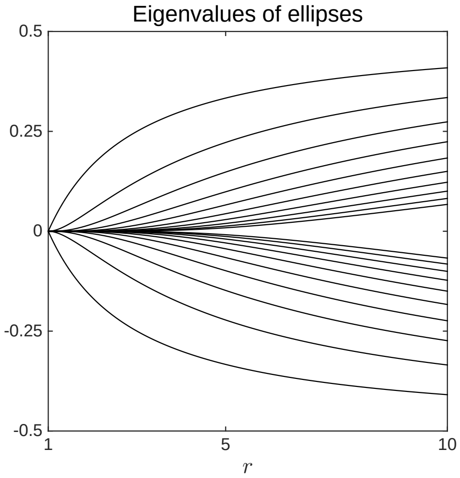


Figure : The 20 largest eigenvalues for ellipses of various aspect ratios r. Thinner ellipses have bigger eigenvalues.

NP operator of Multiple boundaries

$$u(X) = H(X) + \mathcal{S}_{D_1}\varphi_1(X) + \mathcal{S}_{D_2}\varphi_2(X)$$

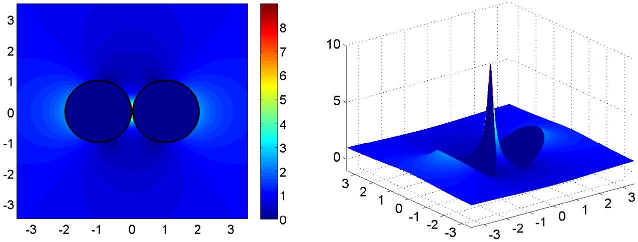
with

$$\begin{cases} (\lambda_1 - \mathcal{K}^*_{D_1})\varphi_1 - \dfrac{\partial}{\partial\nu^{(1)}}\mathcal{S}_{D_2}\varphi_2 = \dfrac{\partial H}{\partial\nu^{(1)}} & \text{on } \partial D_1 \\ -\dfrac{\partial}{\partial\nu^{(2)}}\mathcal{S}_{D_1}\varphi_1 + (\lambda_2 - \mathcal{K}^*_{D_2})\varphi_2 = \dfrac{\partial H}{\partial\nu^{(2)}} & \text{on } \partial D_2 \qquad (\lambda_j = \frac{k_j+1}{2(k_j-1)}), \end{cases}$$

We can rewrite it as

$$(\mathbb{D} - \mathbb{K})\varphi = \mathbf{h}.$$

Stress concentration for two nearly touching inclusions



The operator $\mathbb{K}$ has eigenvalues $|\lambda_n| \approx \frac{1}{2} - c_n\sqrt{\epsilon}$, where c_n is a constant. And $\lambda = \frac{1}{2}$ for $k = \infty$.

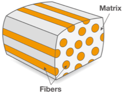
- Bonnetier and Triki(2012). Behavior of the singular values.
- Ammari, Ciraolo, Kang, Lee and Yun(2013). Symmetrization of $\mathbb{K}$ and the characterization of the blow-up terms in terms of the singular function with the one corresponding to two disks osculating to the inclusions.
- Lim and Yu(2015). Asymptotic of the solution for general k.

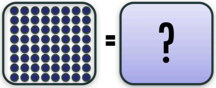
Stress concentration for two nearly touching inclusions

- The generic rate of gradient blow-up is $|\epsilon \ln \epsilon|^{-1}$ in three dimensions (Bao-Li-Yin 2010, L-Yun 2009, Kang-L-Yun 2014, L-Yu, etc.).

- It is $\epsilon^{-1/2}$ in two dimensions
(Keller 63, Budiansky-Carrier 84, Ammari-Kang-L 2005, Ammari-Kang-Lee-Lee-L 2007, Yun 07, 09 etc.).

- Two dimensional problem can be considered as the anti-plane elasticity of fiber reinforced composite materials.





Plasmon resonance

Metallic nano particle and light interaction causes a strong excitation of the collective electrons oscillations(plasmons) in the metallic particle. For noble metals such as gold and silver this resonance happens at the visible frequency. The color of the resonant frequency is absorbed.

Figure : Lycurgus cup, 4th-century Roman glass; Tiny gold particles are embedded in the glass

Green -reflected light.
Red- transmitted light; It is due to tiny gold particles embedded in the glass, which have an absorption peak at around 520 nm

Plasmon resonance

At certain frequency metallic particles has the negative relative permittivity $\epsilon(\omega)$ (Drude model). For wave length much longer than the dimension of the particle, the quasi-static regime is valid.

For such negative permittivity case, $\lambda = \frac{\epsilon(\omega)+1}{2(\epsilon(\omega)-1)}$ can be the eigenvalue of $\mathcal{K}_{\partial D}^*$. We call it plasmon eigenvalues. The eigenvalues correspond to the resonant frequencies.

ALR

Cloaking by Anomalous Localized Resonance.

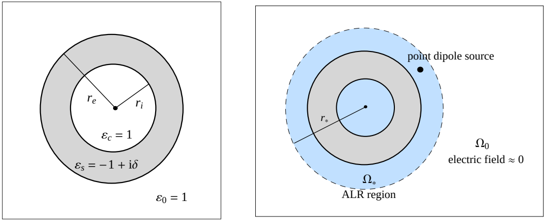


Figure : Cloaking due to the anomalous localized resonance: the concentric case

Anomalous localized resonance occurs at the accumulation point of eigenvalues.
Milton et al. (2006, 2007), Ammari-Ciraolo-Kang-Lee-Milton (2013, Spectral analysis)

Questions and recent approaches

- Eigenvalue distribution for general smooth domain and the spectral decomposition of Lipschitz domain
- Designing the shape for the specific spectrum
- Validation of the quasi-static approximation for nano-scale particle and relative long wave length
- Dependence of the plasmon resonance on the geometry of D
- Full maxwell equation and elastic system.
- Application: Analysis on the focusing effect. Mathematical imaging and focusing in resonant media...

Spectral decomposition of Lipschitz domain?

Intersecting disks: Lipschitz domain example

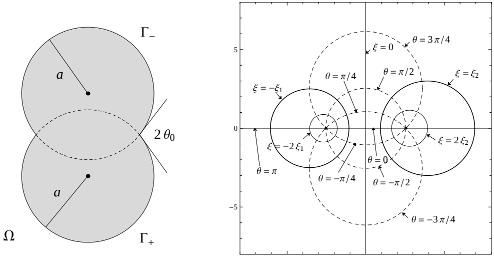


Figure : The left figure is the intersecting disks Ω. Level coordinate curves of bipolar coordinate.

Theorem (Kang-L.-Yu)

Let $\sigma_{ac}(\mathcal{K}^*_{\partial\Omega})$, $\sigma_{sc}(\mathcal{K}^*_{\partial\Omega})$ and $\sigma_{sc}(\mathcal{K}^*_{\partial\Omega})$ be the absolutely continuous, singularly continuous, and pure point spectrum of $\mathcal{K}^*_{\partial\Omega}$ on $\mathcal{H}^*$, respectively. Then we have

$$\sigma_{ac}(\mathcal{K}^*_{\partial\Omega}) = [-b, b], \quad \sigma_{sc}(\mathcal{K}^*_{\partial\Omega}) = \emptyset, \quad \sigma_{pp}(\mathcal{K}^*_{\partial\Omega}) = \emptyset.$$

The spectral bounds is given by

$$b = \frac{1}{2} - \frac{\pi - \theta_0}{\pi} = \frac{\theta_0}{\pi} - \frac{1}{2}.$$

This bound coincides the essential spectrum bound of the NP operator on curvilinear polygonal domains obtained by Perfekt-Putinar 2014.

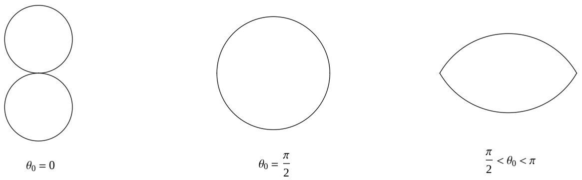


Figure : Disks of various intersecting angles

Resonance and eigenvalues

Consider the following problem:

$$\begin{cases} \nabla \cdot \epsilon \nabla u = f & \text{in } \mathbb{R}^2, \\ u(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases}$$

where the distribution of the dielectric constant is given by

$$\epsilon = (\epsilon_0 + i\delta)\chi(\Omega) + 1\chi(\mathbb{R}^2 \setminus \overline{\Omega}).$$

A typical such source functions are polarized dipoles, namely,

$$f(x) = a \cdot \nabla \delta_z(x)$$

for some $z \in \mathbb{R}^2 \setminus \overline{\Omega}$, where a is a constant vector and δ_z is the Dirac mass.

- The solution u_δ can be represented as

$$u_\delta(x) = q(x) + \mathcal{S}_{\partial\Omega}[\varphi_\delta](x), \quad x \in \mathbb{R}^2,$$

where $q(x) = \int_{\mathbb{R}^d} \Gamma(x-y) f(y) dy$, $(\lambda I - \mathcal{K}^*_{\partial\Omega})[\varphi_\delta] = \partial_\nu q$ on $\partial\Omega$, and

$$\lambda := \frac{\epsilon_0 + 1 + i\delta}{2(\epsilon_0 - 1) + 2i\delta} = s + i\delta.$$

(We denote the imaginary part again as δ for notational simplicity).

- For a given ϵ_c resonance is characterized by the fact

$$\|\nabla(u_\delta - q)\|_{L^2(\Omega)} \to \infty \quad \text{as } \delta \to 0.$$

- We have

$$C_1 \|\varphi_\delta\|_{\mathcal{H}^*} \le \|\nabla(u_\delta - q)\|_{L^2(\Omega)} \le C_2 \|\varphi_\delta\|_{\mathcal{H}^*}$$

for some positive constants C_1 and C_2.

- Remind that $\mathcal{K}^*_{\partial\Omega} = \int_{-b}^{b} t\, d\mathcal{E}_t$. So the boundary density function becomes

$$\varphi_\delta = \int_{-b}^{b} \frac{1}{\lambda - t}\, d\mathcal{E}_t[\partial_\nu q].$$

We have

$$\|\varphi_\delta\|^2_{\mathcal{H}^*} = \int_{-b}^{b} \frac{1}{(s-t)^2 + \delta^2}\, d\langle \partial_\nu q, \mathcal{E}_t[\partial_\nu q]\rangle_{\mathcal{H}^*}. \qquad \text{(Poisson integral)}$$

- Suppose that the spectral measure $\mu(t)dt := d\langle \partial_\nu q, \mathcal{E}_t[\partial_\nu q]\rangle_{\mathcal{H}^*}$ is absolutely continuous near t then

$$\lim_{\delta\to 0} \delta \|\varphi_\delta\|^2_{\mathcal{H}^*} = \lim_{\delta\to 0}\left(\delta^{1/2}\|\varphi_{t,\delta}\|_*\right)^2 = \frac{\pi}{2}(\mu(t+) + \mu(t-)).$$

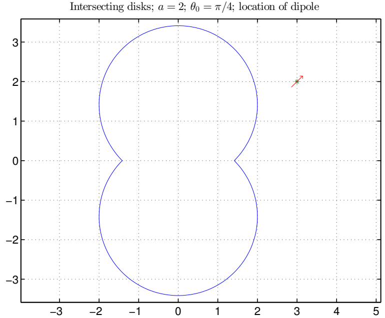
- Define

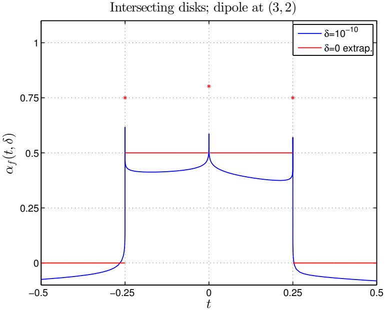
$$\alpha_f(t) := \sup\left\{ \alpha \;\middle|\; \limsup_{\delta\to 0} \delta^{\alpha}\|\varphi_{t,\delta}\|_* = \infty \right\}, \quad t \in (-1/2, 1/2). \qquad (2)$$

Characterization of the purely point spectrum (isolated) and singularly continuous spectrum (not isolated) may be achieved by $\alpha_f(t) = 1$.

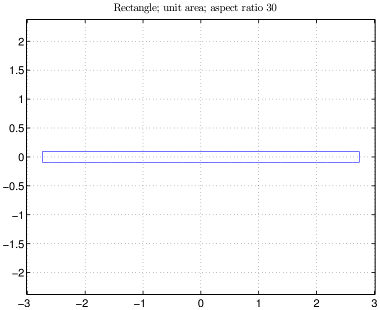
Numerical Computation: joint work with H. Kang and J. Helsing

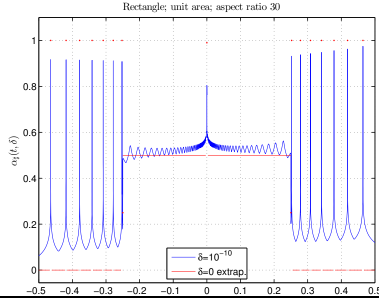
The absolutely continuous spectrum appears:

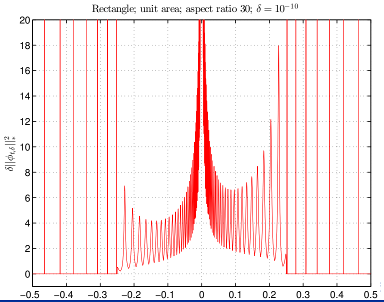


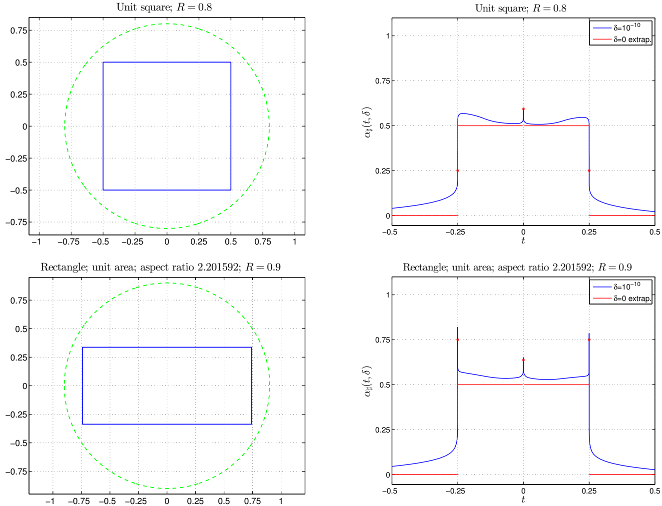


The absolutely continuous and singularly continuous, and pure point spectrum appears:








Unit square; $R = 0.8$
Unit square; $R = 0.8$
δ=10^{-10}
δ=0 extrap.
$\alpha_{\sharp}(t,\delta)$
t
Rectangle; unit area; aspect ratio 2.201592; $R = 0.9$
Rectangle; unit area; aspect ratio 2.201592; $R = 0.9$
δ=10^{-10}
δ=0 extrap.
$\alpha_{\sharp}(t,\delta)$
t

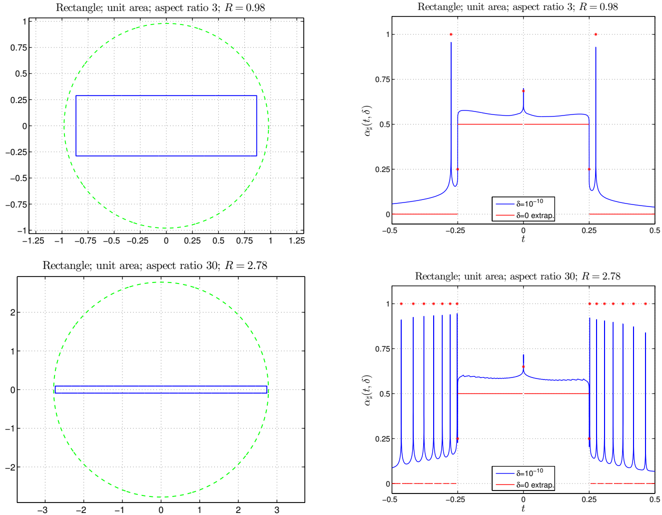


Figure : Rectangles with various aspect ratios (Left column) and the corresponding spectra (Right column). The second row exhibits a rectangle with the special aspect ratio such that eigenvalues just about to emerge at the two ends of the continuous spectrum interval.

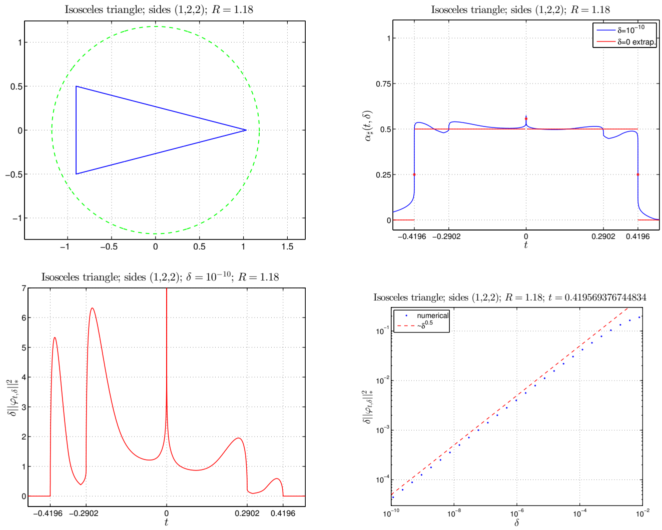


Figure : Spectrum of the isosceles triangle with sides 1, 2 and 2. The values of $0.5(1 - \theta/\pi)$ for interior angles, say θ, are approximately 0.4196 and 0.2902. The larger number 0.4196 bounds the essential spectrum. While the indicator function $\alpha_\sharp(t)$ changes only at zero and 0.4196, the functions $\alpha_\sharp(t,\delta)$ and $\delta\|\varphi_{t,\delta}\|_*^2$ for $\delta = 10^{-10}$ show dynamic changes near 0.2902 as well.

The absolutely continuous and singularly continuous appears:

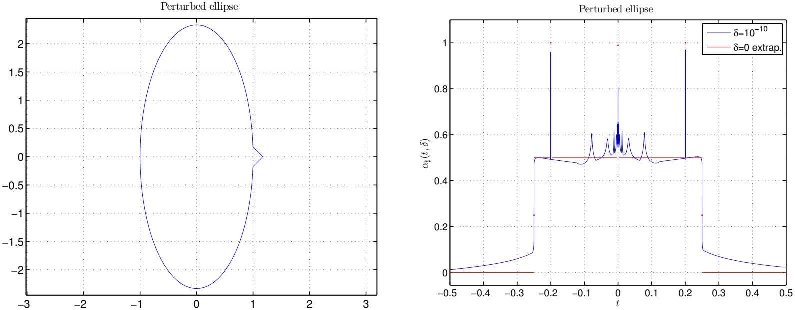


Figure : Perturbed ellipse.

Thank you!