Asymptotic behavior of the interior transmission eigenvalues

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Evolution Equations on Singular Spaces, CIRM, April 25-29, 2016

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1. Definition of the transmission eigenvalues

Let $\Omega \subset \mathbf{R}^d$, $d \geq 2$, be a bounded, connected domain with a C^∞ smooth boundary $\Gamma = \partial\Omega$. A complex number $\lambda \in \mathbf{C}$, $\lambda \neq 0$, will be said to be a transmission eigenvalue if the following problem has a non-trivial solution:

$$\begin{cases} (\nabla c_1(x)\nabla + \lambda n_1(x))\, u_1 = 0 & \text{in} \quad \Omega, \\ (\nabla c_2(x)\nabla + \lambda n_2(x))\, u_2 = 0 & \text{in} \quad \Omega, \\ u_1 = u_2, \ \ c_1\partial_\nu u_1 = c_2\partial_\nu u_2 & \text{on} \quad \Gamma, \end{cases} \tag{1}$$

where ν denotes the exterior Euclidean unit normal to Γ, $c_j, n_j \in C^\infty(\overline{\Omega})$, $j = 1, 2$ are strictly positive real-valued functions. The transmission eigenvalues can be viewed as the eigenvalues of the non-symmetric operator $\mathcal{A}$ defined by

$$\mathcal{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{n_1(x)}\nabla c_1(x)\nabla u_1 \\ -\frac{1}{n_2(x)}\nabla c_2(x)\nabla u_2 \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \Big\{(u_1, u_2) \in \mathcal{H} : \nabla c_1(x)\nabla u_1 \in L^2(\Omega),\ \nabla c_2(x)\nabla u_2 \in L^2(\Omega),$$

$$u_1 = u_2,\ c_1\partial_\nu u_1 = c_2\partial_\nu u_2 \quad \text{on} \quad \Gamma\Big\}$$

where $\mathcal{H} = H_1 \oplus H_2$, $H_j = L^2(\Omega, n_j(x)dx)$. Then the transmission eigenvalues are the poles of the resolvent of $\mathcal{A}$ (if it forms a meromorphic family) and the multiplicity of a pole λ_k is defined by

$$\mathrm{mult}(\lambda_k) = \mathrm{rank}(2\pi i)^{-1}\int_{|\lambda-\lambda_k|=\varepsilon}(\lambda - \mathcal{A})^{-1}d\lambda = \mathrm{tr}(2\pi i)^{-1}\int_{|\lambda-\lambda_k|=\varepsilon}(\lambda - \mathcal{A})^{-1}d\lambda.$$

Our goal is to study the asymptotic behaviour of the counting function $N(r) = \#\{\lambda - \mathrm{trans.\,eig.} : |\lambda| \leq r^2\}$, $r > 1$. We will see that it is closely related to the localization of the transmission eigenvalues on the complex plane.

2. The Dirichlet-to-Neumann map.

The Dirichlet-to-Neumann map, $N_j(\lambda) : H^1(\Gamma) \to L^2(\Gamma)$, associated to the pair (c_j, n_j) is defined by

$$N_j(\lambda)f = \partial_\nu u_j|_\Gamma,$$

where u_j solves the equation

$$\begin{cases} (\nabla c_j(x)\nabla + \lambda n_j(x))\, u_j = 0 & \text{in} \quad \Omega, \\ u_j = f & \text{on} \quad \Gamma. \end{cases} \tag{2}$$

Denote by G_j, $j = 1, 2$, the Dirichlet self-adjoint realization of the operator $-n_j^{-1}\nabla c_j \nabla$ on the Hilbert space H_j. It is well-known that $N_j(\lambda)$ is meromorphic with poles the eigenvalues of G_j. Introduce the operator

$$T(\lambda) = c_1 N_1(\lambda) - c_2 N_2(\lambda).$$

We have the following trace formula.

Lemma 1

Suppose that the inverse $T(\lambda)^{-1}$ exists as a meromorphic function. Then the resolvent of the operator $\mathcal{A}$ is meromorphic, too, and we have the formula

$$M(\gamma) = M_1(\gamma) + M_2(\gamma) + \mathrm{tr}(2\pi i)^{-1} \int_\gamma \frac{dT(\lambda)}{d\lambda} T(\lambda)^{-1} d\lambda \tag{3}$$

where γ is a simple, positively orientied, piecewise smooth, closed curve in the complex plane, which avoids the poles of $T(\lambda)^{-1}$ and the eigenvalues of G_1 and G_2, $M(\gamma)$ is the number of the transmission eigenvalues inside γ, and $M_j(\gamma)$ is the number of the eigenvalues of the operator G_j inside γ.

3. Weyl asymptotics for the counting function

Theorem 1 (Petkov-V., J. Spectral Theory 2016)

Suppose either the condition

$$c_1(x) = c_2(x), \partial_\nu c_1(x) = \partial_\nu c_2(x), n_1(x) \neq n_2(x), \quad \forall x \in \Gamma, \quad (\textit{isotropic case}) \tag{4}$$

or the condition

$$c_1(x) \neq c_2(x), \quad \forall x \in \Gamma. \quad (\textit{anisotropic case}) \tag{5}$$

Suppose also that the operator $T(\lambda)$ is invertible in a region of the form

$$\left\{ \lambda \in \mathbf{C} : \ |\mathrm{Im}\,\lambda| \geq C\,(|\mathrm{Re}\,\lambda| + 1)^{1-\frac{\kappa}{2}} \right\}, \quad C > 0,\, 0 < \kappa \leq 1, \tag{6}$$

and satisfies there the bound

$$\left\| T(\lambda)^{-1} \right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C_0 |\lambda|^{M_0}, \quad C_0, M_0 > 0.$$

Then we have the asymptotics

$$N(r) = (\tau_1 + \tau_2) r^d + O_\varepsilon(r^{d-\kappa+\varepsilon}), \quad \forall\, 0 < \varepsilon \ll 1, \tag{7}$$

where

$$\tau_j = \frac{\omega_d}{(2\pi)^d} \int_\Omega \left(\frac{n_j(x)}{c_j(x)} \right)^{d/2} dx,$$

ω_d *being the volume of the unit ball in* $\mathbf{R}^d$.

Known results. In the isotropic case when $c_1 \equiv c_2 \equiv 1$, $n_2 \equiv 1$, $n_1(x) > 1$ on Ω, the asymptotic for $N(r)$ with a remainder term $o(r^d)$ is proved by M. Faierman, SIAM J. Math. Anal. 2014, and by L. Robbiano, preprint 2013.

Idea of the proof. It is inspired by the paper F. Cardoso, G. Popov and G. Vodev, CPDE 2001, where Weyl type asymptotics have been proved for the counting function of the resonances associated to an exterior transmission problem. We can get an asymptotic for $N(r) - N(r/2)$ by using the trace formula (3), the Weyl asymptotics for the counting functions of the eigenvalues of G_1 and G_2, and the Theorems of Caratheodory and Jensen. We use in an essential way that $\dim \Gamma = d - 1$.

4. Eigenvalue-free regions

Theorem 2 (CMP 2015)

Isotropic case. Assume the condition

$$c_1(x) = c_2(x), \quad \partial_\nu c_1(x) = \partial_\nu c_2(x), \quad n_1(x) \neq n_2(x), \quad \forall x \in \Gamma. \tag{8}$$

Then there are no transmission eigenvalues in $\Lambda_+ \cup \Lambda_-$*, where*

$$\Lambda_+ = \left\{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > 0,\ |\mathrm{Im}\,\lambda| \geq C_\epsilon \left(\mathrm{Re}\,\lambda + 1\right)^{\frac{3}{4}+\epsilon}\right\}, \quad \forall 0 < \epsilon \ll 1,$$

$$\Lambda_- = \{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda \leq -C_1\} \cup \{\lambda \in \mathbf{C} : -C_1 \leq \mathrm{Re}\,\lambda \leq 0,\ |\mathrm{Im}\,\lambda| \geq C_2\}, \quad C_1, C_2 > 0.$$

In this case the asymptotic (7) holds with $\kappa = 1/2$.

Theorem 3 (CMP 2015)

Anisotropic case. Assume the condition

$$(c_1(x)-c_2(x))(c_1(x)n_1(x)-c_2(x)n_2(x))<0, \quad \forall x\in\Gamma. \tag{9}$$

Then there are no transmission eigenvalues in $\Lambda_+\cup\Lambda'_-$*, where* Λ_+ *is as above and*

$$\Lambda'_-=\left\{\lambda\in\mathbf{C}:\mathrm{Re}\,\lambda\le 0,\ |\mathrm{Im}\,\lambda|\ge C_N\left(|\mathrm{Re}\,\lambda|+1\right)^{-N}\right\},\quad \forall N\gg 1.$$

In this case the asymptotic (7) holds with $\kappa=1/2$*.*
Assume the condition

$$(c_1(x)-c_2(x))(c_1(x)n_1(x)-c_2(x)n_2(x))>0, \quad \forall x\in\Gamma. \tag{10}$$

Then there are no transmission eigenvalues in $\Lambda'_+\cup\Lambda_-$*, where* Λ_- *is as above and*

$$\Lambda'_+=\left\{\lambda\in\mathbf{C}:\mathrm{Re}\,\lambda>0,\ |\mathrm{Im}\,\lambda|\ge C\left(\mathrm{Re}\,\lambda+1\right)^{\frac{4}{5}}\right\}.$$

In this case the asymptotic (7) holds with $\kappa=2/5$*.*

Moreover, if in addition to (10) we assume either the condition

$$\frac{n_1(x)}{c_1(x)} \neq \frac{n_2(x)}{c_2(x)}, \quad \forall x \in \Gamma, \tag{11}$$

or the condition

$$\frac{n_1(x)}{c_1(x)} = \frac{n_2(x)}{c_2(x)}, \quad \forall x \in \Gamma, \tag{12}$$

then there are no transmission eigenvalues in $\Lambda_+ \cup \Lambda_-$.

Remark 1

One can show that under the condition (9) there are infinitely many transmission eigenvalues in $\{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda < 0\}$ *and that their counting function,* $N^-(r)$*, satisfies an asymptotic of the form*

$$N^-(r) = \tau_0 r^{d-1} + O(r^{d-2}).$$

Known results. In the isotropic case when $c_1 \equiv c_2 \equiv 1$, $n_2 \equiv 1$, $n_1(x) > 1$ on Ω, it was proved by M. Hitrik, K. Krupchyk, P. Ola and L. Päivärinta, Math. Res. Lett. 2011, that there are no transmission eigenvalues in $\Lambda''_+ \cup \Lambda_-$, where

$$\Lambda''_+ = \left\{ \lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > 0,\ |\mathrm{Im}\,\lambda| \ge C\,(\mathrm{Re}\,\lambda + 1)^{\frac{24}{25}} \right\}.$$

To prove the above theorems we make our problem semi-classical by putting $h = |\mathrm{Re}\,\lambda|^{-1/2}$, $z = h^2\lambda = \pm 1 + i\mathrm{Im}\,z$, if $|\mathrm{Re}\,\lambda| \ge |\mathrm{Im}\,\lambda|$, and $h = |\mathrm{Im}\,\lambda|^{-1/2}$, $z = h^2\lambda = \mathrm{Re}\,z + i$, if $|\mathrm{Re}\,\lambda| \le |\mathrm{Im}\,\lambda|$. The proof of Theorems 2 and 3 is based on the following semi-classical properties of the Dirichlet-to-Neumann map $\mathcal{N}_j(z,h) = -ihN_j(\lambda)$.

Theorem 4 (CMP 2015)

For every $0 < \epsilon \ll 1$, $0 < h \ll 1$, $|\mathrm{Im}\,z| \ge h^{1/2-\epsilon}$, the Dirichlet-to-Neumann map $\mathcal{N}_j(z,h)$ is an $h - \Psi$ DO of class $OPS^1_{1/2-\epsilon}(\Gamma)$ with a principal symbol $\rho_j(x,\xi) = \sqrt{-r_0(x,\xi) + m_j(x)z}$ with $\mathrm{Im}\,\rho_j > 0$, where m_j denotes the restriction on Γ of the function n_j/c_j, and r_0 is the principal symbol of the Laplace-Beltrami operator $-\Delta_\Gamma$, Γ being considered as a Riemannian manifold equipped with the Riemannian metric induced by the Euclidean one.

Recall that $a \in S^k_\delta(\Gamma)$, $0 \le \delta < 1/2$, if $a \in C^\infty(T^*\Gamma)$ satisfies the bounds

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha,\beta} h^{-\delta(|\alpha|+|\beta|)} \langle\xi\rangle^{k-|\beta|}.$$

Set $\chi_j(x,\xi) = \phi((r_0(x,\xi) - m_j(x))/h^{\epsilon/2})$, where $\phi \in C_0^\infty(\mathbf{R})$, $\phi(t) = 1$ for $|t| \le 1$, $\phi(t) = 0$ for $|t| \ge 2$. That is, $\chi_j(x,\xi) = 1$ in an $O(h^{\epsilon/2})$ neighbourhood of the glancing region $\Sigma_j = \{(x,\xi) \in T^*\Gamma : r_0(x,\xi) - m_j(x) = 0\}$. Theorem 4 implies the following

Corollary 1

For every $0 < \epsilon \ll 1$, $0 < h \ll 1$, $h^{1/2-\epsilon} \le |\mathrm{Im}\, z| \le h^\epsilon$, we have the bound

$$\|N_j(z,h)\mathrm{Op}_h(\chi_j)\|_{L^2(\Gamma)\to L^2(\Gamma)} \le Ch^{\varepsilon/4}. \tag{13}$$

5. The case of strictly concave domains

We can improve (13) for strictly concave domains. More precisely, we have the following

Theorem 5 (Math. Ann. 2016)

Assume that Γ is strictly concave with respect to the Riemannian metric $g_j = \frac{n_j(x)}{c_j(x)} g_E$ in Ω, where g_E denotes the Euclidean metric in $\mathbf{R}^d$. Then, for every $0 < \epsilon \ll 1$, $0 < h \ll 1$, $h^{1-\epsilon} \le |\mathrm{Im}\, z| \le h^{\epsilon}$, we have the bound

$$\|N_j(z,h)\mathrm{Op}_h(\chi_j)\|_{L^2(\Gamma)\to L^2(\Gamma)} \le C h^{\varepsilon/4}. \tag{14}$$

We derive (14) from the following property of the Airy function $\mathrm{Ai}(z)$ and its first derivative $\mathrm{Ai}'(z)$.

Lemma 2 (Melrose-Taylor)

There exists a constant $C > 0$ such that for all $z \in \mathbf{C} \setminus (-\infty, 0)$ we have the bound

$$\left|\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}\right| \le C|z|^{1/2} + C|\mathrm{Im}\, z|^{-1}. \tag{15}$$

Using Theorem 5 we can prove the following

Theorem 6 (Math. Ann. 2016)

Assume that Γ is strictly concave with respect to both Riemannian metrics $g_j = \frac{n_j(x)}{c_j(x)} g_E$ in Ω, $j = 1, 2$. Assume also either the condition (8) or the condition (9). Then there are no transmission eigenvalues in

$$\left\{ \lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > 0,\ |\mathrm{Im}\,\lambda| \geq C_\epsilon \left(\mathrm{Re}\,\lambda + 1\right)^{\frac{1}{2}+\epsilon} \right\}, \quad \forall 0 < \epsilon \ll 1.$$

In this case the asymptotic (7) holds with $\kappa = 1$.

Thus getting eigenvalue-free regions is reduced to inverting the operator $T(z,h) = c_1 N_1(z,h) - c_2 N_2(z,h)$ with a principal symbol

$$c_1\rho_1 - c_2\rho_2 = \frac{\widetilde{c}(x)(c_0(x) r_0(x,\xi) - z)}{c_1\rho_1 + c_2\rho_2} \tag{16}$$

where $\widetilde{c}$ and c_0 are the restrictions on Γ of the functions

$$c_1 n_1 - c_2 n_2 \quad \text{and} \quad \frac{c_1^2 - c_2^2}{c_1 n_1 - c_2 n_2}$$

respectively. In the isotropic case we have $c_0 \equiv 0$ on Γ, while in the anisotropic case we have $c_0(x) \neq 0$, $\forall x \in \Gamma$. Under the condition (9) we have $c_0(x) \leq 0$, $\forall x \in \Gamma$.

6. Parametrix construction in the glancing region.

Known results. A parametrix for the DN map has been constructed by Sjöstrand, Mémoire de la SMF, 2014, (see Section 11), when $C_1 h^{2/3} \leq |\mathrm{Im}\, z| \leq C_2 h^{2/3}$, $C_2 > C_1 > 0$ being arbitrary, independent of h, without using the Airy function.

Idea of the proof of Theorem 5. We build a parametrix for the operator $N_j(z,h)\mathrm{Op}_h(\chi_j)$. To this end, we use the global symplectic normal form proved by Popov and Vodev, CMP 1999, valid in an $O(h^\epsilon)$ neigbourhood of the glancing region, which alows a complete separation of the normal and tangential variables. More precisely, after a suitable symplectic change of the variables, we are led to study a model operator of the form

$$P_0 = \mathcal{D}_t^2 + t + \mathcal{D}_{y_1} + i\mu q(y, \mathcal{D}_y) + h\widetilde{q}(y, \mathcal{D}_y; h, \mu), \quad t > 0,$$

where $\mu = \mathrm{Im}\, z$ satisfies $h^{1-2\varepsilon} \leq |\mu| \leq h^\varepsilon$, $\mathcal{D}_t = -ih\partial_t$, $\mathcal{D}_y = -ih\partial_y$, $y \in Y$, Y being a bounded manifold without boundary, $\dim Y = d-1$. The function $q \in C^\infty(T^*Y)$, $q \in S_0^0$, is real-valued and does not depend on t, h and μ, satisfying $0 < C_1 \leq q \leq C_2$, C_1 and C_2 being constants, $\widetilde{q} \in S_0^0$ uniformly in h and μ. Let $\eta = (\eta_1, \eta')$ be the dual variables of $y = (y_1, y')$. Then in these coordinates the glancing region is defined by $\eta_1 = 0$.

Given any integer $M \geq 1$ and any function $f \in L^2(Y)$, $\|f\| = 1$, we construct a function $\widetilde{u}(t,y) = \widetilde{u}_M(t,y;h,\mu)$ such that

$$P_0\widetilde{u} = O\left(h^{(M+1)\varepsilon/2}\right), \quad \widetilde{u}(0,y) = \mathrm{Op}_h\left(\phi(\eta_1|\mu|/h^{1+\varepsilon})\right)f + O(h^\infty)$$

where $\phi \in C_0^\infty(\mathbf{R})$, $\phi(\sigma) = 1$ for $|\sigma| \leq 1$, $\phi(\sigma) = 0$ for $|\sigma| \geq 2$. We will be looking for $\widetilde{u}$ in the form

$$\widetilde{u} = \phi(t/h^\varepsilon)\mathrm{Op}_h(A(t))g$$

where $g \in L^2(Y)$ is determined such that $\|g\|_{L^2(Y)} \leq O(1)\|f\|_{L^2(Y)}$, and

$$A(t) = \sum_{k=0}^{M} a_k(y,\eta;h,\mu)\psi_k(t,y,\eta;h,\mu),$$

$$\psi_k = h^{k/3}\frac{\mathrm{Ai}^{(k)}\left((t+\eta_1+i\mu q(y,\eta))h^{-2/3}\right)}{\mathrm{Ai}\left((\eta_1+i\mu q(y,\eta))h^{-2/3}\right)}.$$

Using the properties of the Airy functions we can prove the following

Proposition 1

For $t = 0$, all $k \geq 0$ and multi-indices α, we have the bound

$$\left|\partial_y^\alpha \psi_k\right| \leq C_{k,\alpha}\, p_1^k. \tag{17}$$

For all $t > 0$, $k \geq 0$ and multi-indices α, we have the bound

$$\left|\partial_y^\alpha \psi_k\right| \leq C_{k,\alpha}\, h^{-1/3} p_1^k. \tag{18}$$

Here

$$p_1 = |\eta_1|^{1/2} + |\mu|^{1/2} + \frac{h}{|\mu|} < 1.$$

The functions a_k do not depend on t, $a_0 = \phi(\eta_1|\mu|/h^{1+\varepsilon})$, and satisfy relationships of the form

$$\partial_y^\alpha a_{k+1} = \sum_{|\alpha_1|=0}^{|\alpha|+1} O(p_2^2)\partial_y^{\alpha_1} a_{k-1} + \sum_{|\alpha_2|=0}^{|\alpha|} O(p_2)\partial_y^{\alpha_2} a_k + \sum_{\ell=0}^{k} \sum_{|\beta|=0}^{k+|\alpha|} O(1)\partial_y^\beta a_\ell \tag{19}$$

for every multi-index α, where

$$p_2 = \frac{|\mu| p_1}{h} + \sqrt{\frac{|\mu|}{h}} > 1.$$

By induction we can deduce from (19) that the functions a_k satisfy the bounds

$$\left|\partial_y^\alpha a_k\right| \leq C_{k,\alpha} p_2^k. \tag{20}$$

We have

$$p_1 p_2 \leq O(h^{\varepsilon/2}) \tag{21}$$

as long as

$$|\mu|\left(|\mu| + |\eta_1|\right) \leq h^{1+\varepsilon}. \tag{22}$$

Using (21) we can conclude that the function $\widetilde{u}$ provides a parametrix in the region $|\eta_1| \leq h^{1+\varepsilon}/|\mu|$. The parametrix construction in the region $h^{1+\varepsilon}/|\mu| \leq |\eta_1| \leq h^\varepsilon$ is easier and can be done as in the hyperbolic region, showing that in our case the solutions of the corresponding eikonal and transport equations belong to better classes.

7. The case of a ball.

Better eigenvalue-free regions can be obtained if the functions c_j, n_j are constants and Ω is a ball. In this case we can use the properties of the Bessel functions to prove the following

Theorem 7 (Petkov-V., preprint 2016)

Assume that $\Omega = \{x \in \mathbf{R}^d : |x| \leq 1\}$ and the functions $n_j(x)$ and $c_j(x)$ are constants in a neighbourhood of the boundary Γ, $j = 1, 2$. Assume also either the condition (8) or the condition (9). Then there are no transmission eigenvalues in

$$\left\{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > 0,\ |\mathrm{Im}\,\lambda| \geq C\,(\mathrm{Re}\,\lambda + 1)^{\frac{1}{2}} \log\,(\mathrm{Re}\,\lambda + 2)\right\}.$$

If the functions $n_j(x)$ and $c_j(x)$ are constants everywhere in Ω, then there are no transmission eigenvalues in

$$\left\{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > 0,\ |\mathrm{Im}\,\lambda| \geq C\,(\mathrm{Re}\,\lambda + 1)^{\frac{1}{2}}\right\}. \tag{23}$$

We conjecture that the eigenvalue-free region (23) is optimal, but this is hard to prove even in the case of a ball.

Thank you !