Twisted equivariant K-theory and topological phases

Yosuke KUBOTA

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Spectral Theory Novel Materials April 19, 2016

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- $U: G = \mathbb{Z}^d \curvearrowright \mathcal{H}$: unitary representation

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Assumption

The Hamiltonian H has a spectral gap at $\mu \in \mathbb{R}$.

We say that H_1 and H_2 are in the same topological phase if $E_{\leq \mu}(H_1) \cong E_{\leq \mu}(H_2)$ as vector bundles.

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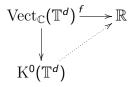
K-theory and topological phases

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Example: The first Chern number for d = 2;

$$c_1(E_{\leq \mu}(H)) := \frac{-1}{2\pi i} \int_{\mathbb{T}^2} \operatorname{tr}(p_x[\nabla_1, p_x][\nabla_2, p_x]) dx$$

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(p_x : orthogonal projection onto $E_{\leq \mu}(H)_x$). Rem. In 2d IQHE, it is related to the Hall conductance by the TKNN formula.

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- ◎ (K'15) The uniform Roe algebra $C_u^*(X)$: the closure of { $T \in \mathbb{B}(\ell^2 X) \mid \exists R > 0 \text{ s.t. } T_{xy} = 0 \text{ if } d(x, y) > R$ }

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Anyway, there is a group homomorphism

$$\mathsf{ind}\colon \mathrm{K}_0(\mathcal{A}) o \mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$$

Generalizations

Equivariant K-theory

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-graded C*-algebras i.e. $\mathbb{Z}_2 \curvearrowright A$.

Example: The Clifford algebra $C\ell_{n,m}$.

- $\mathfrak{H}:=\ell^2(\mathbb{Z}^2,\mathbb{C}^n)$,
- $U: \mathbb{Z}^d \to \mathcal{U}(\mathcal{H})$: the regular representation (i.e. U_g : shift operator),
- *H* ∈ B(ℋ)_{sa}: Z^d-invariant, (*H_k*)_{k∈T²}: continuous, *H* has a spectral gap at μ.

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In addition, we assume that $\exists T : \mathcal{H} \to \mathcal{H}$ s.t.

- T is antilinear, $U_g T = T U_g$, $T^2 = -1$,
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Definition

 $[E_{\leq \mu}(H)] \in \operatorname{KR}_0(C(\mathbb{T}^d, \mathbb{M}_n), \operatorname{Ad} T) (= \operatorname{KQ}^0(\mathbb{T}^2, \tau)) \cong \mathbb{Z}_2$ is called the *Kane-Mele invariant*.

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$$\begin{split} & \mathcal{H}: \ \mathbb{Z}_2\text{-graded separable Hilbert space.} \\ & \to \mathbb{P}\mathcal{H}:= (\mathcal{H}\setminus\{0\})/\mathbb{T}\text{: the space of states.} \\ & \text{It is equipped with the function} \end{split}$$

$$\Phi(\neg, \neg): \mathbb{PH} \times \mathbb{PH} \to \mathbb{R}_{>0}, \Phi([\xi], [\eta]) = rac{|\langle \xi, \eta \rangle|}{\|\xi\| \|\eta\|}.$$

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The group of symmetries in quantum mechanics:

 $\mathsf{Aut}_{\mathsf{qtm}}(\mathbb{PH}) := \{ f : \mathbb{PH} \to \mathbb{PH} \mid f^* \Phi = \Phi, f\gamma = \gamma f \}$

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Theorem (Wigner's theorem)

$$\mathsf{Aut}_{\mathsf{qtm}}(\hat{\mathbb{P}}\mathcal{H})\cong\mathsf{Aut}_{\mathsf{qtm}}(\mathcal{H})/\mathbb{T}$$

where

 $\mathsf{Aut}_{\mathsf{qtm}}(\mathcal{H}) := (\mathsf{linear}/\mathsf{antilinear} \text{ and } \mathsf{even}/\mathsf{odd} \text{ unitaries on } \mathcal{H}).$

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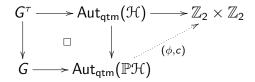
$$\operatorname{Aut}_{qtm}(\mathcal{H}) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\downarrow$$

$$G \longrightarrow \operatorname{Aut}_{qtm}(\mathbb{P}\mathcal{H})$$

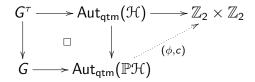
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Theorem (Freed-Moore'13, K.)

The data (ϕ, c, τ) is classified by the set

$$\bigsqcup_{\phi\in\check{H}^1(G;\mathbb{Z}_2)}\check{H}^1(G;\mathbb{Z}_2)\ltimes_{\epsilon}\check{H}^2(G;\mathbb{T}).$$

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K-theory and topological phases

The twisted equivariant K_0 -group

- G: finite group, (ϕ, c, τ) : twist on G,
- A: ϕ -twisted (\mathbb{Z}_2 -graded) G-C*-algebra i.e. $G \curvearrowright A$ s.t. α_g is linear/antilinear if $\phi(g) = 0/1$.

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We define the twisted equivariant K-functor

$${}^{\phi}\mathrm{K}^{\mathsf{G}}_{*,\boldsymbol{c},\tau} \colon {}^{\phi}\mathfrak{Calg}^{\mathsf{G}}_{\mathbb{Z}_{2}} \to \mathfrak{Ab}$$

as a canonical generalization of $\mathrm{K}^{\mathsf{G}}_*$, $\mathrm{KR}^{\mathsf{G}}_*$.

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It classifies topological phases with the symmetry given by (G, ϕ, c, τ) . Assume the \mathbb{Z}_2 -grading of A is trivial.

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$$\mathfrak{F}^{\mathsf{G}}_{\mathsf{c}, \mathbb{V}}(\mathsf{A}) := \{ \mathsf{s} \in \mathsf{A} \, \hat{\otimes} \, \mathbb{K}(\mathbb{V})_{\mathrm{sa}} \mid \mathsf{s}^2 = 1, lpha_{\mathsf{g}}(\mathsf{s}) = (-1)^{\mathsf{c}(\mathsf{g})} \mathsf{s} \}$$

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Theorem

$${}^{\phi}\mathrm{K}^{\mathcal{G}}_{0,c, au}(\mathcal{A}) = \bigcup_{\mathcal{V}} \mathfrak{F}^{\mathcal{G}}_{c,\mathcal{V}}(\mathcal{A}) / \sim_{\mathrm{homotopy}}$$

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H satisfies
$$HU_g = (-1)^{c(g)}U_gH \Rightarrow [H|H|^{-1}] \in {}^{\phi}\mathrm{K}^{\mathcal{G}}_{0,c,\tau}(A).$$

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K-theory and topological phases

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The *twisted crossed product* is the \mathbb{R} -*-algebra defined to be

$$G \ltimes_{c,\tau}^{\phi} A := \{\sum_{g \in G} a_g u_g \mid a_g \in A\}$$

with

$$(\sum a_g u_g)(\sum b_h u_h) = \sum \tau(g, h) a_g \alpha_g(b_h) u_{gh}$$
$$(\sum a_g u_g)^* = \sum \tau(g, g) \alpha_{g^{-1}}(a_g^*) u_{g^{-1}}$$

(identified with the Real C*-algebra $(G \ltimes_{c,\tau}^{\phi} A) \otimes_{\mathbb{R}} \mathbb{C}).$

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Theorem

$${}^{\phi}\mathrm{K}^{\mathsf{G}}_{0,c,\tau}(A) \cong \mathrm{KR}(\mathsf{G} \ltimes_{c,-\overline{\tau}}^{\phi} A)$$

(Here
$$\overline{\tau} = \tau + \epsilon(c, c)$$
.)

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- the index of the twisted $C\ell_{0,d}$ -Dirac operator $(A = C(\mathbb{T}^d))$,
- the Kasparov product $(A = \mathbb{Z}^d \ltimes C(\Omega))$ or

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- G: finite group, (ϕ, c, τ) : a twist on G,
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- the index of the twisted $C\ell_{0,d}$ -Dirac operator $(\mathcal{A} = \mathcal{C}(\mathbb{T}^d))$,
- the Kasparov product $(A = \mathbb{Z}^d \ltimes C(\Omega))$ or
- the coarse Baum-Connes isomorphism $(A = C_u^*(X))$, we get the group homomorphism

$$\mathsf{ind}\colon {}^{\phi}\!\mathrm{K}^{\mathcal{G}}_{0,c,\tau}(\mathcal{A}) \to {}^{\phi}\!\mathrm{K}^{\mathcal{G}}_{0,c,\tau}(\mathrm{C}\ell_{0,d}).$$

Example: CT-symmetries

We consider the case that $(\phi, c) : \mathcal{A} \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is injective. Choices of (\mathcal{A}, τ) are classified by

$$C^1 = \pm 1$$
 and $T^2 = \pm 1$

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 $(C, T \in \mathcal{A}^{\tau} \text{ are lifts of } (1, 1), (1, 0) \in \mathcal{A} \text{ s.t. } (CT)^2 = 1).$ There are 10 choices and ${}^{\phi}C^*_{c,\tau}\mathcal{A} := \mathcal{A} \ltimes_{c,\tau}^{\phi} \mathbb{R}$ is classified by

A	1	Р	T		(S	9				
<i>C</i> ²	\backslash				1	-1	1	1	-1	-1	
T^2	\backslash		1	-1		\backslash	1	$^{-1}$	1	-1	
${}^{\phi}C^*_{c,\tau}\mathcal{A}$	\mathbb{C}	$\mathbb{C}\ell_1$	$\mathbb{M}_2(\mathbb{R})$	H	$C\ell_{0,2}$	$\mathrm{C}\ell_{2,0}$	$\mathrm{C}\ell_{1,2}$	$\mathrm{C}\ell_{0,3}$	$C\ell_{2,1}$	Cℓ _{3,0}	
${}^{\phi}\mathrm{K}^{\mathcal{A}}_{0,\boldsymbol{c},\tau}$	K ₀	K ₁	KR ₀	KR_4	KR ₂	KR_{6}	KR_1	KR_3	KR_7	KR_{5}	
Cartan	A	All	AI	All	D	С	BDI	DIII	CI	CII	

Table: The 10-fold way and Clifford algebras

dim	Α	AIII	AI	BDI	D	DIII	All	CII	С	CI
0	\mathbb{Z}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}	0	0	0
1	0	\mathbb{Z}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}	0	0
2	\mathbb{Z}	0	0	0	Z	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}	0
3	0	\mathbb{Z}	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}

Table: Kitaev's periodic table

cf. Bott periodicity

$$\pi_i(U) \cong \begin{cases} \mathbb{Z} & i = 2n+1 \\ 0 & i = 2n \end{cases}, \pi_i(O) \cong \begin{cases} \mathbb{Z} & i = 8n-1, 8n+3 \\ \mathbb{Z}_2 & i = 8n, 8n+1 \\ 0 & \text{otherwise} \end{cases}$$

Example: reflection-invariant systems

 $G=\mathcal{A}\times \mathfrak{R},$ where $\mathfrak{R}\cong \mathbb{Z}_2$ acting on the material as a reflection.

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 $(P := CT, R \text{ is the lift of the generator of } \mathcal{R} \text{ s.t. } R^2 = 1).$ It is not difficult to determine the finite-dimensional algebras $G \ltimes_{c,\tau}^{\phi} C\ell_{0,d}$ and we get

$${}^{\phi}\mathrm{K}^{\mathcal{G}}_{0,c,\tau}(\mathrm{C}\ell_{0,d}) \cong \begin{cases} {}^{\phi}\mathrm{K}^{\mathcal{A}}_{d-1,c,\tau}(\mathbb{R}) & \text{if } (\epsilon,\nu) = (+,+), \\ {}^{\phi}\mathrm{K}^{\mathcal{A}}_{d+1,c,\tau}(\mathbb{R}) & \text{if } (\epsilon,\nu) = (+,-), \\ {}^{\phi}\mathrm{K}^{\mathcal{A}}_{d,c,\tau}(\mathbb{R})^{2} & \text{if } (\epsilon,\nu) = (-,+), \\ \mathrm{K}_{d,c,\tau}(\mathbb{R}) & \text{if } (\epsilon,\nu) = (-,-). \end{cases}$$

where $RP = \epsilon PR$ and $RT = \nu TR$.

Reflection	Class	C_q or R_q	d = 0	d = 1	d=2	d=3	d=4	d=5	d=6	d=7
R	Α	C_1	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
R^+	AIII	C_0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0
R^{-}	AIII	C_1	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
	AI	R_1	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2
	BDI	R_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0
	D	R_3	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}
R^{+}, R^{++}	DIII	R_4	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0
	AII	R_5	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0
	CII	R_6	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0
	\mathbf{C}	R_7	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
	CI	R_0	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2
	AI	R_7	0	0	0	\mathbb{Z}	0	" \mathbb{Z}_2 "	\mathbb{Z}_2	\mathbb{Z}
	BDI	R_0	\mathbb{Z}	0	0	0	\mathbb{Z}	0	" \mathbb{Z}_2 "	\mathbb{Z}_2
	D	R_1	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	" \mathbb{Z}_2 "
$R^{-}, R^{}$	DIII	R_2	" \mathbb{Z}_2 "	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0
	AII	R_3	0	" \mathbb{Z}_2 "	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}
	CII	R_4	\mathbb{Z}	0	" \mathbb{Z}_2 "	\mathbb{Z}_2	\mathbb{Z}	0	0	0
	\mathbf{C}	R_5	0	\mathbb{Z}	0	" \mathbb{Z}_2 "	\mathbb{Z}_2	\mathbb{Z}	0	0
	\mathbf{CI}	R_6	0	0	\mathbb{Z}	0	" \mathbb{Z}_2 "	\mathbb{Z}_2	\mathbb{Z}	0
R^{+-}	BDI	R_1	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2
R^{-+}	DIII	R_3	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}
R^{+-}	CII	R_5	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0
R^{-+}	CI	R_7	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
R^{-+}	BDI, CII	C_1	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
R^{+-}	DIII, CI	C_1	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}

Classification of reflection invariant topological phases

Takahiro Morimoto and Akira Furusaki, Topological classification with additional symmetries from Clifford algebras, Phys. Rev. B 88, 125129.

Yosuke KUBOTA (Univ. Tokyo)

K-theory and topological phases

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$$\mathrm{K}^{0}_{\mathfrak{R}}(\mathbb{T}^{1}) \to \mathrm{K}^{\mathfrak{R}}_{0}(\mathrm{C}\ell_{0,1}) \cong \mathbb{Z}.$$

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The simplest vector bundle with nontrivial index is $E \to \mathbb{T}^1$ s.t. $E|_0 \cong V_+$ and $E|_{\pi} \cong V_-$ ($V_{\pm} \cong \mathbb{C}$ with the \mathbb{Z}_2 -action given by ± 1).

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The corresponding Hamiltonian is

$$H:=rac{1}{2}egin{pmatrix} s+s^*&i(s-s^*)\ i(s-s^*)&-(s+s^*) \end{pmatrix}\in\mathbb{B}(\ell^2(\mathbb{Z};\,V_+\oplus\,V_-)),$$

where s is the shift operator.

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where s is the shift operator.

cf.) the clean Kitaev chain (a 1D type BDI systems):

$$H = rac{1}{2} egin{pmatrix} s+s^*+2\mu & -i(s-s^*) \ -i(s-s^*) & -(s+s^*+2\mu) \end{pmatrix},$$

(μ : chemical potential).