# Regularity of linear waves at the Cauchy horizon of black hole spacetimes

Peter Hintz
joint with András Vasy

Luminy

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(subextremal) Reissner-Nordström-de Sitter spacetime

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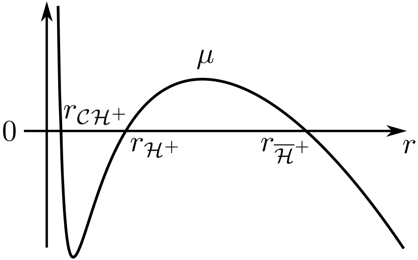
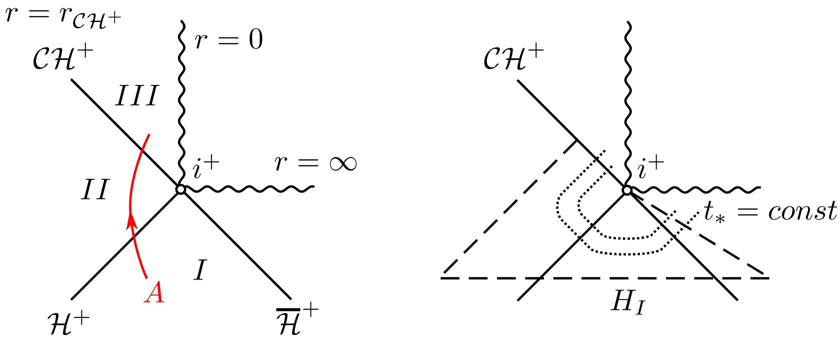
(subextremal) Reissner-Nordström-de Sitter spacetime

- solution of Einstein-Maxwell system,
- black hole mass $M > 0$, charge $Q > 0$,
- cosmological constant $\Lambda > 0$,
- topology: $\mathbb{R}_{t_*} \times (0, \infty)_r \times \mathbb{S}^2$,
- metric: $g = \mu(r)\, dt^2 - \mu(r)^{-1}\, dr^2 - r^2\, d\sigma^2$; $t_* = t - F(r)$.



Penrose diagram:



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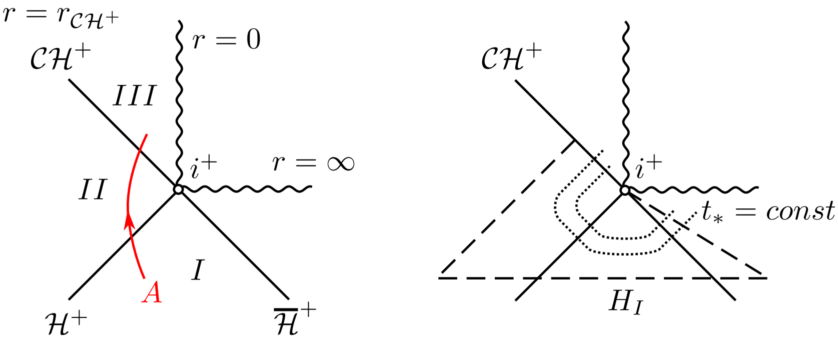
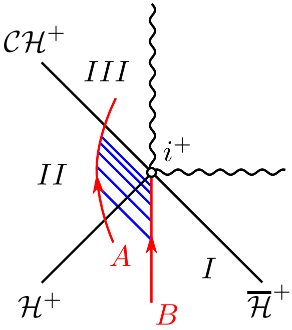



Cauchy horizon $\mathcal{CH}^+$: boundary of domain of uniqueness of solution $u$ to wave equation $\Box_g u = 0$ with Cauchy data on $H_I$

# Blue-shift effect and strong cosmic censorship

(Simpson–Penrose '73.)




Observer $A$ crosses $\mathcal{CH}^+$ in finite time.

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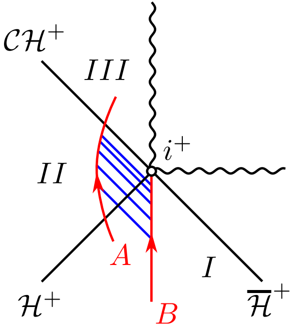


Observer $A$ crosses $\mathcal{CH}^+$ in finite time.

Observer $B$ lives forever.

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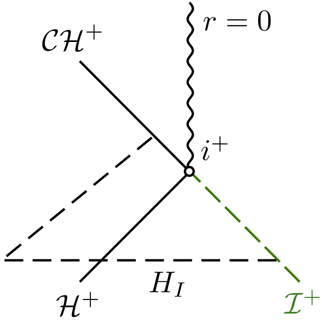
Related work:

- Christodoulou (. . . ,'99, '08),
- Dafermos ('03, '05, '13),
- ongoing work by Dafermos–Luk, Luk–Oh.

# Toy model: linear wave equation

Cauchy problem for $\Box_g u = 0$ on Reissner–Nordström.



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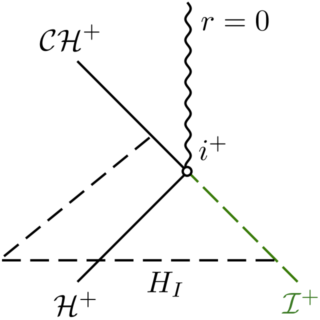


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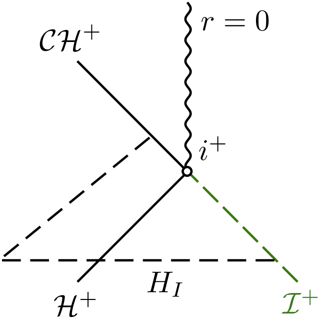


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**Theorem (Luk–Oh, '15)**

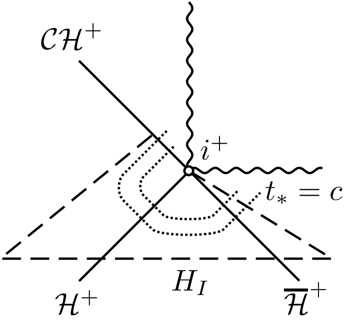
*For generic $\mathcal{C}^\infty$ initial data, $u$ is not in $H^1_{\mathrm{loc}}$ near any point of $\mathcal{CH}^+$.*

Theorem (H.–Vasy, '15)

*For $\mathcal{C}^\infty$ initial data on Reissner–Nordström–de Sitter, u solving $\Box_g u = 0$ has a partial asymptotic expansion near $i^+$,*

$$u = u_0 + u', \quad u_0 \in \mathbb{C}, \ |u'(t_*)| \lesssim e^{-\alpha t_*},$$

*where $\alpha > 0$ depends only on the spacetime.*



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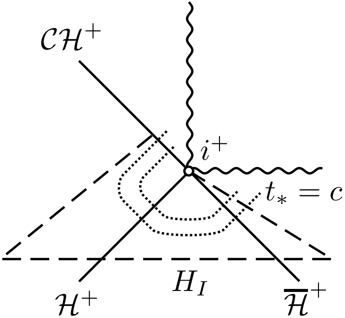

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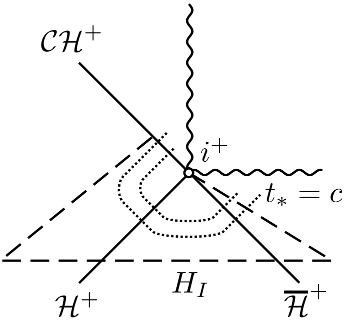


$\kappa > 0$: surface gravity of the Cauchy horizon

# Previous work

Microlocal analysis/scattering theory approach:

- Melrose ('93)
- Sá Barreto–Zworski ('97), Bony–Häfner ('08)
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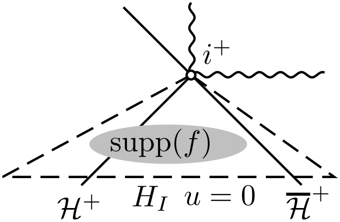
Energy estimates:

- Dafermos–Shlapentokh-Rothman ('15)
- Luk–Sbierski ('15)

# Analysis near the exterior region

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study regularity and decay of $u$.

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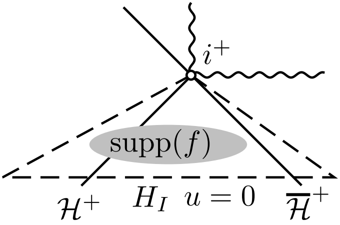


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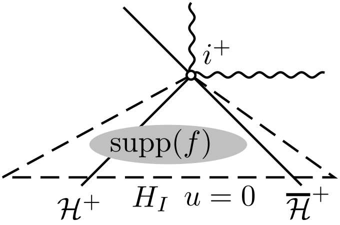


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$u$ is $\mathcal{C}^\infty$. Quantitative bounds as $t_* \to \infty$?

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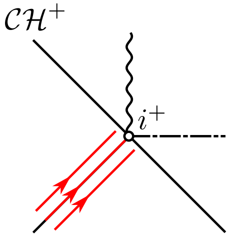


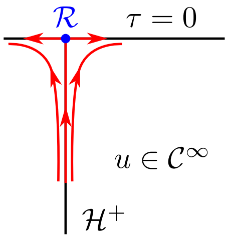

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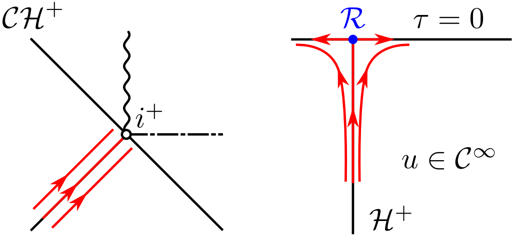


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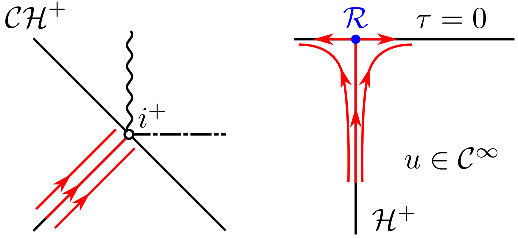


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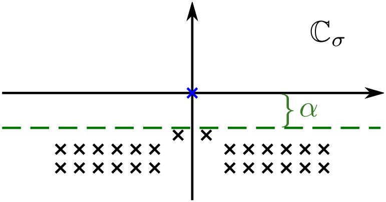

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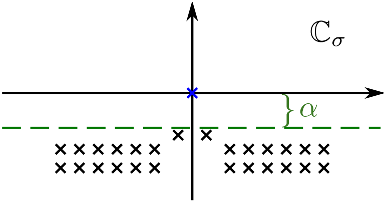
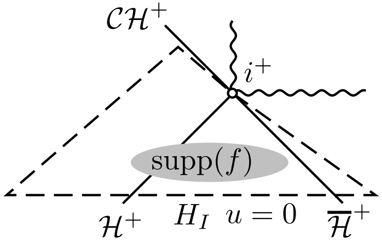



Obtain:

$$u = u_0 + u', \quad u_0 \in \mathbb{C}, \ u' \in e^{-\alpha t_*} H^\infty.$$

This gives asymptotics and decay in $r \geq r_{\mathcal{CH}^+} + \epsilon$, $\epsilon > 0$.



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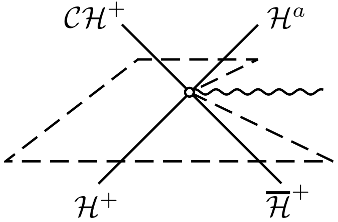

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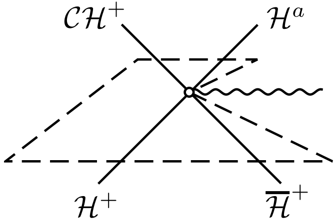
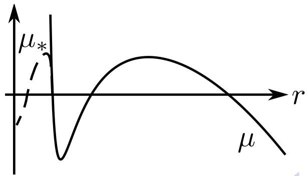
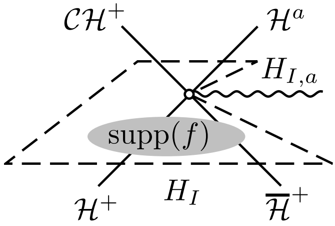



$$\widetilde{g} = \mu_* \, dt^2 - \mu_*^{-1} \, dr^2 - r^2 \, d\sigma^2.$$



# Setup for the extended problem

Study forcing problem $\Box_{\widetilde{g}} u = f$, with $u = 0$ near $H_I \cup H_{I,a}$.



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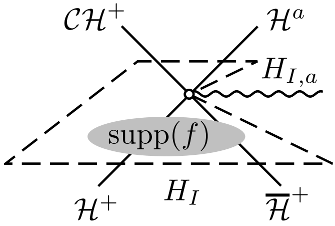


(Add complex absorbing potential $\mathcal{Q} \in \Psi_b^2$ beyond $\mathcal{CH}^+$ to hide additional trapping and $\mathcal{H}^a$. Study $\Box_{\widetilde{g}} - i\mathcal{Q}$.)

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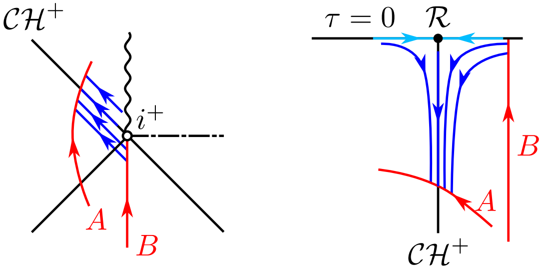

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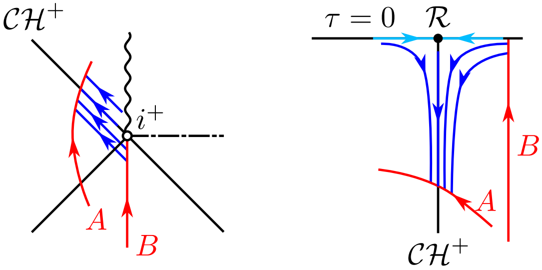
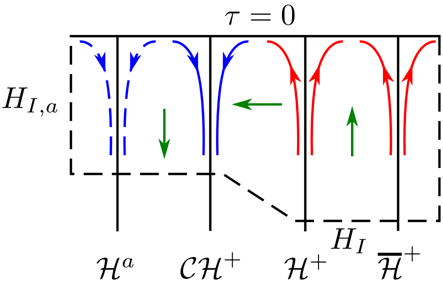



If $u \in e^{-\ell t_*} H^{-\infty}$ near $\mathcal{R}$, and $u \in e^{-\ell t_*} H^s$ in a punctured neighborhood of $\mathcal{R}$ within $\{\tau = 0\}$, then $u \in e^{-\ell t_*} H^s$ near $\mathcal{R}$, provided $s < 1/2 + \ell/\kappa$.

# Microlocal analysis of the extended problem




Green arrows: Future directed timelike vectors.

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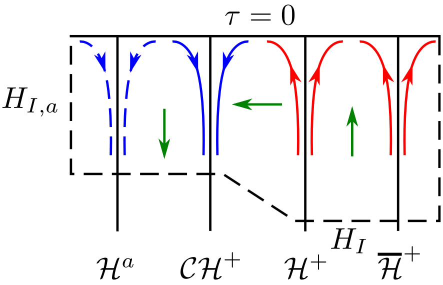


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Control $u$ solving $\Box_{\widetilde{g}} u = f$ using standard hyperbolic theory near dashed surfaces, and microlocal elliptic regularity and propagation of singularities.

Obtain

$$\|u\|_{e^{-\ell t_*}H^s} \lesssim \|\Box_{\widetilde{g}} u\|_{e^{-\ell t_*}H^{s-1}} + \|u\|_{e^{-\ell t_*}H^{s_0}},$$

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Get solvability of extended problem $\Box_{\widetilde{g}} u = f$, and control of $u$ near $\mathcal{CH}^+$.

# Resonance expansion

Solution $u$ of $\Box_{\widetilde{g}} u = f$ has partial expansion

$$u|_{r>r_{\mathcal{CH}^+}} = u_0 + u', \quad u_0 \in \mathbb{C},\ u' \in e^{-\alpha t_*} H^{1/2+\alpha/\kappa-0} \text{ near } \mathcal{CH}^+.$$

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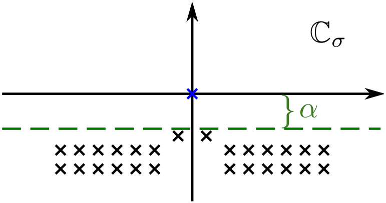
$\alpha > 0$: spectral gap of $\Box_g$.



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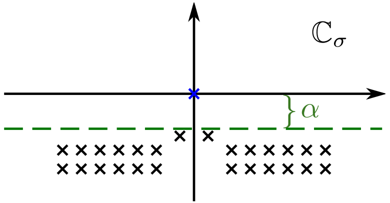


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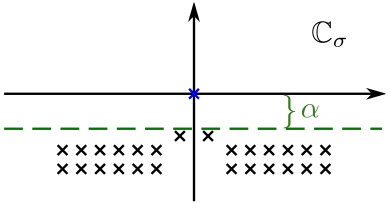


$u'(t_*, r, \omega)$ is smooth in $t_*, \omega$. (Haber-Vasy '13.)
$H^{1/2+0}(\mathbb{R}_r) \hookrightarrow L^\infty(\mathbb{R}_r)$ yields $|u'(t_*)| \lesssim e^{-\alpha t_*}$.

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Only potential issue for large $a$: resonances in $\operatorname{Im} \sigma \geq 0$ ('mode stability,' see Whiting '89, Shlapentokh-Rothman '14 for Kerr)

# Outlook

Shallow resonances. Mode solution $\Box_g(e^{-i\sigma t_*}v(x)) = 0$ has $v \in H^{1/2-\operatorname{Im}\sigma/\kappa-0}$ at $\mathcal{CH}^+$; could be $\mathcal{C}^\infty$ in principle. Study location and regularity properties of shallow resonances.

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$\Lambda = 0$. Localize analysis near black hole region; in the exterior, polynomial decay according to Price's law (Tataru '13, Metcalfe–Tataru–Tohaneanu '12). Get polynomial decay near $\mathcal{CH}^+$. (Luk–Sbierski '15, H. '15)

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Nonlinear problems. Einstein's field equations. Work in progress by Luk–Rodnianski, Dafermos–Luk, Luk–Oh.