A Polyakov formula for sectors

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Polyakov's formula

- Closed surface (M, g)
- Smooth conformal family of metrics $\{g_t = e^{2\sigma(t)} g\}$
- $\partial_t \log \det(\Delta_{g_t}) = -\frac{1}{24\pi} \int_M \sigma'(t)\, \mathrm{Scal}_t \, dA_{g_t} + \partial_t \log \mathrm{Area}(M, g_t)$
- Polyakov *Quantum geometry of bosonic strings,* (1981); Alvarez, *Theory of strings...*(1983); Osgood, Phillips & Sarnak *Extremals...,* (1988) and *Compact isospectral sets of surfaces* (1988).

Polyakov's formula on surfaces with conical singularities

- Alexey Kokotov, *On the Spectral theory...* (2013).
- The boundary of a connected polyhedron in $\mathbb{R}^3$ is a polyhedral surface which has the structure of a complex manifold. Near vertex local parameter $\zeta = z^{2\pi/\alpha}$, α is sum of angles adjacent to the vertex.
- This is a Riemann surface with a conformal metric which is flat and has conical singularities at the vertices.
- Two *smooth* conformal metrics m_1 and m_2 on such a surface. Kokotov proved that

$$\frac{\det \Delta^{m_1}}{\det \Delta^{m_2}} = C \frac{\mathrm{Area}^{m_1}}{\mathrm{Area}^{m_2}} \frac{\prod_{l=1}^{M} |g_l|^{b_1/6}}{\prod_{k=1}^{N} |f_k|^{a_k/6}}.$$

Computing the uncomputable: Alexey & co.

- Yulia Klochko & Alexey Kokotov *Genus one ...* (2007); Alexey (all genus) Polyhedral surfaces and ... *(2013)*

$$\det \Delta^m = C \, \mathrm{Area}^m(X) \det \Im B |\tau(X, w)|^2 \frac{\prod_{l=1}^{2g-2} |g_l|^{1/6}}{\prod_{k=1}^{N} |f_k|^{b_k/6}}.$$

Further results:

- *ESCSs giving ration of determinants via S-matrix (Luc Hillairet & Alexey Kokotov, 2013).*
- *Comparison formula for determinants (Luc & Alexey 2015).*
- *Spectral determinants on Mandelstam diagrams (Luc & Alexey 2015).*
- *Our inspiration comes from these works (and others!), spiritual inspiration from Aurell & Salomonson 1994.*

Differentiating the determinant with respect to angular variation

Theorem (Clara & moi)

Let $\{S_\gamma\}_{\gamma \in (0,\pi)}$ be a family of finite circular sectors in $\mathbb{R}^2$, where S_γ has opening angle γ and unit radius. Let Δ_γ be the Euclidean Dirichlet Laplacian on S_γ. Then for any $\alpha \in (0,\pi)$

$$\frac{\partial}{\partial \gamma}\left(-\log(\det(\Delta_\gamma))\right)\Big|_{\gamma=\alpha}$$

$$= \mathrm{p.f.}_{t=0} \int_{S_\alpha} \frac{2}{\alpha}(1+\log(r)) H_{S_\alpha}(t,r,\phi)\, r\, dr\, d\phi, \tag{1}$$

where H_{S_α} denotes the heat kernel on the finite angular sector S_α.

Exercise: Determine how the determinant depends on variation of the radius of the sector.

The corner contribution

Theorem (Clara & moi)

Let $S_{\pi/2} \subset \mathbb{R}^2$ be a circular sector of opening angle $\pi/2$ and radius one. Then the corner at the origin gives a purely local contribution to (1), in the sense that this contribution is identical for a circular sector of opening angle $\pi/2$ and any radius. *This corner contribution is*

$$-\frac{1}{4\pi} - \frac{\gamma_e}{4\pi},$$

where γ_e is the Euler-Mascheroni constant.

A conjecture vs. computing the incomputable

Conjecture (Clara & moi)

Amongst all convex n-gons of fixed area, the regular one maximizes the determinant.

Exercise

Compute the eigenvalues of a regular n-gon.

Hint: Ask Alex Strohmaier & Ville Uski, *An Algorithm...* (2013) to help you.

Conjecture (Alex)

Clara & J's conjecture is false. It holds for some n but not all.

A computable example

Proposition (Clara & moi)

Let R be a rectangle of dimensions $L \times L^{-1}$. Then the zeta regularized determinant is uniquely maximized for $L = 1$, and tends to 0 as $L \to 0$ or equivalently as $L \to \infty$.

In search of the determinant

Our ultimate goal is to compute $\zeta'(0)$ for a polygonal domain, in the spirit of $\zeta(0)$. The first baby steps have been made.

- Compute the corner contribution for a general corner of opening angle α. Two ways to do this: Sommerfeld or Kantorovich-Lebedev/Inverse Laplace transform.
- Use Schwarz-Christoffel mapping and spiritual inspiration of Aurell & Salomonson, *On functional determinants ...* (1994) to determine the variation due to the side lengths.
- Obtain a formula for $\dot{\zeta}'(0)$. Integrate it.
- Use computable example to determine constant of integration: $\det \Delta_L = e^{-\zeta_L'(0)} = \frac{|\eta(i/L^2)|^2}{2L}$, on rectangle of dimensions $L \times L^{-1}$. (Dedikind η function).

Merci à la CIRM, Jared, Luc, Dean, et vous tous!!

Avez-vous des questions?

All realities, all dimensions are open to me!