Stability conditions: Applications to AG

Goal: Derived categories / with stab. cond. / wall-xing

$\overset{!}{\Rightarrow}$
- New proof of BN theorem
- Birat'l geometry of mod. spaces

§1 Definition / Construction of stability conditions

Setting: X proj. $/k$, $D^b(X) = D^b(\mathrm{Coh}\,X)$

Fix: $ch : K(D(X)) \to \Lambda$ finite rank lattice,

e.g. $\Lambda = \mathrm{Im}\, ch \subset H^*_{alg}(X, \mathbb{Q})$

Def.: A stability condition on X is a pair $(Z, \mathcal{A})$ where

- $\mathcal{A} \subset D^b(X)$ abelian subcategory, "heart of a bdd t-structure"
- $Z : \Lambda \to \mathbb{C}$ group homomorphism.

① $E \in \mathcal{A} \Rightarrow \overset{Z(ch(E))=}{Z(E)} \in$  $\subset \mathbb{C}$

i.e. "$\mathrm{rk}\, E$" $= \mathrm{Im}\, Z(E) \geq 0$ and "$\deg E$" $= -\mathrm{Re}\, Z(E)$ with $\mathrm{rk} = 0 \Rightarrow \deg > 0$.

$\rightsquigarrow$ notion of slope $\mu_Z = \dfrac{-\mathrm{Re}\, Z}{\mathrm{Im}\, Z}$

$\rightsquigarrow E \in \mathcal{A}$ slope-semistable if $A \hookrightarrow E$ (in $\mathcal{A}$) $\Rightarrow \mu_Z(A) < \mu_Z(E)$

② HN filtrations exist: all $E \in \mathcal{A}$ have $0 = E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_m = E$, E_i/E_{i-1} μ_Z-semistable

$\mu_Z(E_1/E_0) > \dots > \mu_Z(E_m/E_{m-1})$

③ Support property: exists quadratic form $Q: \Lambda \to \mathbb{R}$ s.t.

- $Q|_{\ker Z} < 0$
- E semistable $\Rightarrow Q(ch(E)) \geq 0$.

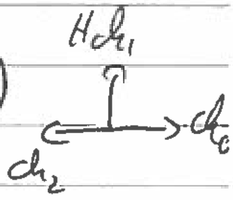
Example: $X =$ proj. curve/k, $\mathcal{A} = Coh X$,
$Z(E) = -\deg(E) + i\, rk(E)$.

1.2 Stability conditions on surfaces

$X =$ smooth proj. surface/k, char $k = 0$
$H =$ polarization, $\Lambda = \operatorname{Im} ch \subset H^*(X)$

Observation: Cannot achieve ① with $\mathcal{A} = Coh X$!

Ansatz: $Z(E) = -ch_2(E) + iH ch_1(E) + \frac{H^2}{2} ch_0(E)$



Step 1: $E \in Coh X$

$$\rightsquigarrow \mu_H(E) := \begin{cases} +\infty & ch_0(E) = 0 \\ \dfrac{H ch_1(E)}{H^2 ch_0(E)} & \text{else} \end{cases}$$

$\rightsquigarrow \mu_H$-stability, HN-filtrations exist.

Theorem (B-G): E μ_H-stable
$\Rightarrow \Delta(E) = ch_1(E)^2 - 2 ch_0(E) ch_2(E) \geq 0$.

Step 2:

$$\mathrm{Coh}_H^{>0} X = \{ E \in \mathrm{Coh}\, X \mid \text{all HN-factors of } E \text{ have } \mu_H(E) > 0 \} \iff \mu_{\min}(E) > 0$$

$$= \langle E \text{ semistable}, \mu_H(E) > 0 \rangle$$

$\langle \ldots \rangle$ = extension-closure

$$\mathrm{Coh}_H^{\leq 0} X = \{ \quad \ldots \quad \leq 0 \}$$

$$= \langle \quad \ldots \quad \leq 0 \rangle$$

Def/Prop:

let $\mathrm{Coh}^{H,0} X := \langle \mathrm{Coh}_H^{>0} X, \mathrm{Coh}_H^{\leq 0} X[1] \rangle \subset D^b(X)$

$$= \{ E^{-1} \xrightarrow{d} E^0 \mid \ker d \in \mathrm{Coh}_H^{\leq 0} X, \operatorname{cok} d \in \mathrm{Coh}_H^{>0} X \}$$

This is the heart of a bounded t-structure, in particular an abelian category!

Rmk:

- Proof needs general criterion (HRS) + HN filtrations + $\mathrm{Hom}(\mathrm{Coh}_H^{>0} X, \mathrm{Coh}_H^{\leq 0} X) = 0$ ("torsion pair")
- Say $E \in \mathrm{Coh}_H^{>0} X \subset \mathrm{Coh}^{H,0} X$. Then
  $A \hookrightarrow E$ (in $\mathrm{Coh}^{>0} X$)
  $\xleftrightarrow{1:1} A \in \mathrm{Coh}_H^{>0} X, f : A \to E$ s.t. ~~$\operatorname{cok} f \in \mathrm{Coh}^{>0}$~~ $\ker f \in \mathrm{Coh}_H^{\leq 0} X$
- $\mathrm{Coh}^{H,0} X$ by itself already gives new methods:
  say want to prove $H^1(X, M) = 0$ some M
  & have $H^1(Y, M) = \mathrm{Ext}^1(B, A)$, $A \in \mathrm{Coh}^{\leq 0}$ $B \in \mathrm{Coh}^{>0}$
  $\Rightarrow j \in H^1(M)$ becomes morphism $B \to A[1]$ in abelian category!

Rmk.: can generalize all the above: by choose $B \in H^{1,1}(X, \mathbb{R})$:

- $ch \rightsquigarrow ch^B = e^{-B} ch$, $\mu_H \rightsquigarrow \mu_{H,B}$
- $Coh^{H,0} \rightsquigarrow Coh^{H,B}$

$Z \rightsquigarrow Z_{H,B}(E) = -ch_2^B + iH ch_1^B + \frac{H^2}{2} ch_0^B$

Step 3:

Prop: $\sigma_{H,B} = (Coh^{H,B} X, Z_{H,B})$ is a stability condition on X.

Rmk:
- $\mathcal{O}_x$ stable, $x \in X$
- E slope-stable / Gieseker-stable, $E \in Coh_H^{>B}$

$\Leftrightarrow$ E is $\sigma_{H,B}$-stable for $H \gg 0$

Part of proof: ① $E \in Coh^{H,B} X$

$$\Rightarrow Z(E) = Z(\underset{Coh_H^{>B}}{H^0(E)}) - Z(\underset{Coh_H^{\leq 0}}{H^{-1}(E)})$$

$\Rightarrow \operatorname{Im} Z \geq 0$ &

$\operatorname{Im} Z = 0 \Rightarrow H^0(E) =$ 0 dim'l torsion sheaf, $H^{-1}(E)$ slope-~~sta~~ semistable, $\mu_{H,B}(E) = 0$

$B - G \Rightarrow ch_2^B(H^{-1}(E)) \leq 0$

$$\Rightarrow \operatorname{Re} Z(E) = -ch_2^B H^0(E) - \frac{H^2}{2} ch_0^B H^{-1}(E) + ch_2^B H^{-1}(E) < 0.$$

③ Given by (modification) of $B-G$.

§2 Brill-Noether via wall-crossing

2.1 Stability conditions on K3 surfaces

(following Bridgeland)

X = smooth proj. K3/$\mathbb{C}$, $\overset{= N(X)^{\vee}}{\Lambda} = H^*_{alg}(X, \mathbb{Z})$

$v: K(X) \to \Lambda$, $v(E) = ch(E)\sqrt{td_X} = (ch_0, ch_1, ch_2 + ch_0)$

Mukai pairing s.t. $\chi(E, F) = -(v(E), v(F))$

$\mathrm{Pic}\, X = \mathbb{Z} \cdot H$. $\quad H, B \rightsquigarrow \alpha H, \beta H$

$$\sigma_{\alpha,\beta} = (\mathrm{Coh}^{\beta}, Z_{\alpha,\beta}) \quad Z_{\alpha,\beta} = \left(v(E), e^{(-i\alpha - \beta)H}\right)$$

Theorem (Yoshioka, Huybrechts, O'Grady; Bridgeland, Toda)

Pick $v \in$ ~~M~~ $N(X)^{\vee}$, primitive

$M_H(v) :=$ Gieseker moduli space of H-st.

$M_{\sigma_{\alpha,\beta}}(v)$ = moduli space of $E \in \mathrm{Coh}^{\beta}$ $\sigma_{\alpha,\beta}$-stable
α, β generic

Then: • $\dim M_H(v) = \dim M_{\sigma_{\alpha,\beta}}(v) = v^2 + 2$

and non-empty $\Longleftrightarrow$ $v^2 \geq -2$.

Both stronger than B-G & will give ③ = support property

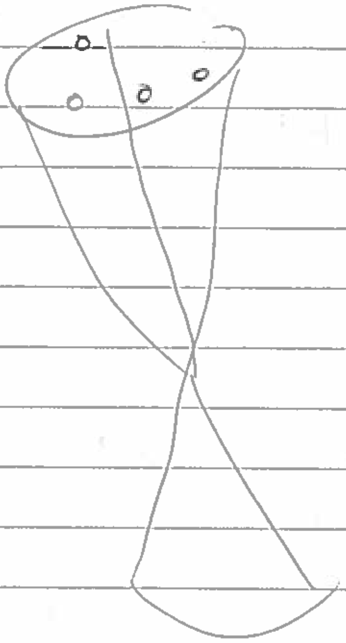
$\mathrm{Stab}(X) :=$ space of stability conditions on X

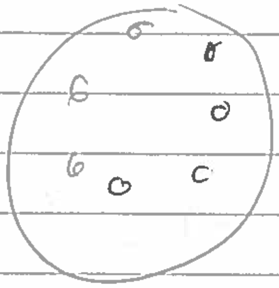
$\overset{\cup}{\mathrm{Stab}}^{+}(X)$ = conn. component containing $\sigma_{\alpha,\beta}$

Theorem (Bridgeland): $\mathrm{Stab}^+(X)$ is a cplx manifold s.t. $\mathcal{Z}: \mathrm{Stab}^+ \to \mathrm{Hom}(\Lambda, \mathbb{C})$, $(Z, \mathcal{A}) \mapsto Z$ is a covering of its image &

Image = Z satisfying

- $(\,,\,)|_{\mathrm{Ker}\, Z} < 0$
- orientation
- $\mathrm{Ker}\, Z$ does not contain -2 class





2.2 Brill-Noether:

C = proj curve$/\mathbb{C}$, $g = \mathrm{genus}(C)$,

Given d, r let

$$V^r_d(C) := \{L \in \mathrm{Pic}^d(C) \mid h^0(L) = r+1\}$$

Q: $\neq \emptyset$? dim = ?)

$$\rho(r,d,g) = g - (r+1)(d-g+r)$$

Theorem (Griffiths-Harris) C generic $\Rightarrow \dim V^r_d(C) = \rho$.

Theorem (Lazarsfeld): X, H K3 surface, $\mathrm{Pic}\, X = \mathbb{Z}\cdot H$,

$C \in |H| \Rightarrow \dim V^r_d(C) = \rho$.

2.3 Moduli Space

Assume from now on $d \leq g-1$, $g = 1 + \frac{H^2}{2}$

$v := (0, H, d+1-g)$

$M_H(v)$ = compactification of relative Picard variety

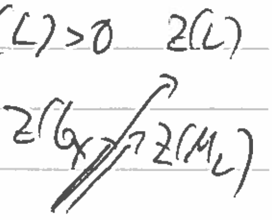
$$\begin{tikzcd}
\mathrm{Pic}^d \arrow[d] & \mathrm{Pic}^d(C) \arrow[d] \\
{|H| \ni} & C
\end{tikzcd}$$

$$V_d^r(|H|) = \{ L \in M_H(v) \mid h^0(L) = r+1 \}$$

$M_H(v) = M_{\sigma_{\alpha,\beta}}(v)$ for $\alpha \gg 0$.

2.4 Wall-crossing

Prop: Assume $0 > \beta \gg -1$. Then first wall as $\alpha \to 0$ is given by $L \in M_H(v)$ becomes semistable $\Leftrightarrow$ $h^0(L) > 0$ w/ destabilizing sequence



$$\mathcal{O}_X^{\oplus h^0(L)} \overset{ev}{\hookrightarrow} L \twoheadrightarrow M_L[1]$$

$M_L = \ker ev$ if L glob. generated.

M_L semistable at wall. σ_{wall}.

Cor.: $V_d^r(|H|) =$ empty if $\rho < 0$

Pf: $v(M_L)^2 \geq -2 \Rightarrow \rho \geq 0$.

More precisely: $w := v - (r+1) \cdot (1,0,1) = v(M_L)$

Lemma1: $\dim M_{\sigma_{wall}}^{stable}(w) = \dim M_{\sigma_{wall}}^{s.stable}(w) = w^2+2$

(all J-H factors in $\langle v, w \rangle_{saturated} \subset \Lambda$

Lemma2: Given $M \in M_\sigma^{stable}(w)$ &

$\mathcal{O}_X^{\oplus r} \hookrightarrow E \to M$ non-trivial extension

$\Rightarrow$ E becomes stable

Cor.:
$$\begin{tikzcd}
V_d^r(|H|) = Gr(r+1, \mathcal{E}) \arrow[d] \\
M_\sigma^{stable}(w)
\end{tikzcd}$$

$$\mathcal{E}|_B = Ext^1(B, \mathcal{O}_X)$$

$\Rightarrow$ dim $V_d^r(|H|) = \rho + g$

2.5 Conclusion

Needs geometry of holomorphic symplectic varieties

$$\begin{tikzcd}
M_H(0,v,-d+1-g) \arrow[drrr, bend left, "F"] \arrow[ddrrr, "\pi"] & & & \\
V_d^r(|H|) \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\supset"] \arrow[d] & V_d^r(C) \arrow[rr] \arrow[dl, "q"] & & \mathrm{Supp}\,F \\
M_W & & & \mathbb{P}^g
\end{tikzcd}$$

Lagrangian fibration

1. Curve C contracted by π $\Rightarrow$ Know $\mathbb{R}_{>0}[C] \in N_1(M)$
lines in Grassmannians not of this class
So $q: V_H(C) \to M_W$ is a (quasi-) finite map

2. Symplectic form on $V_d^r(|H|)$ pulled back from M_W;
$V_d^r(C)$ ~~to~~ isotropic $\Rightarrow$ $q(V_d^r(C))$ is isotropic

$\Rightarrow$ $\dim V_d^r(C) \le \frac{1}{2} \dim M_W = \rho$

§3 Birational Geometry of moduli spaces

(joint w/ Macri)

X = smooth proj. K3 / $\mathbb{C}$.

3.1 Postscriptum

Yesterday: $\mathrm{Pic}\, X = \mathbb{Z} \cdot H$, $d \leq g-1$

$$M_H(0, \underset{v}{H}, d+1-g) \supset \{h \mid h^0(L) > 0\}$$

$$= \bigcup_r \mathrm{Gr}(r+1, \mathcal{E} \downarrow M(v-(r+1)v(\mathcal{O})))$$

$$= \text{BN locus} \quad (\text{Markman})$$

Will see: Can now find birational contraction of $M_H(v)$ that precisely contracts BN-locus.

3.2 Goal:

$v \in N(X) = H^*_{\mathrm{alg}}(X, \mathbb{Z})$ primitive.

Want: complete description of birational geometry of $M_H(v) =: M$

① Description of chamber decomposition of

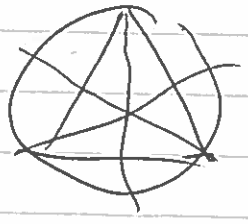
$\mathrm{Pos}(M) \subset$ ~~$H^2(M, \mathbb{R})$~~ $NS(M) \otimes \mathbb{R}$

$\cup$

$\mathrm{Mov}(M) = \langle \text{movable divisors} \rangle$

$\cup$

$\mathrm{Nef}(M)$



Recall: $NS(M) = v^\perp \subset N(X)$ with

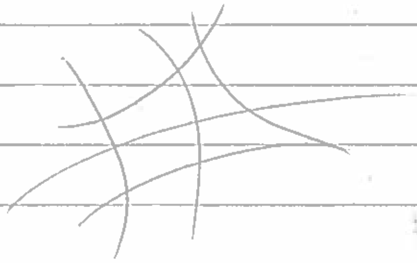
$(,)|_{v^\perp} = q(\ )$ B-B form

② Describe geometry associated to all contractions coming from nef (⇒ glob generated) divisors

"Understand MMP"

3.3 Wall-crossing

Consider $M_\sigma(v)$, v fixed as above, σ varies



Want: complete understanding of walls in Stab(X) wrt v

Earlier observations: wall-crossing follows MMP.

3.4 Connection:

Positivity Lemma (B-Macri):

S family of semistable objects of class v

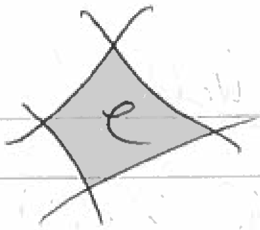
i.e. $\mathcal{E} \in D(S \times X)$, $\mathcal{E}_s$ = stable of class v all closed points $s \in S$.

$\Rightarrow$ get nef divisor ℓ_σ on S with $C \subset S$ curve

$\Rightarrow \ell_\sigma . C \geq 0$ & $\ell_\sigma . C = 0 \Leftrightarrow$ C parametrizes S-equivalent objects

i.e. ~~all~~ all $\mathcal{E}_c$ for $c \in C$ have filtration with filtration quotients stable, of same phase, & independent of $c \in C$.

Cor. Consider chamber $\mathcal{C}$.



Get family of ~~nef~~ ample divisors ~~ft~~ ℓ_σ for $\sigma \in \mathcal{C}$
& nef divisors for $\sigma \in \partial \mathcal{C}$; $\ell : \bar{\mathcal{C}} \to \mathrm{Nef}\, M_{\mathcal{C}}$

3.5 Results

(B-Macri)
Theorem 1: Every wall-crossing $\sigma_{West} \,|\, \sigma_{East}$
induces a birational map $M_{\sigma_{West}}(v) \dashrightarrow M_{\sigma_{East}}(v)$
$\leadsto NS(M_{\sigma_{West}}(v)) = NS(M_{\sigma_{East}}(v))$

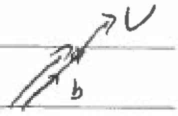
Fix $\sigma_0 \in Stab(X)$.

Theorem 2: The above maps glue to a continuous map
$$\ell : Stab^+(X) \to \cancel{M_{\sigma_0}(v)}\ Mov(M_{\sigma_0}(v))$$
with

- Compatibility: $M_\sigma(v)$ is birational model corr to $\ell(\sigma)$
- ~~Essen~~ Image = $Mov \cap Pos$
- Can use this map to explicitly describe wall-crossing in $Stab(X)$ + chamber decomposition of $Mov(M_{\sigma_0}(v))$ as well as geometry of birational contractions.

3.6 Method

$\sigma \in \text{Wall} \subset \text{Stab}(X) \overset{\text{easy}}{\Longrightarrow} v = a+b, \quad Z(v) \parallel Z(a) \parallel Z(b)$



~~$a^2, b^2 \geq -2$~~ $\quad a^2, b^2 \geq -2$

$\overset{\Longleftarrow}{\text{more difficult}}$

- need to construct extension $A \hookrightarrow E \twoheadrightarrow B$
  $A \in M_\sigma^{\text{stable}}(a), \ B \subset M_\sigma(B)$,

~~diff~~ Difficulty: what if $M_\sigma^{\text{stable}}(a) = \emptyset$?

- This corresponds to wall in $\text{Mov}(M_{\sigma_0}(v))$
  $\Leftrightarrow$ exists a curve of S-equivalent objects, i.e. iff we can vary this extension class for some a, b as above

Key tool: $\langle a, b \rangle_{\text{saturation}} \subset N(X)$

- To overcome difficulty: turns out
  $M_\sigma^{\text{stable}}(a) = \emptyset$ always due to spherical object S with $v(S) \in \langle a,b \rangle_{\text{sat}}$, $v(S)^2 = -2$.
  $\rightsquigarrow$ associated autoequivalence, reflection of $N(X)$

3.7 Applications:

Theorem: $\text{Nef}(M_{H_0}(v))$ cut out by
$v^\perp \cap a^\perp$ for all $a \in N(X)$, $a^2 \geq -2$, $0 \leq (v,a) \leq \frac{v^2}{2}$
Deformation (B-HT)

Theorem: If $D \in NS(M_H(v))$ with $q(D) = 0$ then
$M_H(v) \dashrightarrow M \rightarrow \mathbb{P}^d$ has birat'l Lagrangian fibration.

Other surfaces: