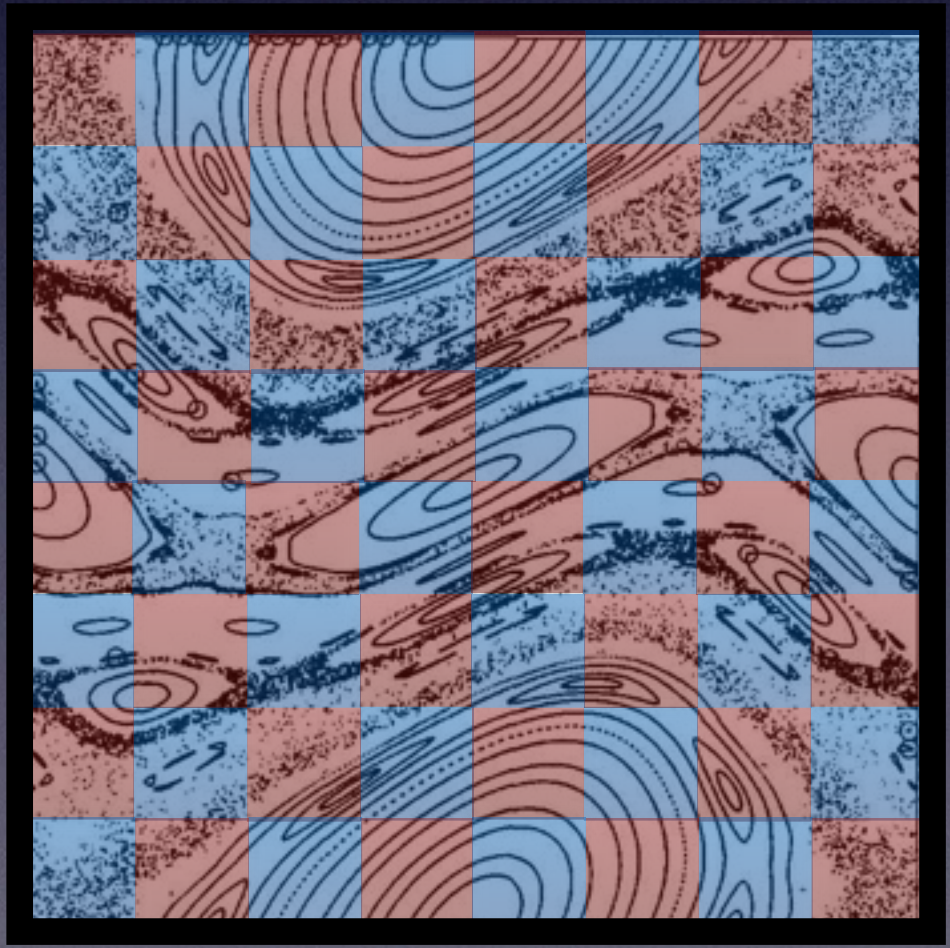
Hamiltonian stability

over discrete spaces

Franco Vivaldi
Queen Mary, University of London



Smooth area-preserving maps

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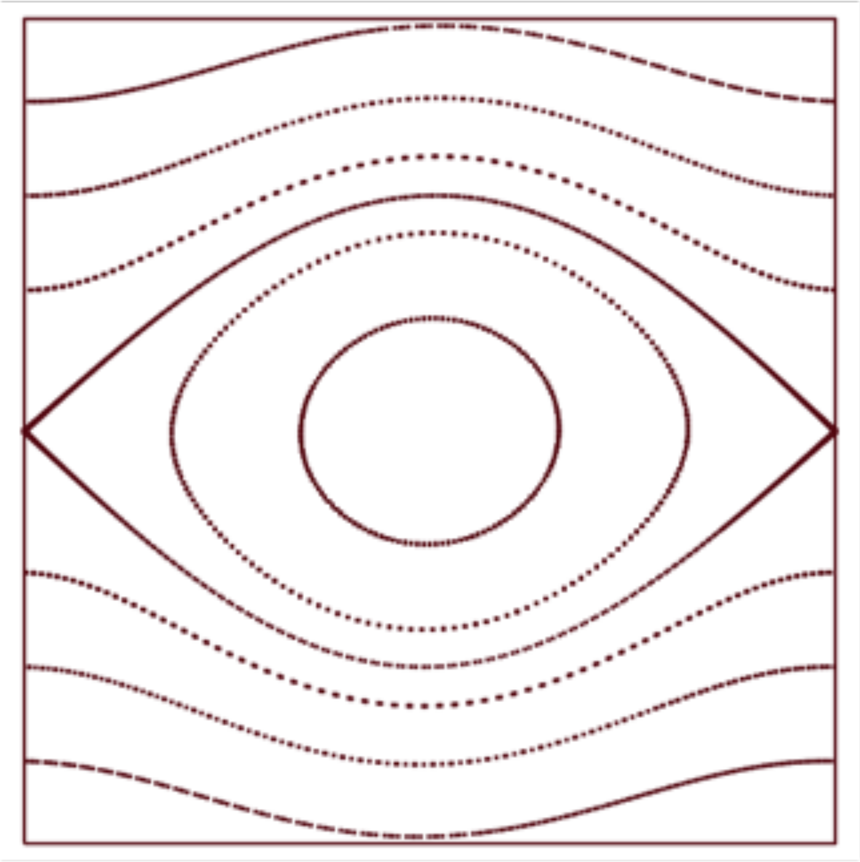
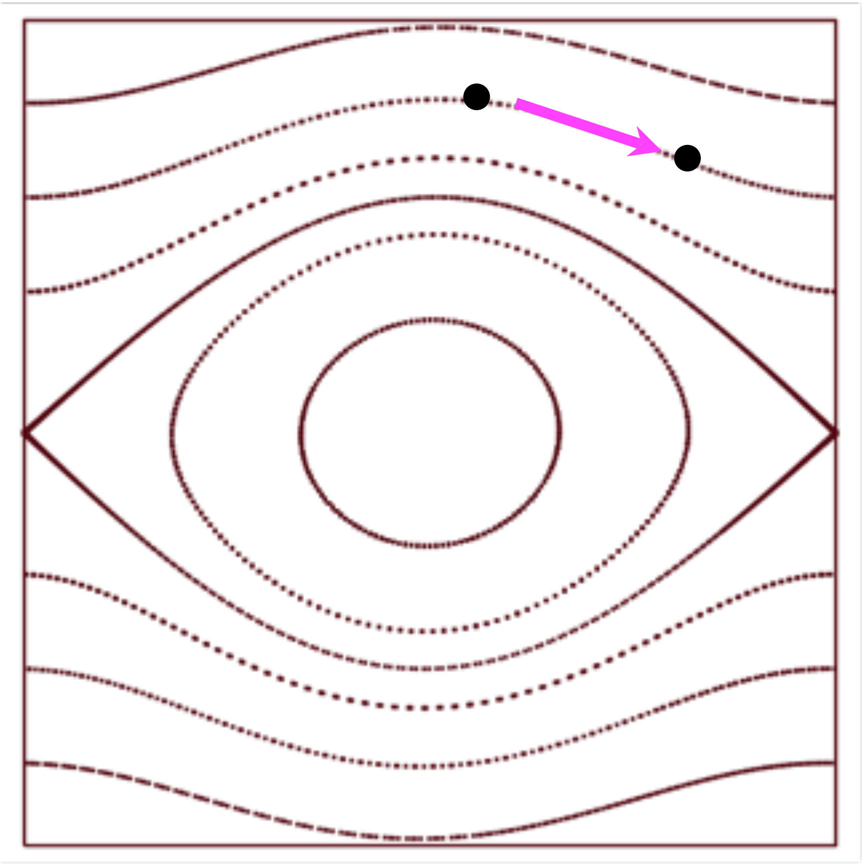


integrable

Smooth area-preserving maps

foliation by
invariant curves



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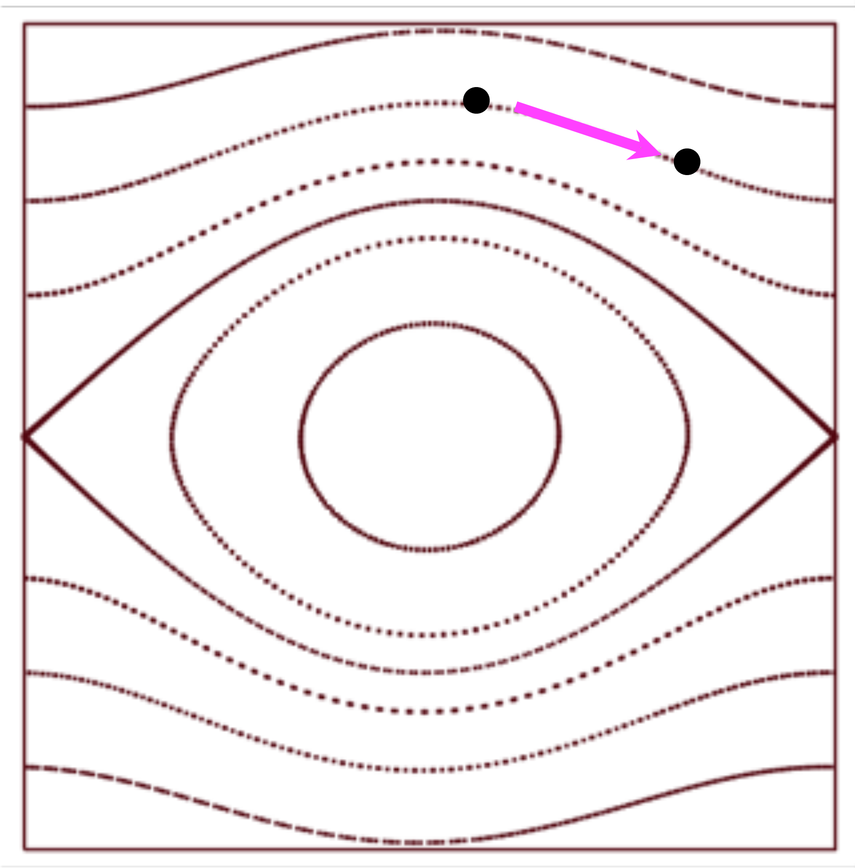

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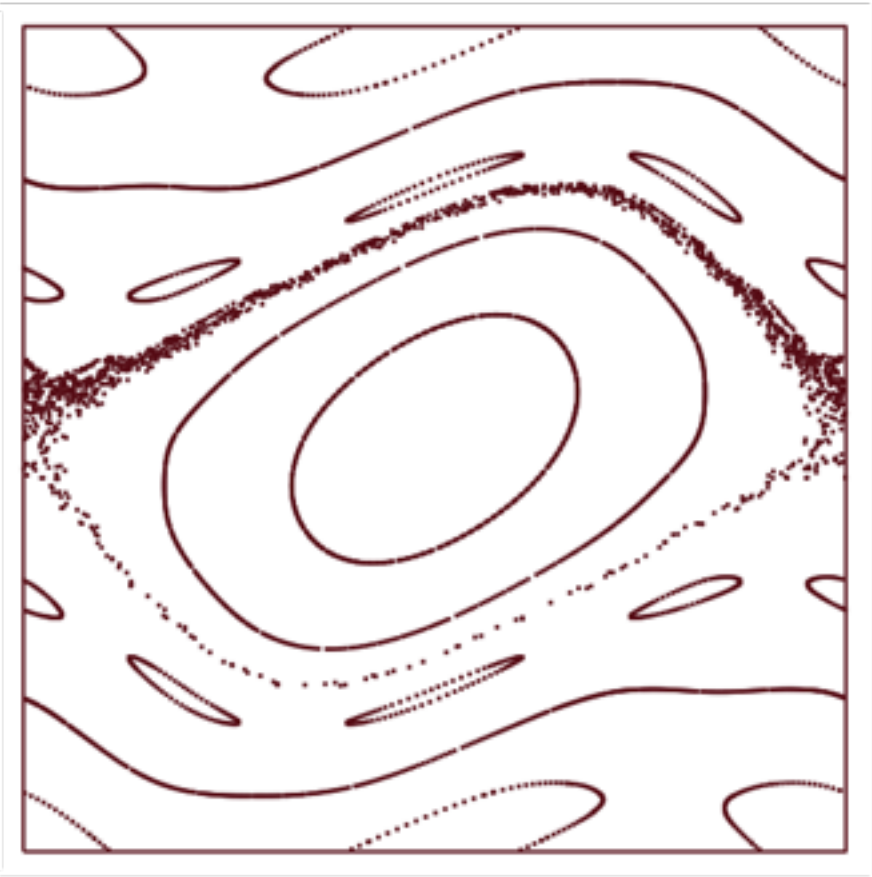

near-integrable: stable

Smooth area-preserving maps

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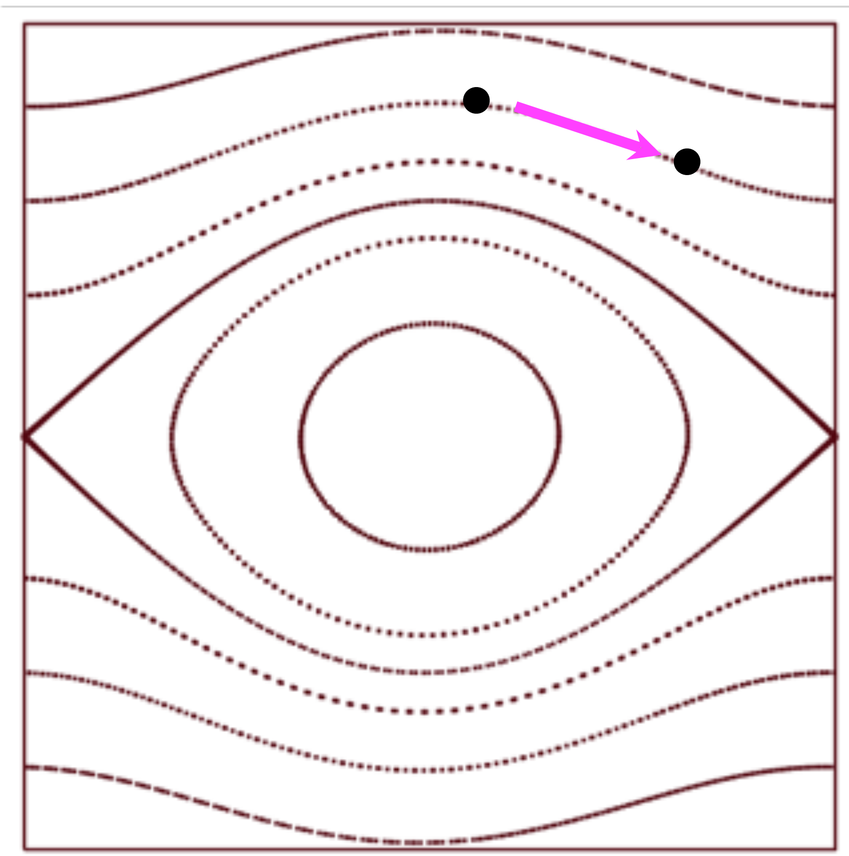
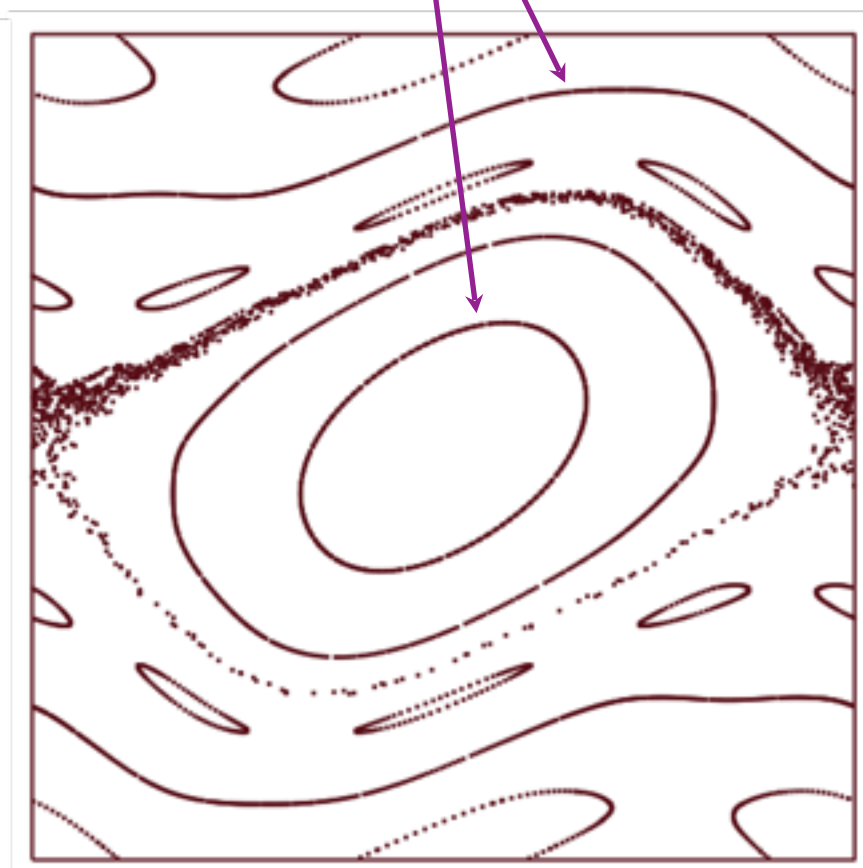
KAM curves





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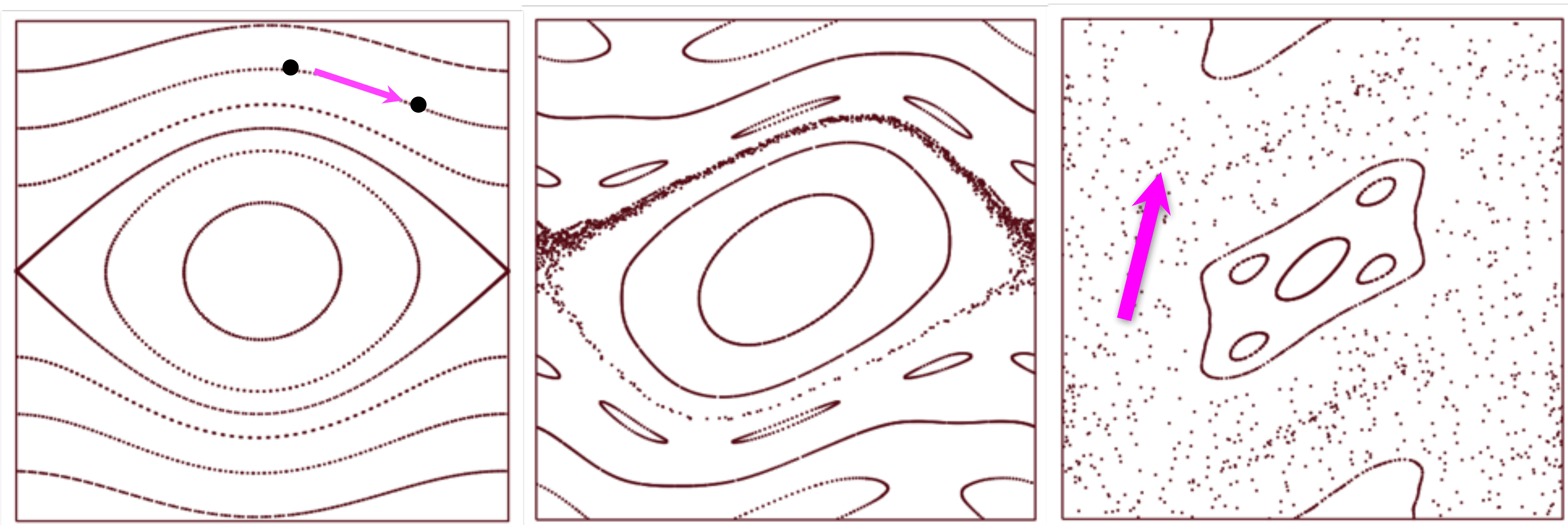
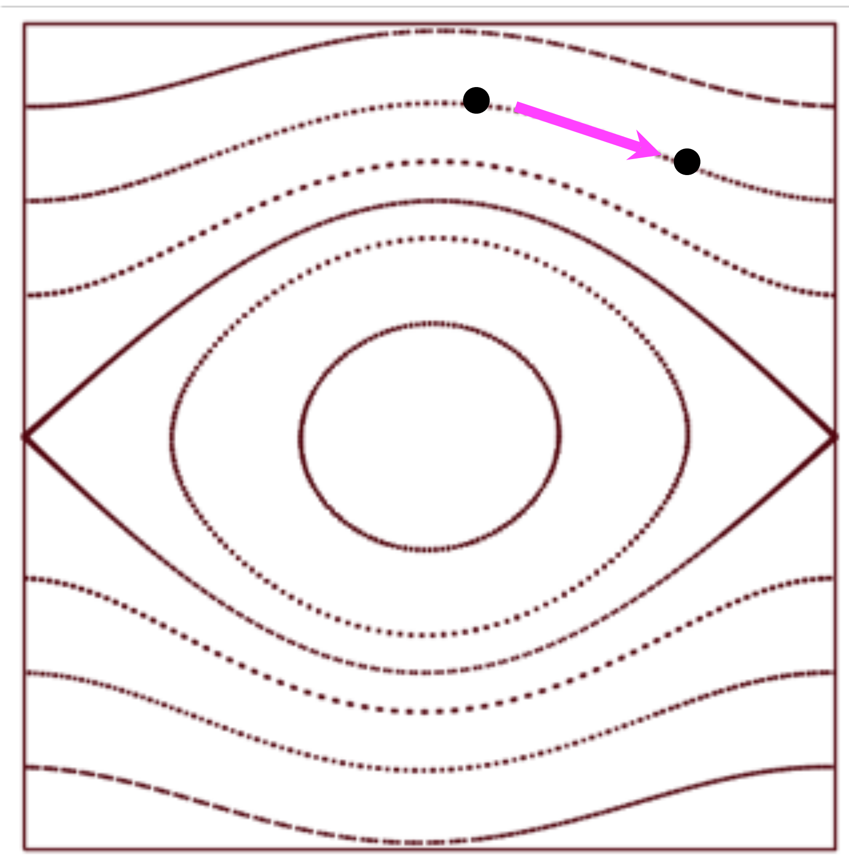


integrable

near-integrable: stable

strong perturbation: unstable

Smooth area-preserving maps



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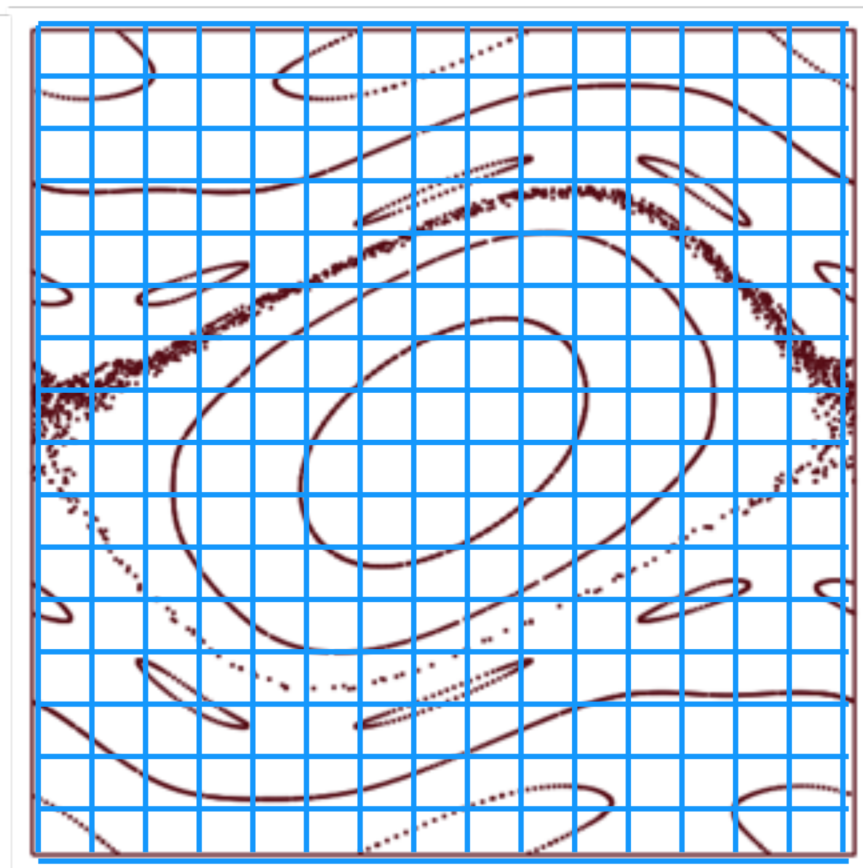

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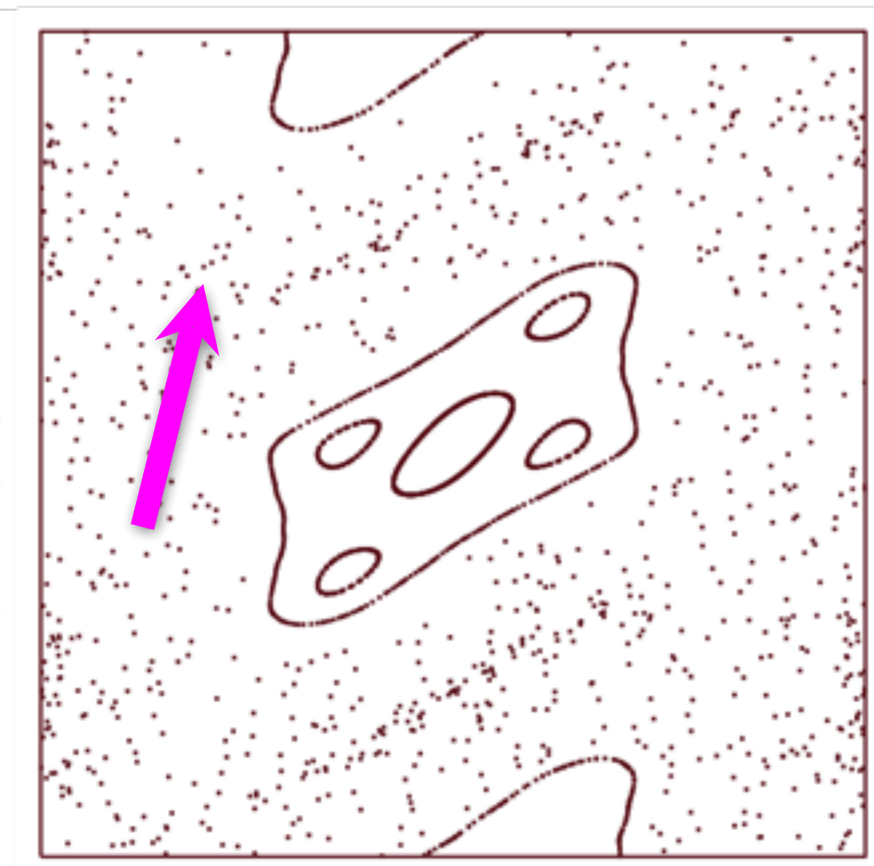


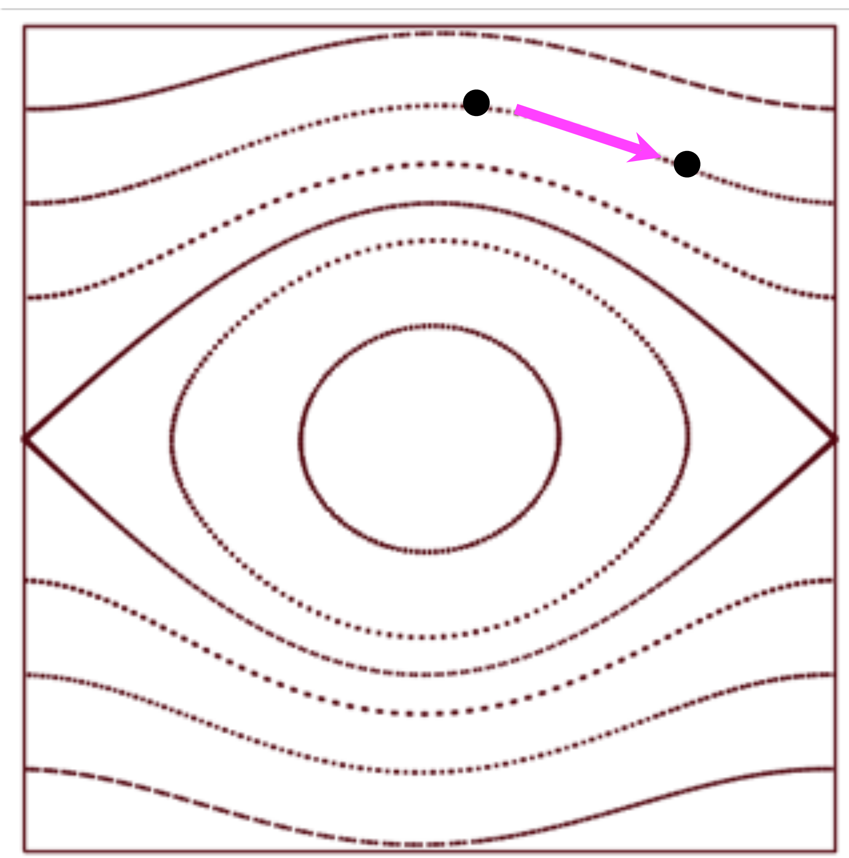
strong perturbation:
unstable

What happens if the space is discrete?

Smooth area-preserving maps

foliation by invariant curves

KAM curves ?



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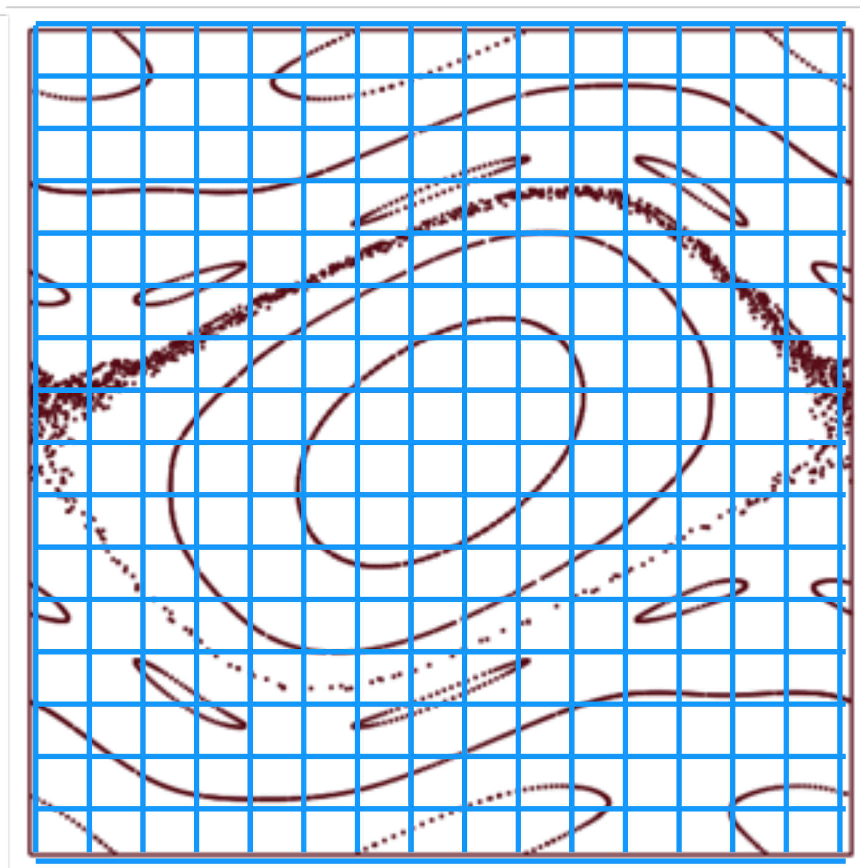

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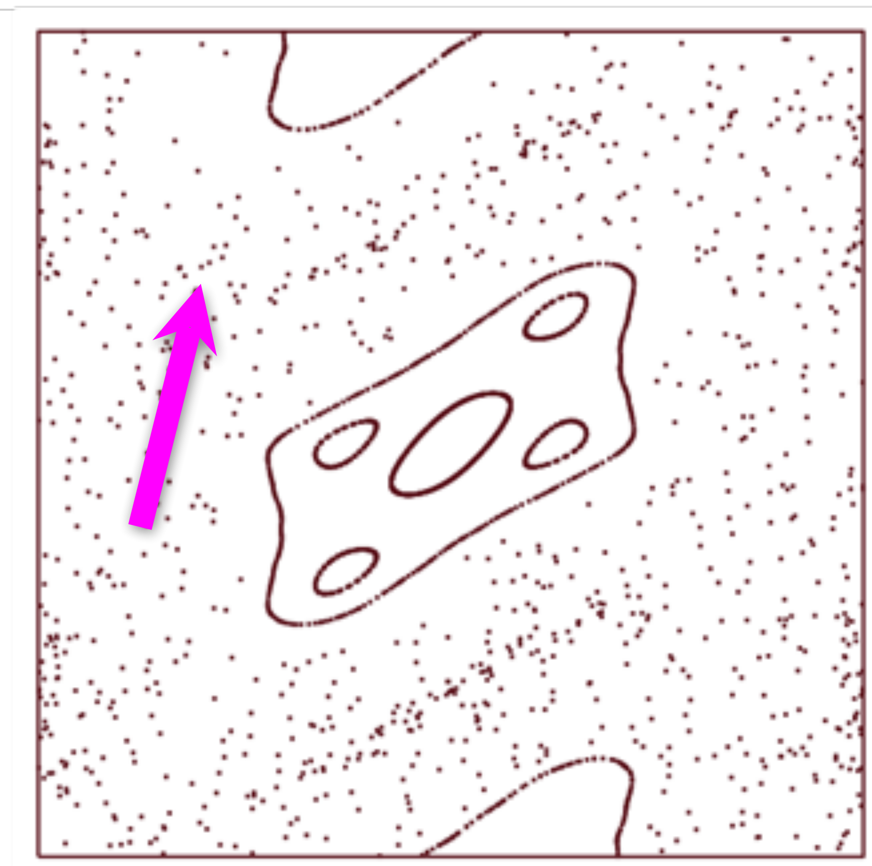

strong perturbation: unstable

What happens if the space is discrete?

Some methods for discretizing space

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- Truncation (computer arithmetic).

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- Restricting coordinates to a discrete field (ring, module).

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- Restricting coordinates to a discrete field (ring, module).
- Geometric discretization.
- Reduction to a finite field.

Early investigations:

F. Rannou (1974)

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Astron. & Astrophys. 31, 289 – 301 (1974)

Numerical Study of Discrete Plane Area-preserving Mappings

F. Rannou
Observatoire de Nice

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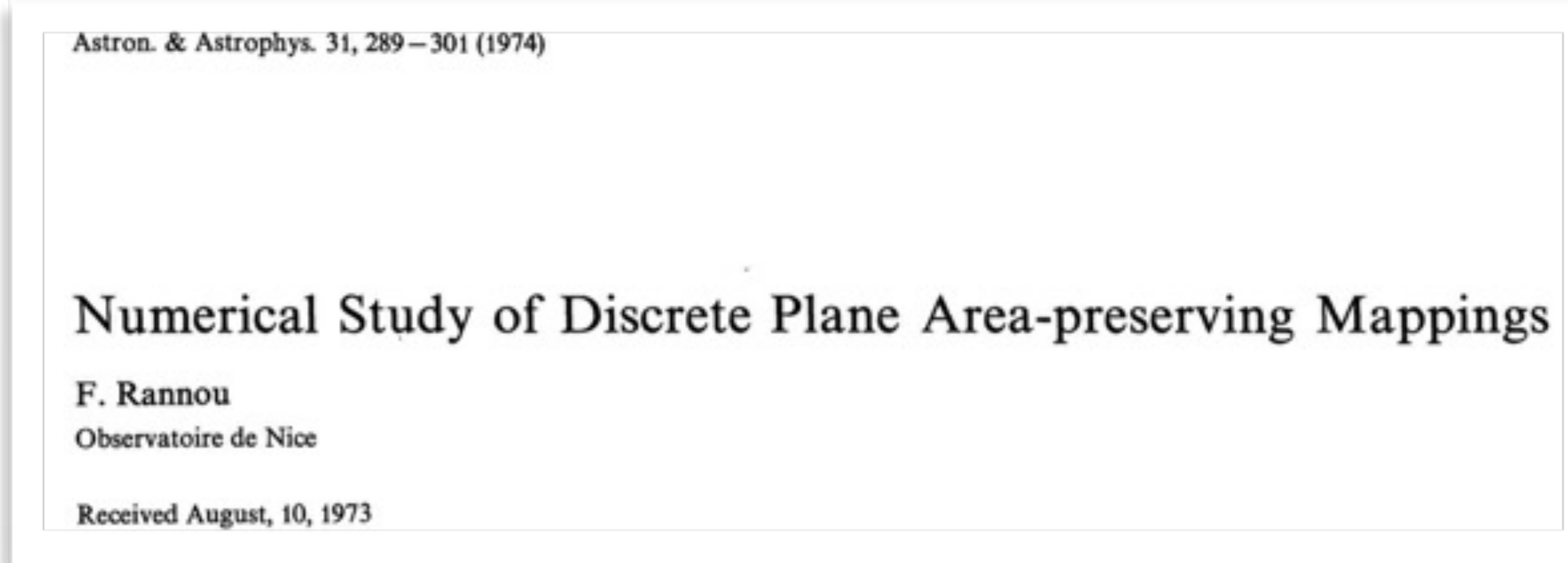

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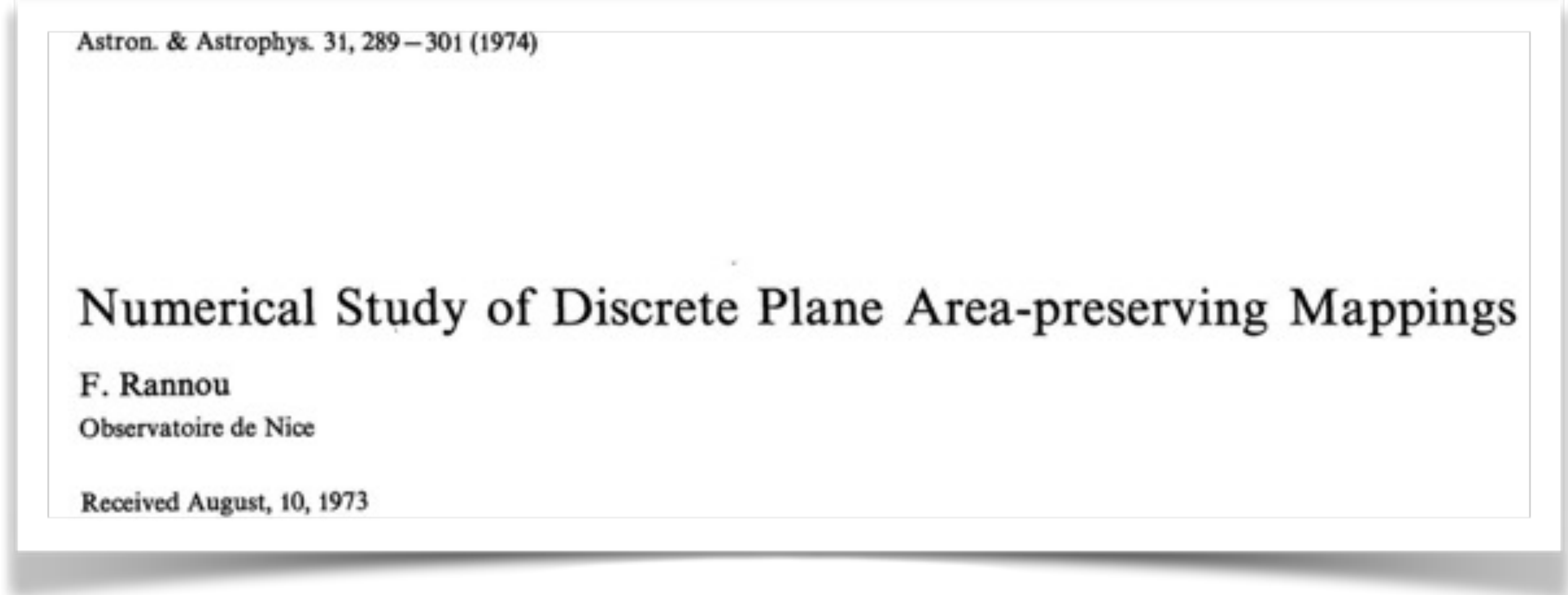
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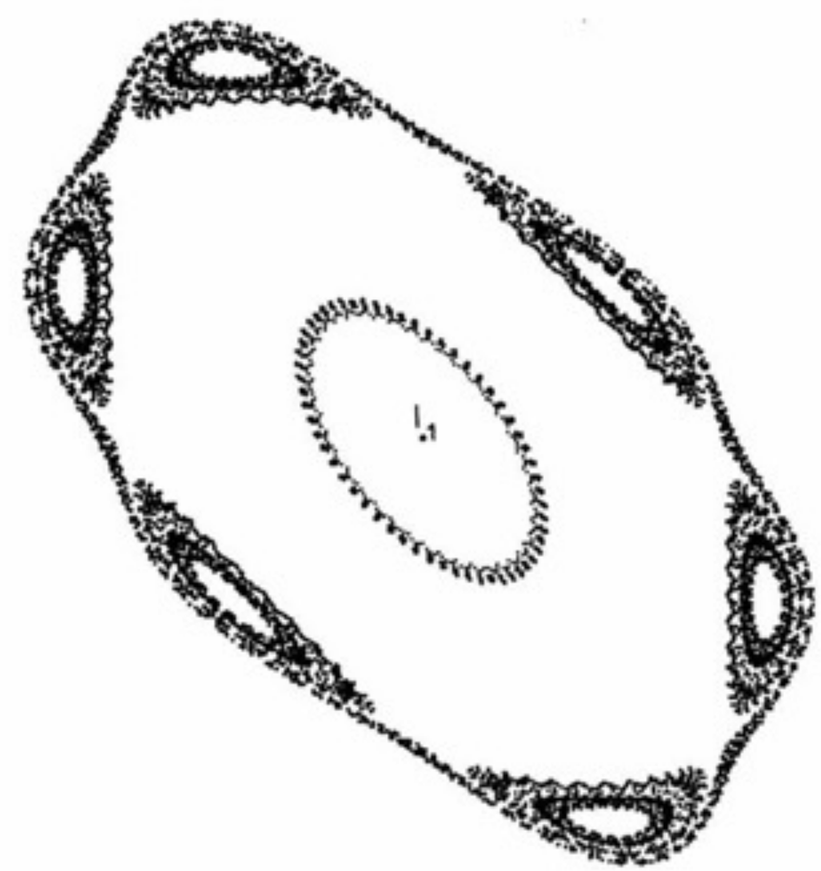
- All orbits are periodic.
- Orbit representing curves develop some "thickness", but remain stable.



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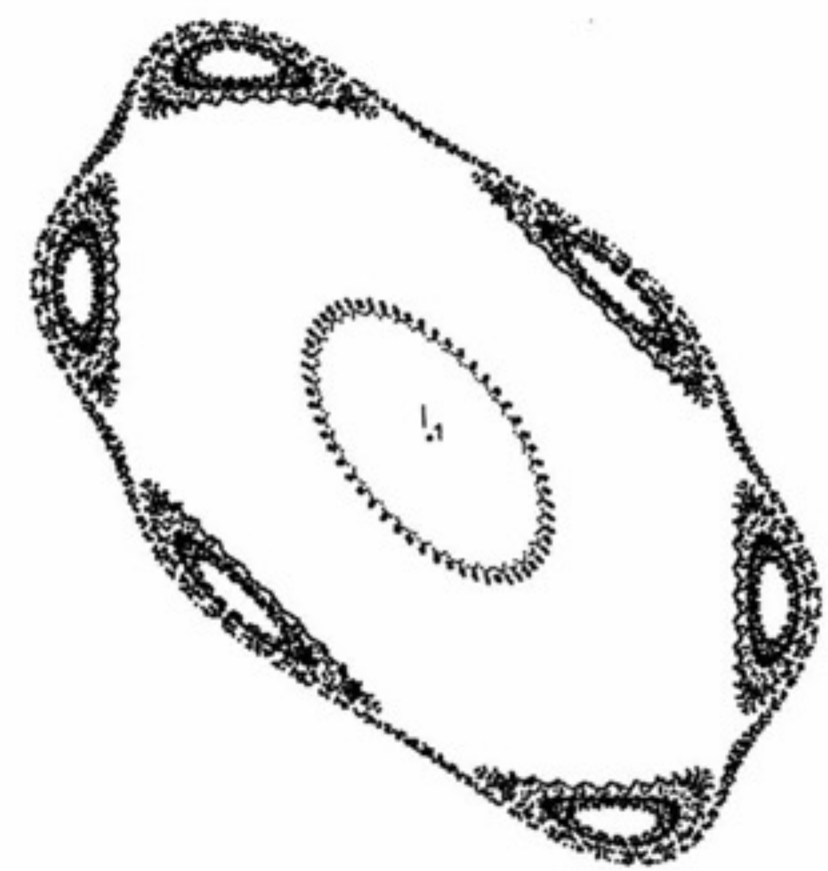
- All orbits are periodic.
- Orbit representing curves develop some “thickness”, but remain stable.
- In the chaotic regions the map behaves like a random permutation.



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- All orbits are periodic.
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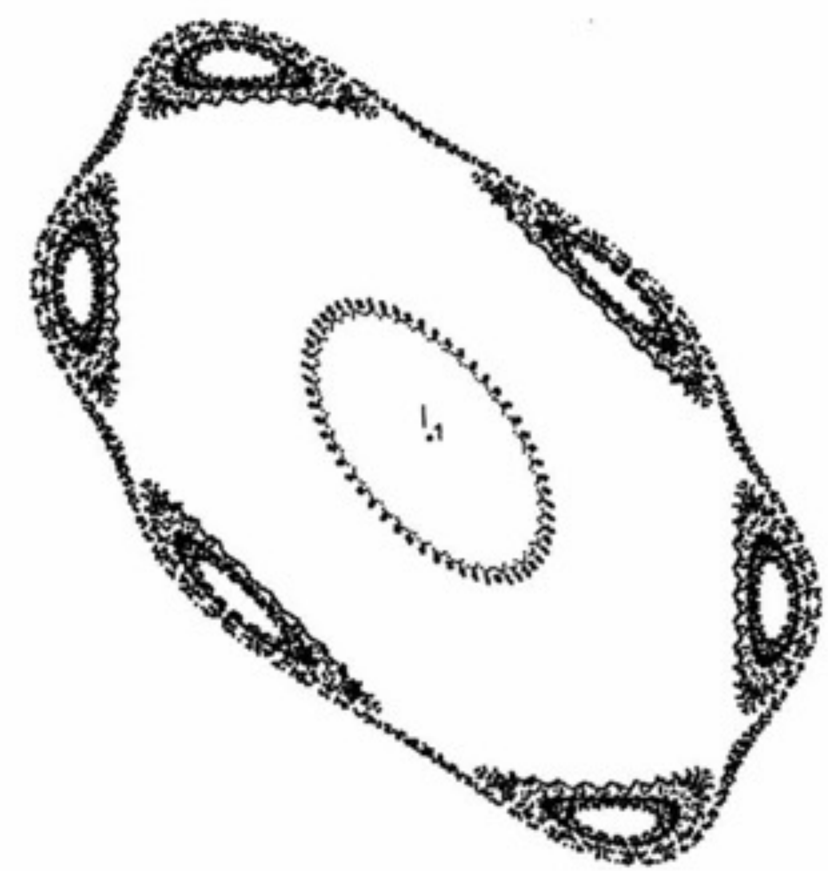

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PPPL–1938 (Nov. 1982)
Physica **8D**(3), 360–380 (Sept. 1983)

LONG-TIME CORRELATIONS IN THE STOCHASTIC REGIME

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Princeton, New Jersey 08544, USA

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An algebraic mapping forced on a lattice via truncation

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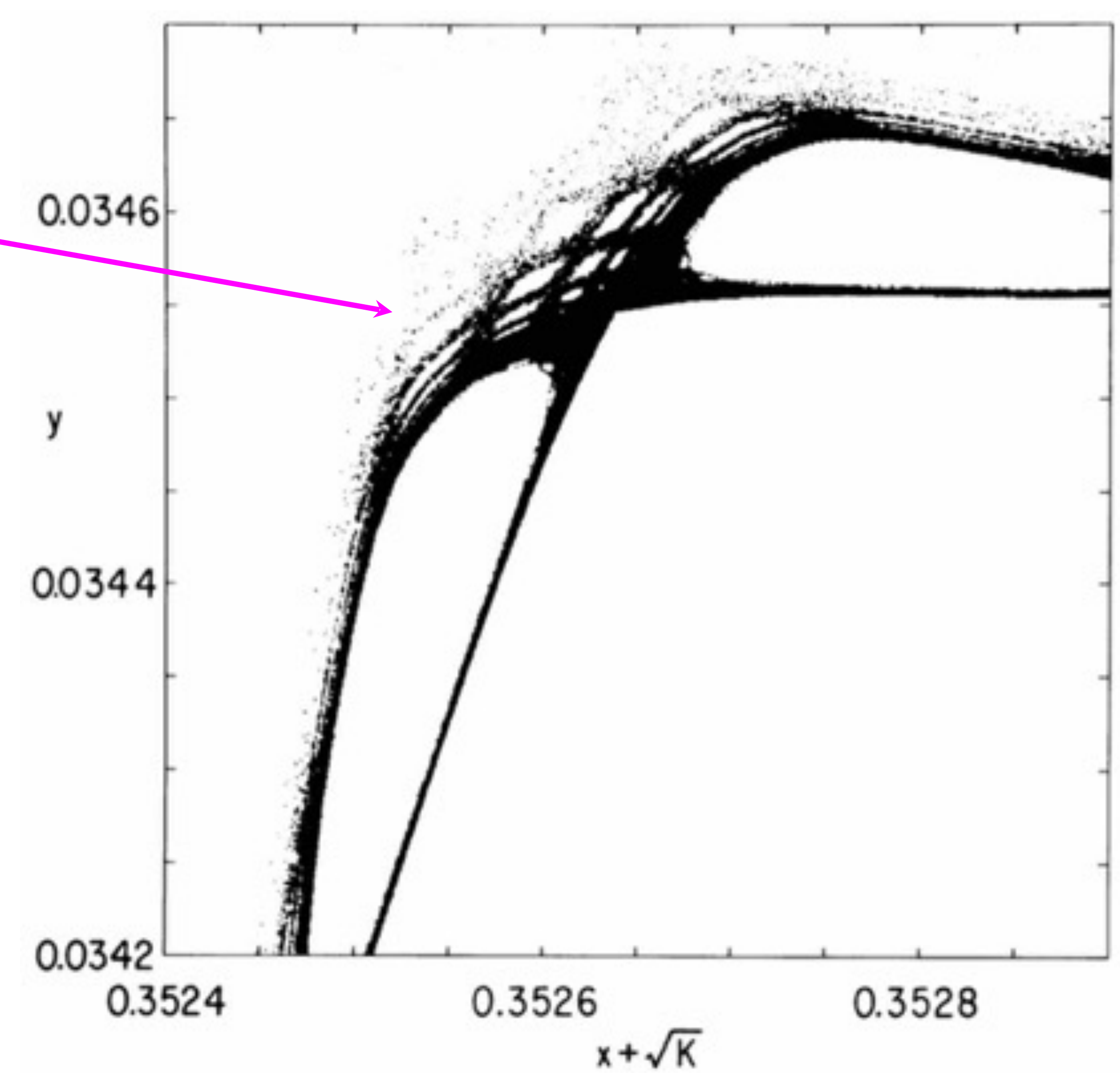
Detailed study of the boundary between order and chaos



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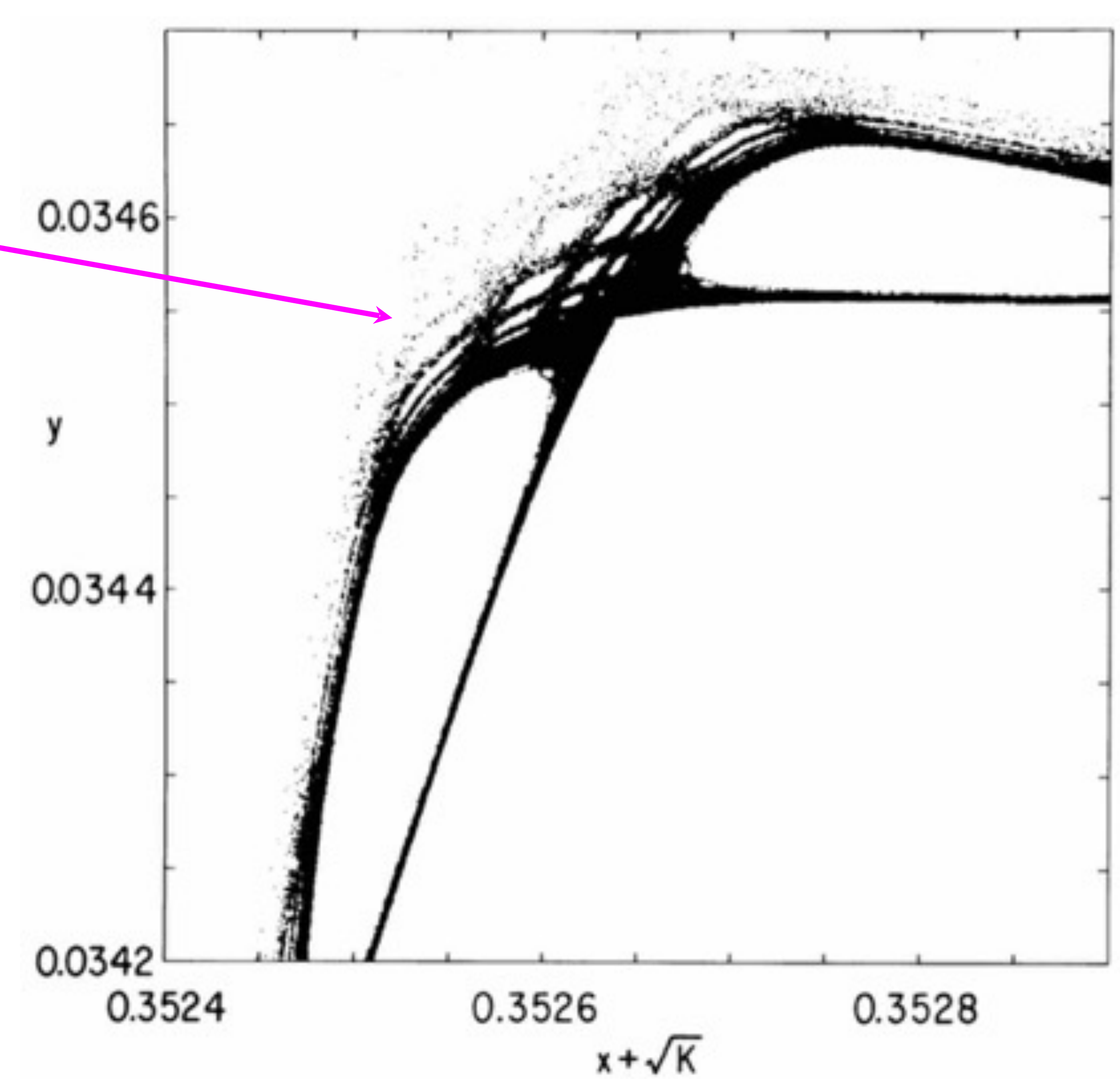

- Very large lattice (2^{48} x 2^{48} points).

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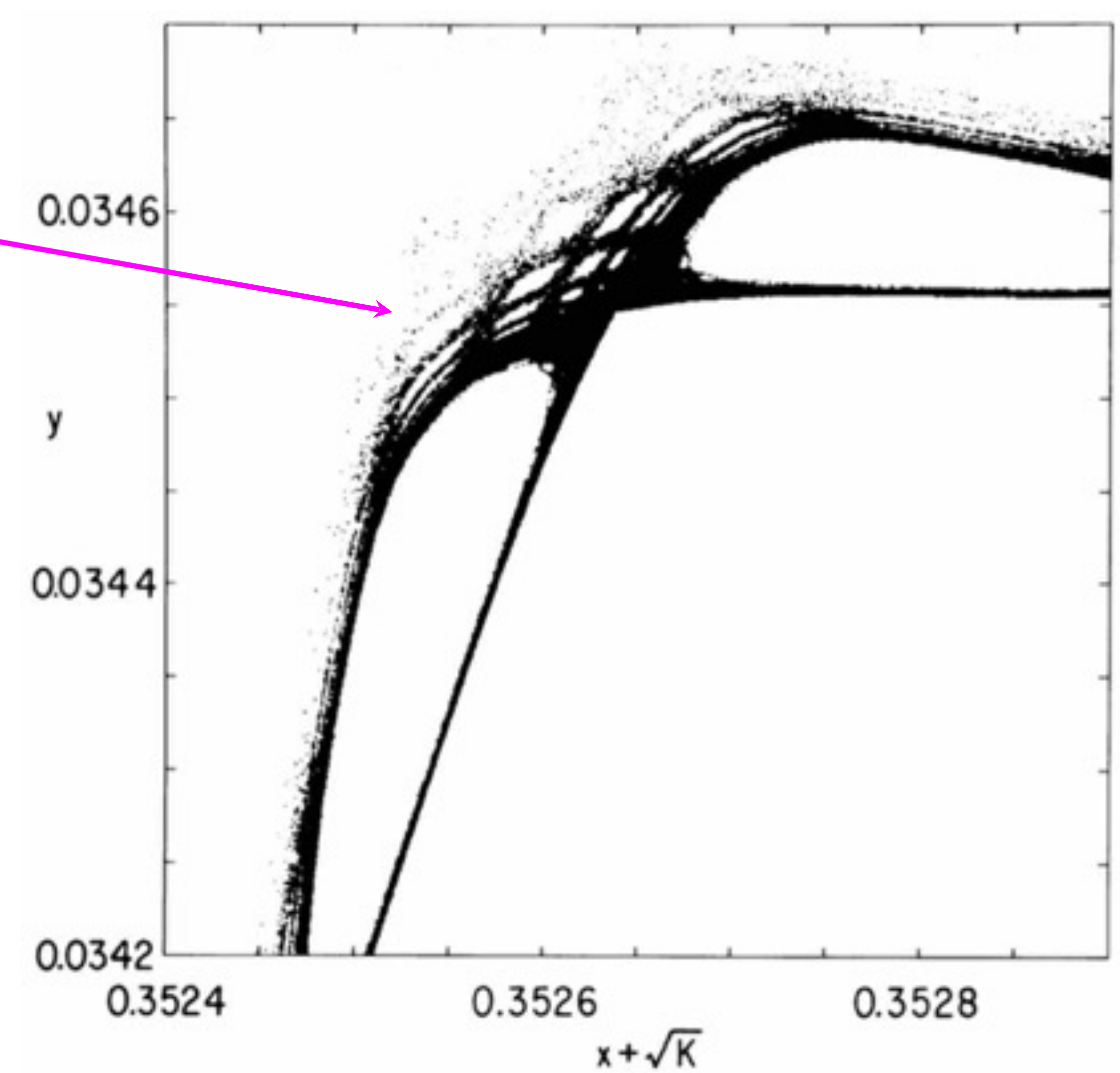

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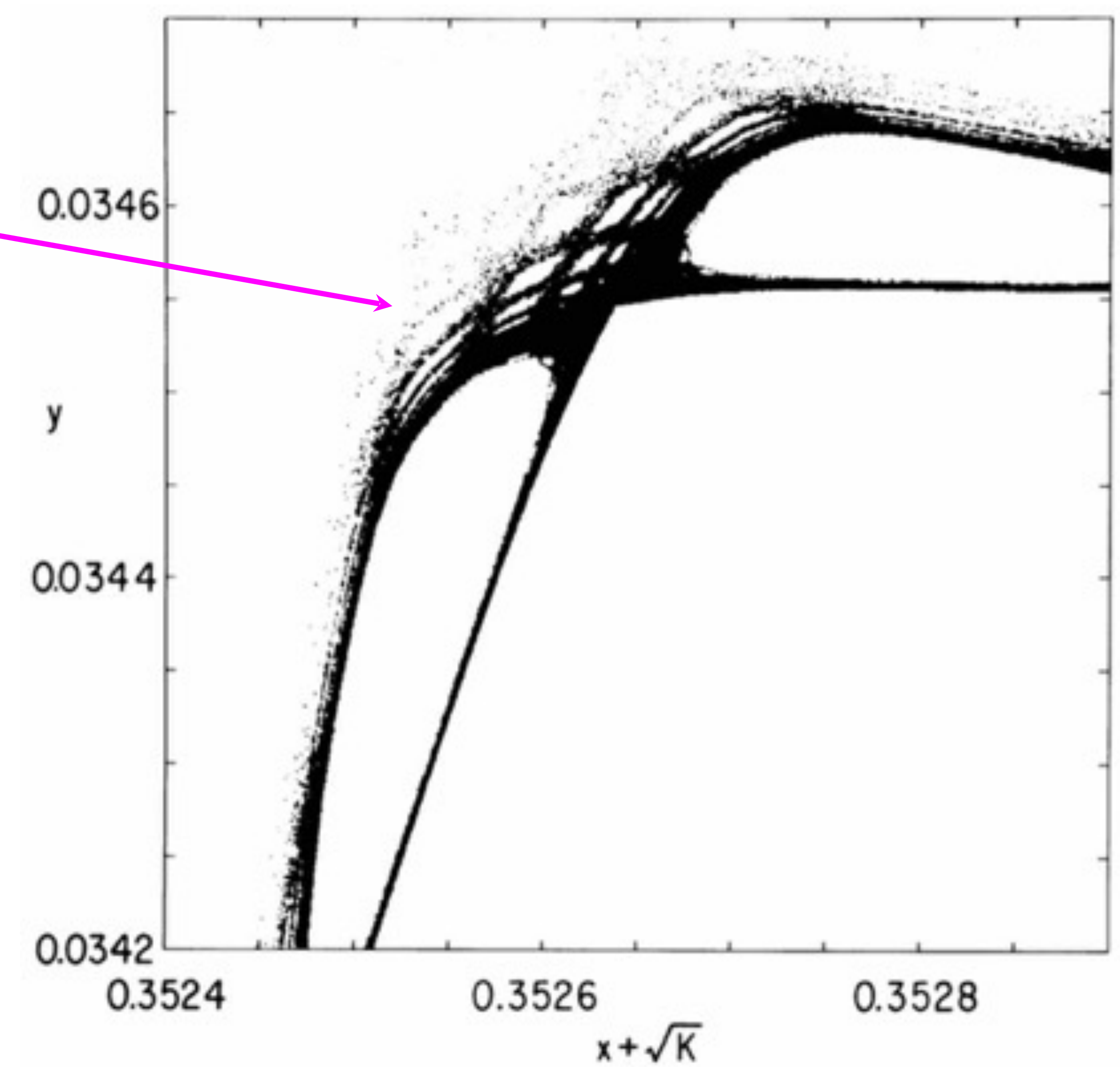

- Very large lattice (2^{48} x 2^{48} points).
- 90 hours of CPU on the Cray-I computer.
- Detected non-exponential correlation decay caused by structure of chaos boundary.

Dynamics with floating-point numbers (Earn and Tremaine, 1992)

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Physica D 56 (1992) 1–22
North-Holland

PHYSICA D

Exact numerical studies of Hamiltonian maps: Iterating without roundoff error

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Institute of Astronomy, Madingley Road, Cambridge CB3 0HA, UK

and

Scott Tremaine
Canadian Institute for Theoretical Astrophysics, University of Toronto, Toronto, Canada M5S 1A1[1]
and Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, CA 91125, USA

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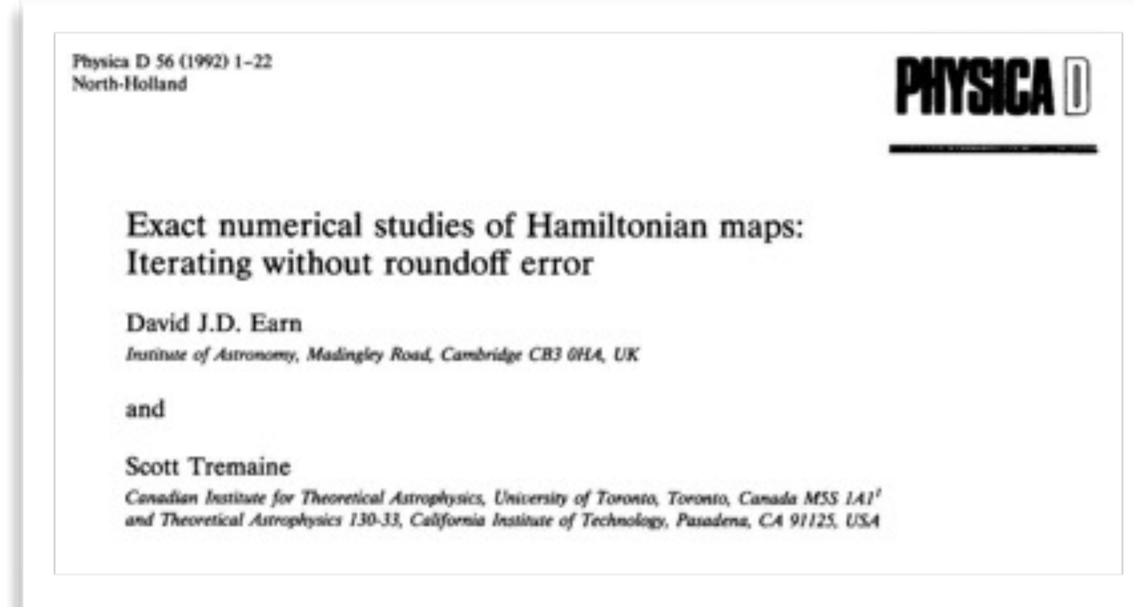
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Floating-point numbers 00110101001101101100101000111010

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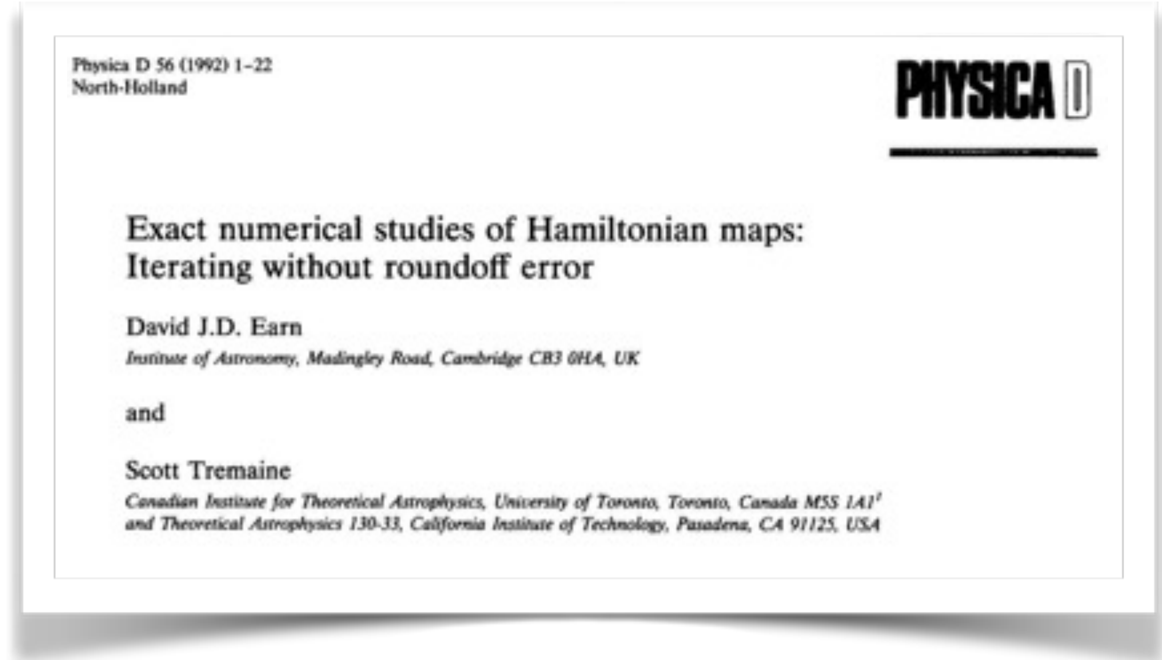
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Floating-point numbers

sign exponent mantissa

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$$\pm 2^{\text{exponent}-q} \times \text{mantissa}$$

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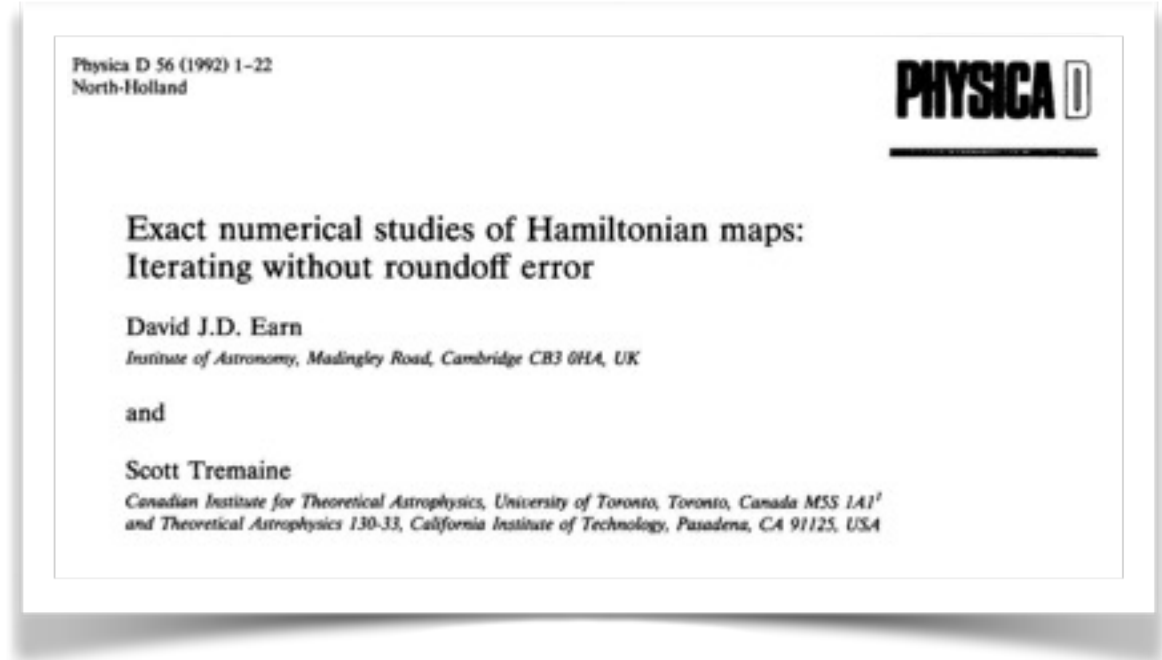

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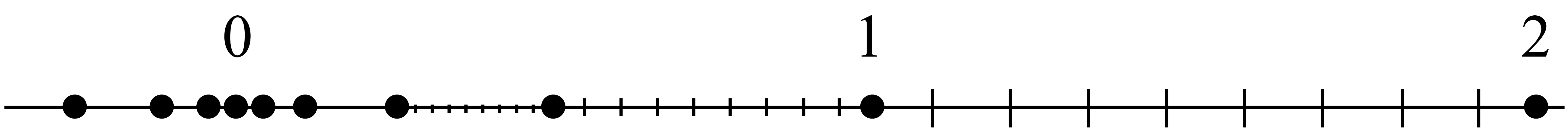

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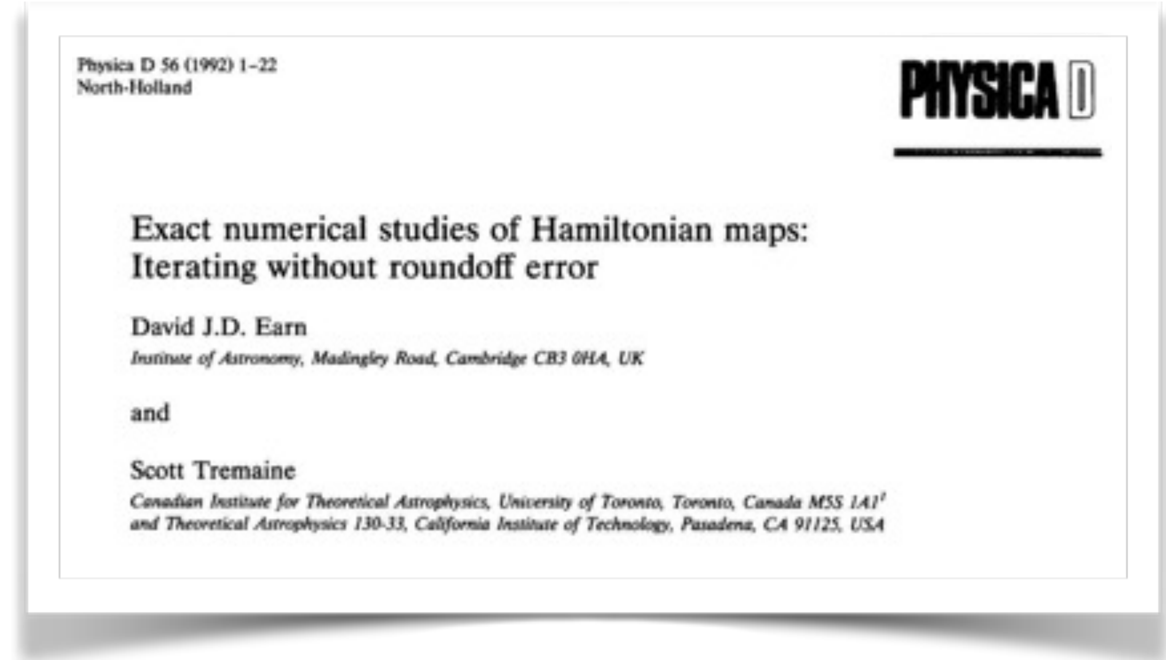
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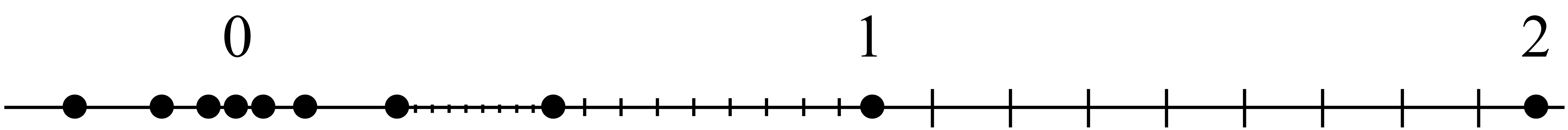
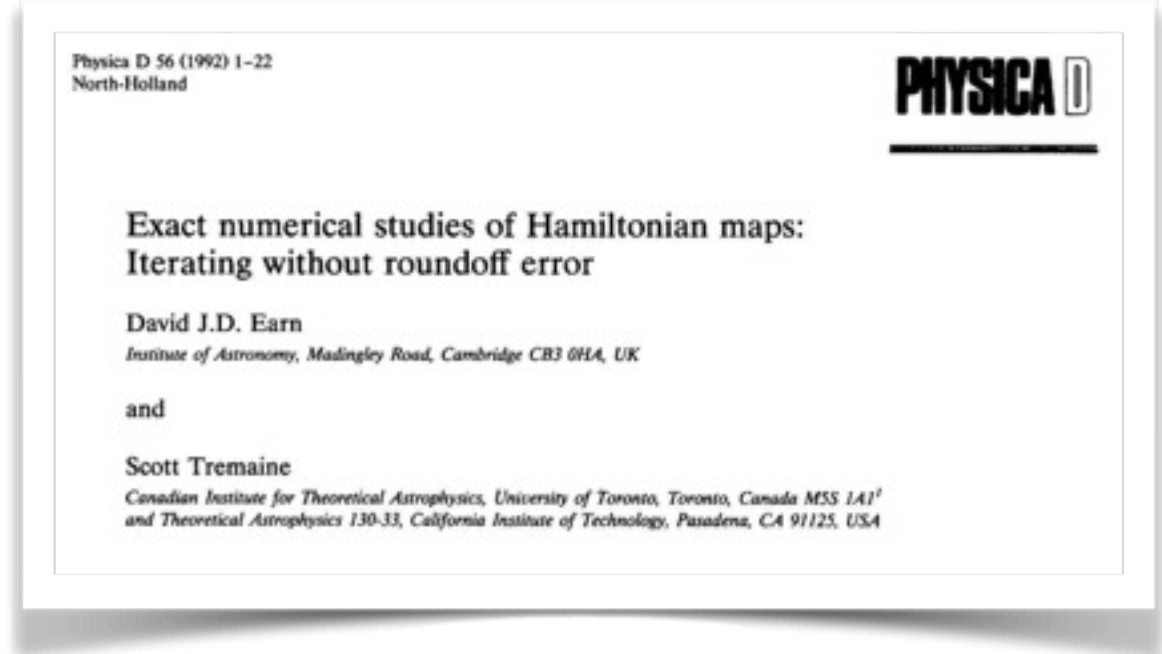


- An idiosyncratic subset of $\mathbb{Z}\left[\frac{1}{2}\right]$

Dynamics with floating-point numbers (Earn and Tremaine, 1992)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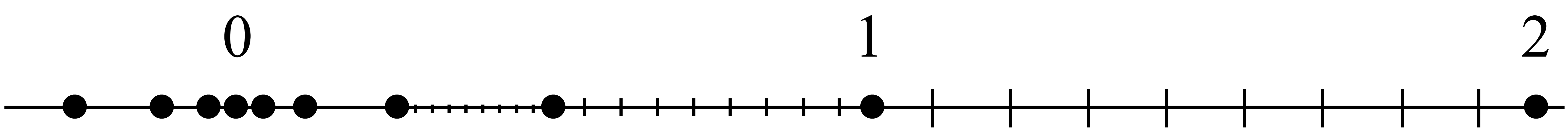
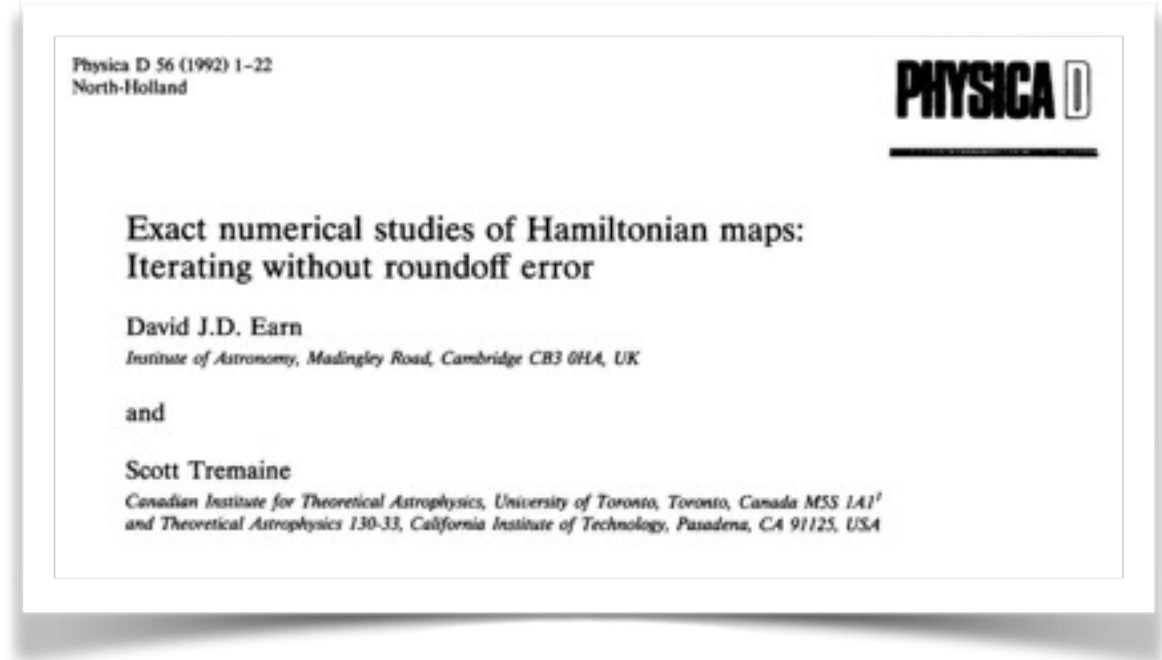


- An idiosyncratic subset of $\mathbb{Z}\left[\frac{1}{2}\right]$
- Same number of points between any two consecutive powers of 2.

Dynamics with floating-point numbers (Earn and Tremaine, 1992)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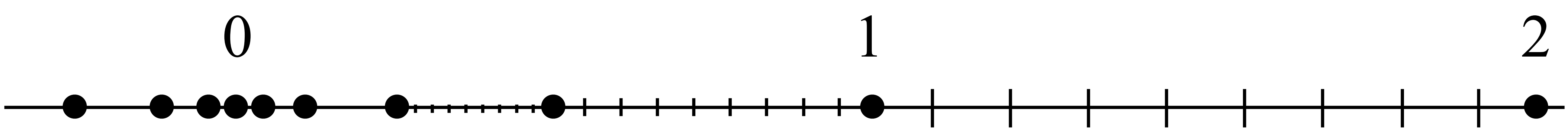

- An idiosyncratic subset of $\mathbb{Z}\left[\frac{1}{2}\right]$
- Same number of points between any two consecutive powers of 2.
- Highly non-uniform distribution.

Floating-point dynamics

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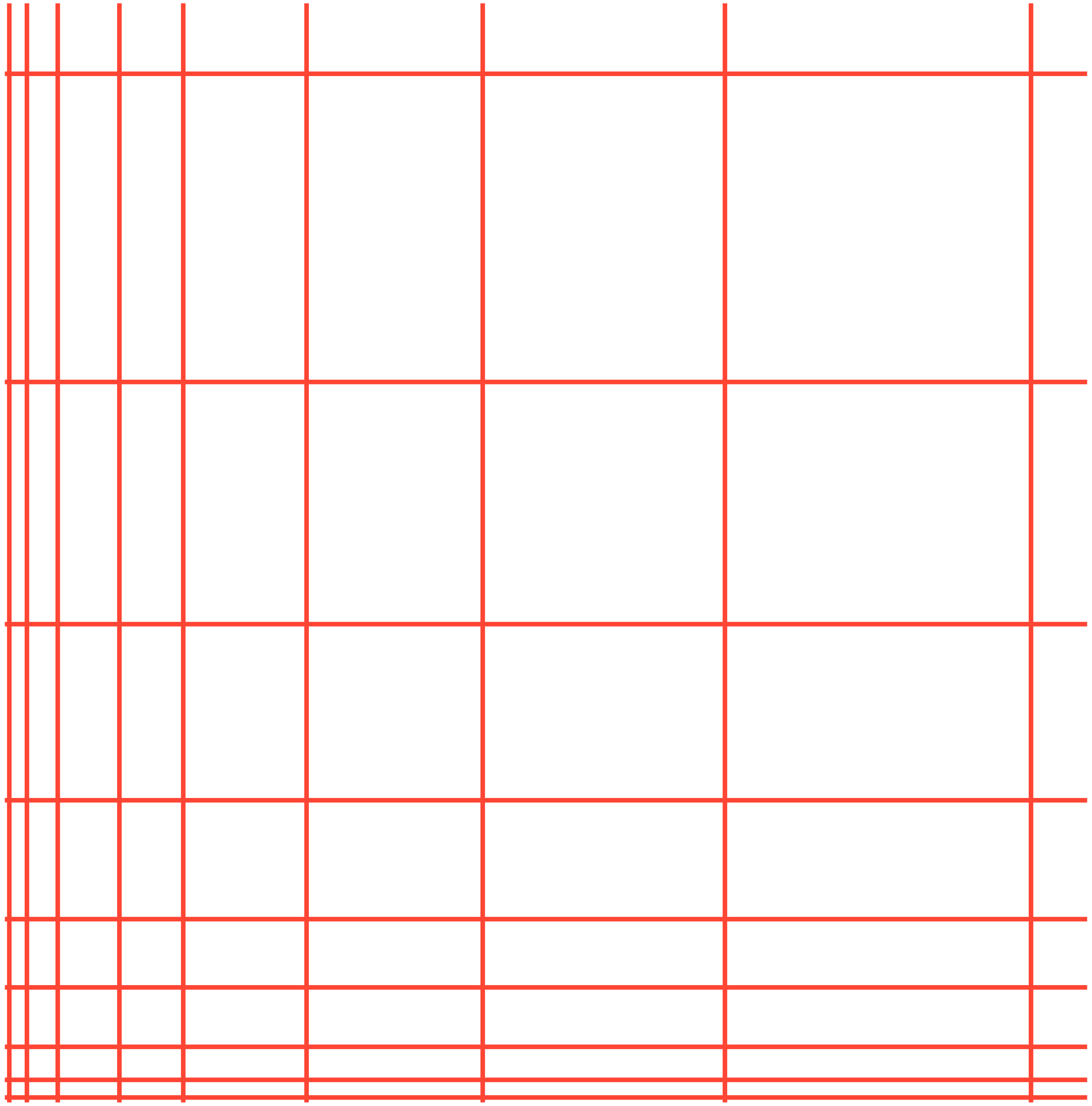

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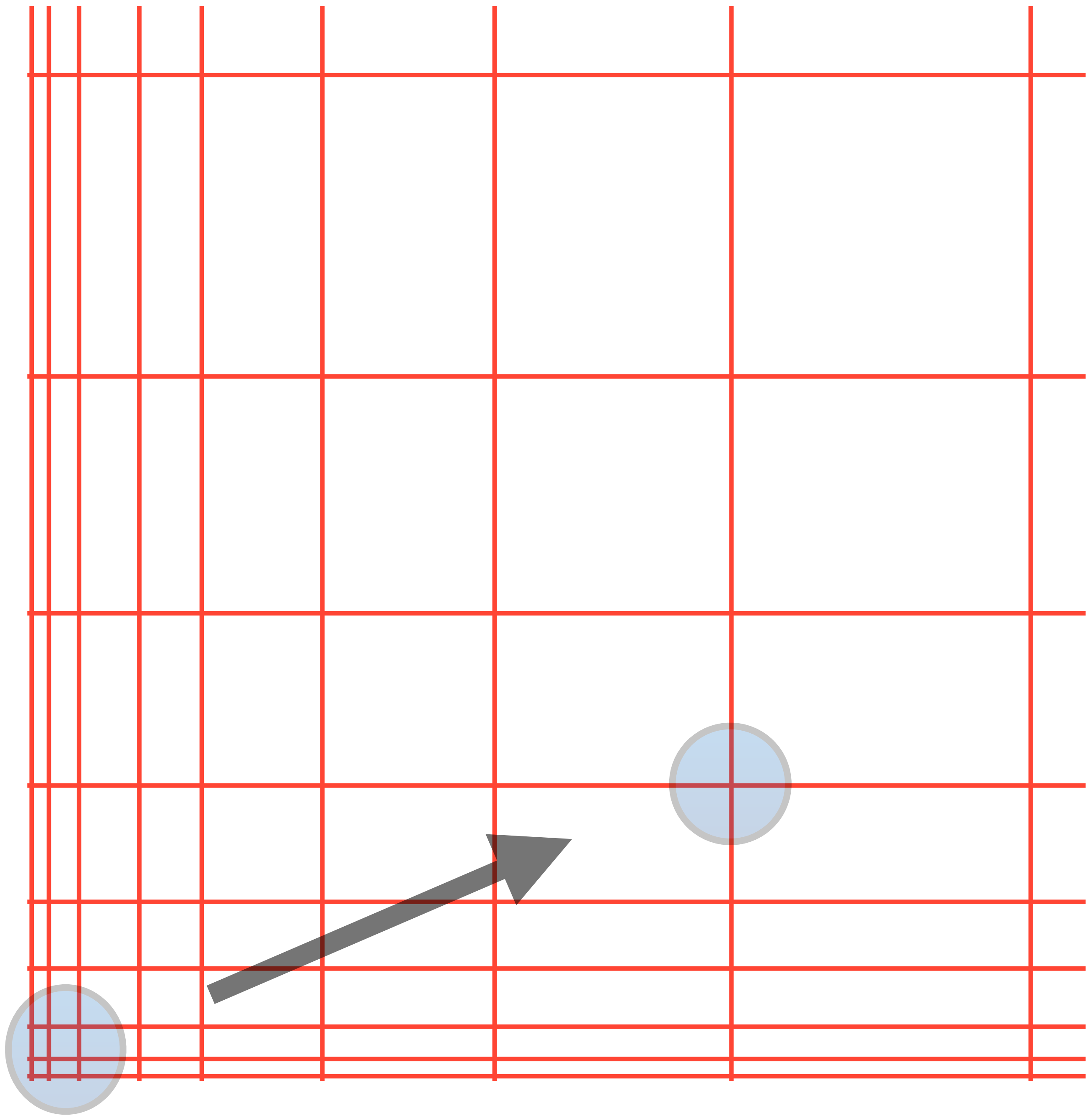

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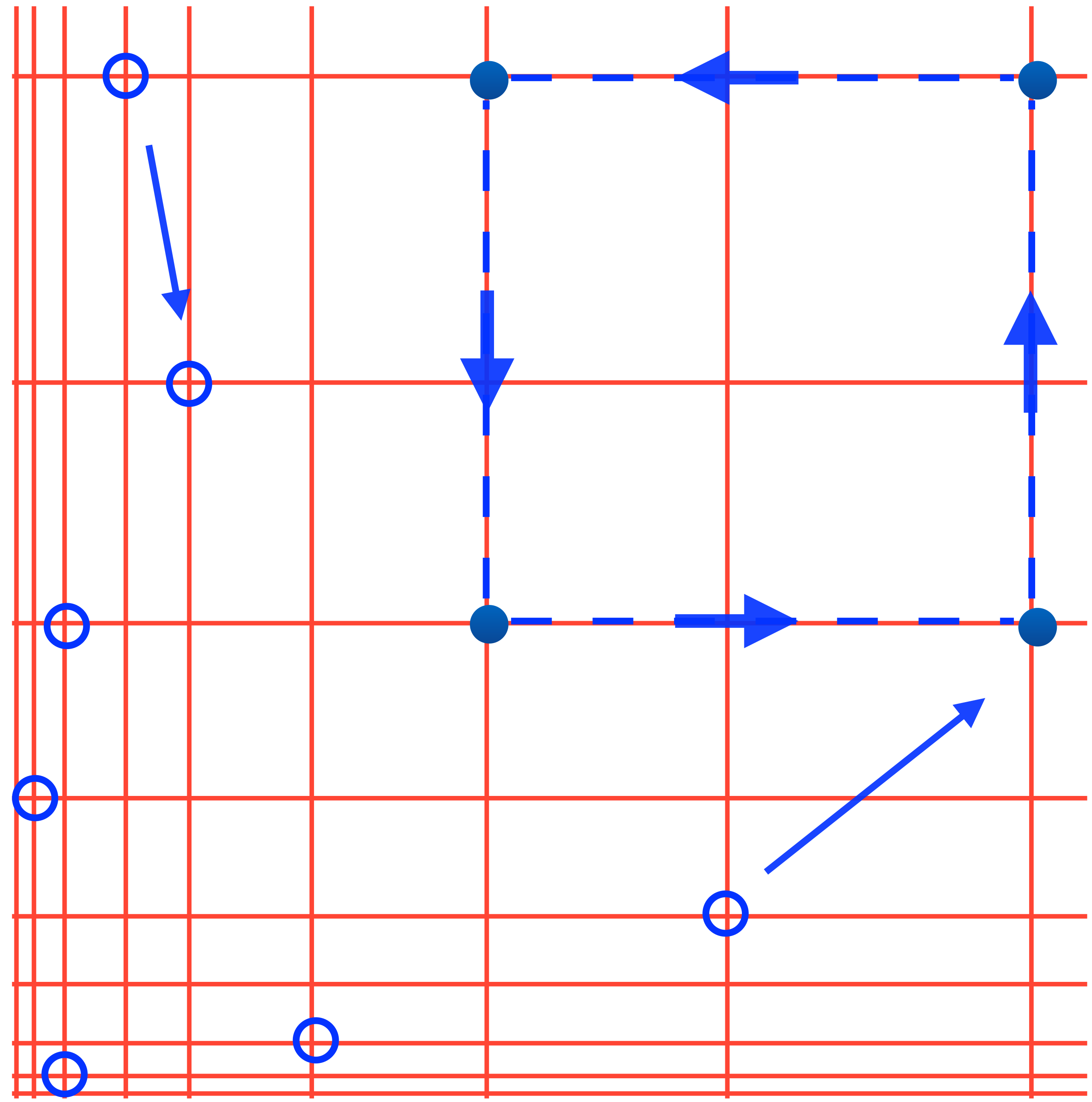

Floating-point dynamics

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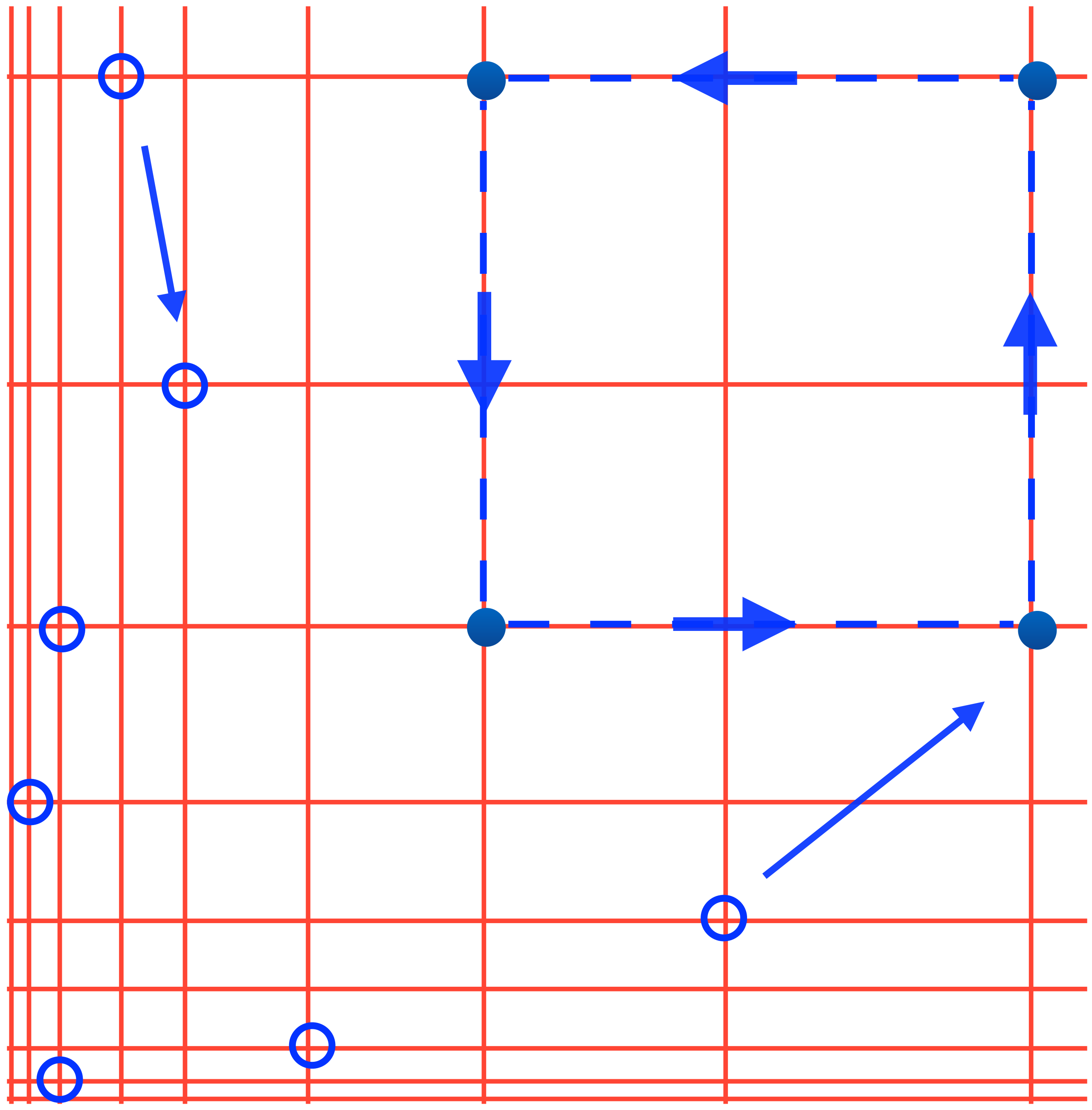
- Points of a limit cycle arrange themselves according to the invariant measure.



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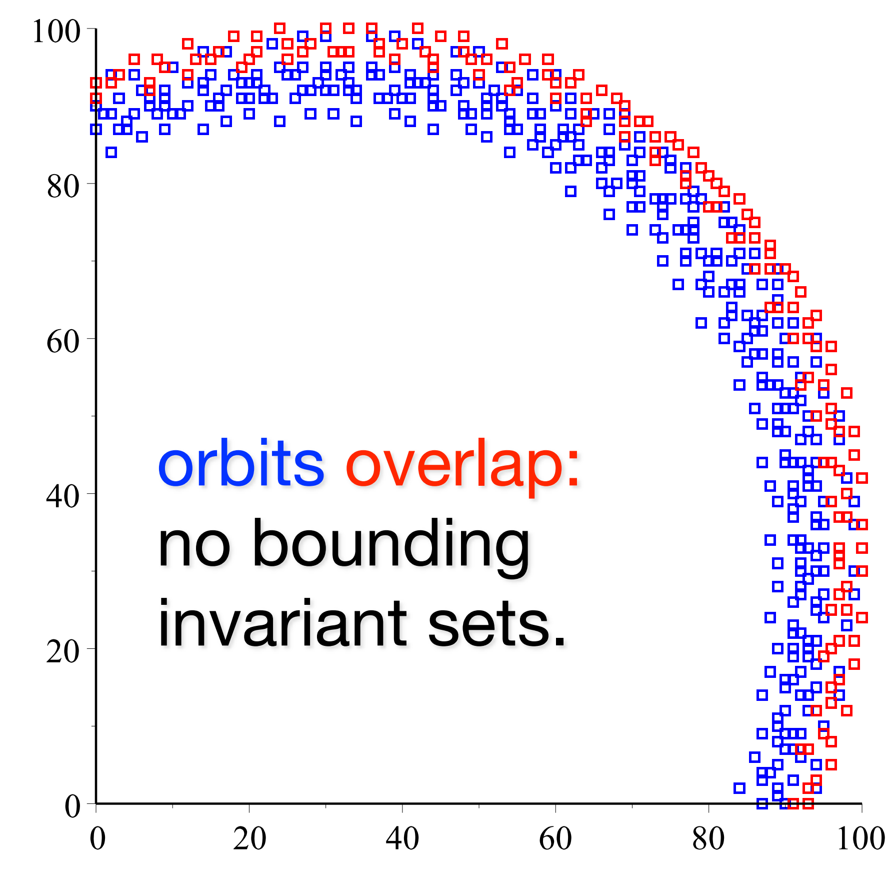

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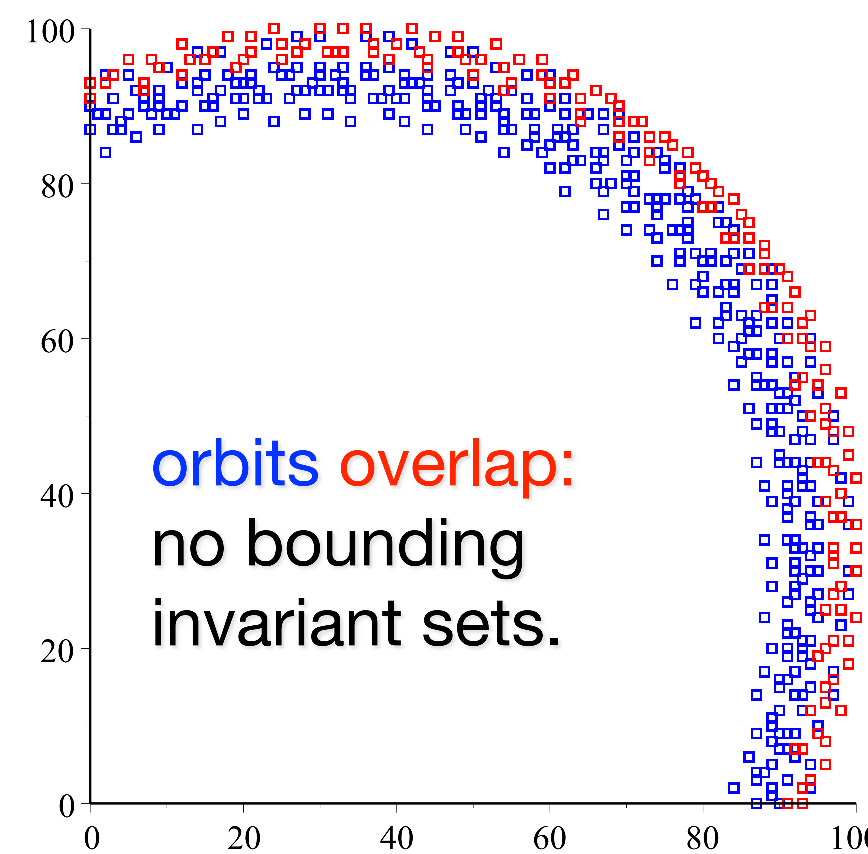
If λ is rational with a single prime divisor p at denominator, then the round-off map may be embedded into a positive-entropy map of the ring of p-adic integers. [Bosio, fv]



The period function (period vs. height)

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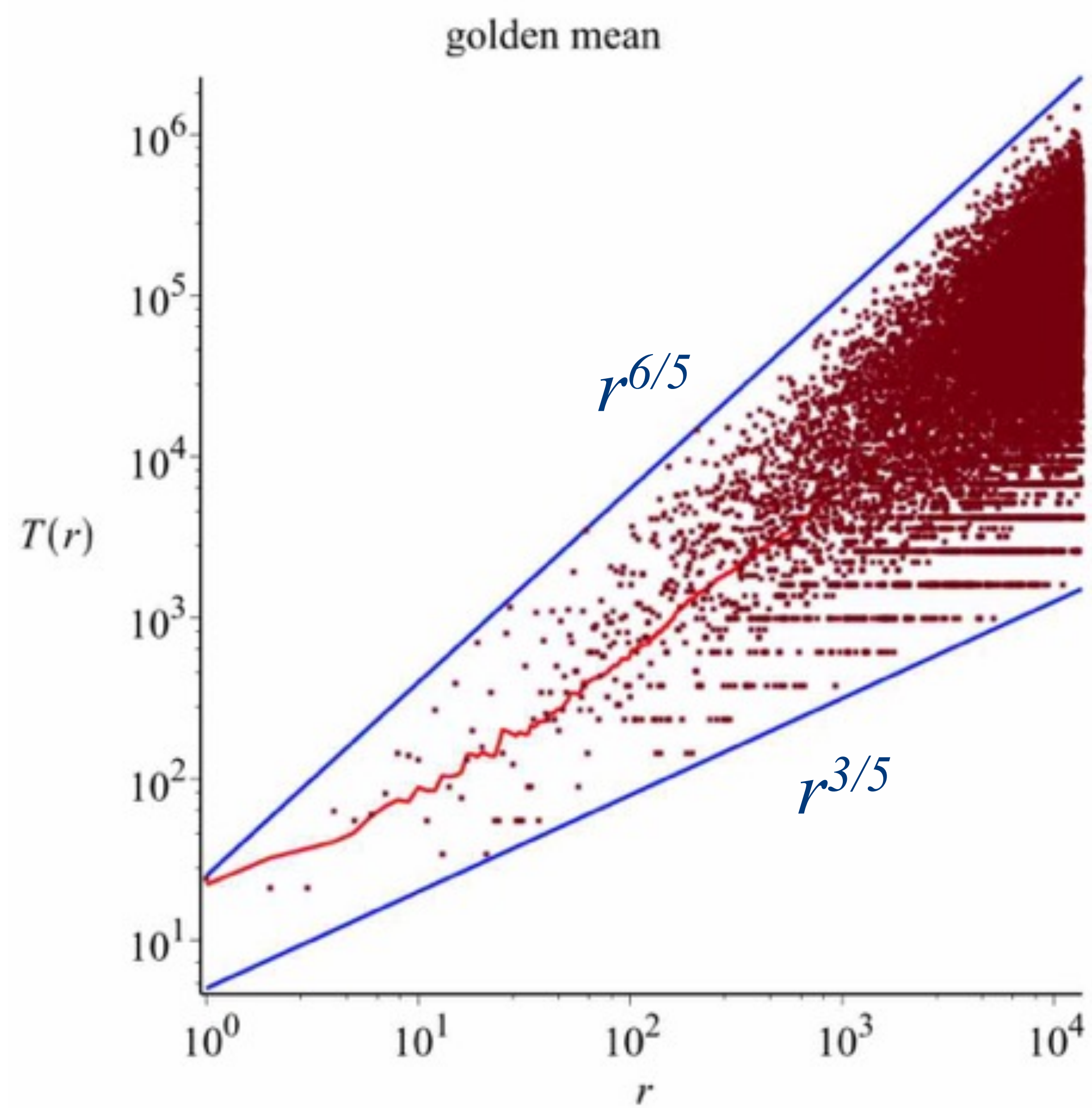

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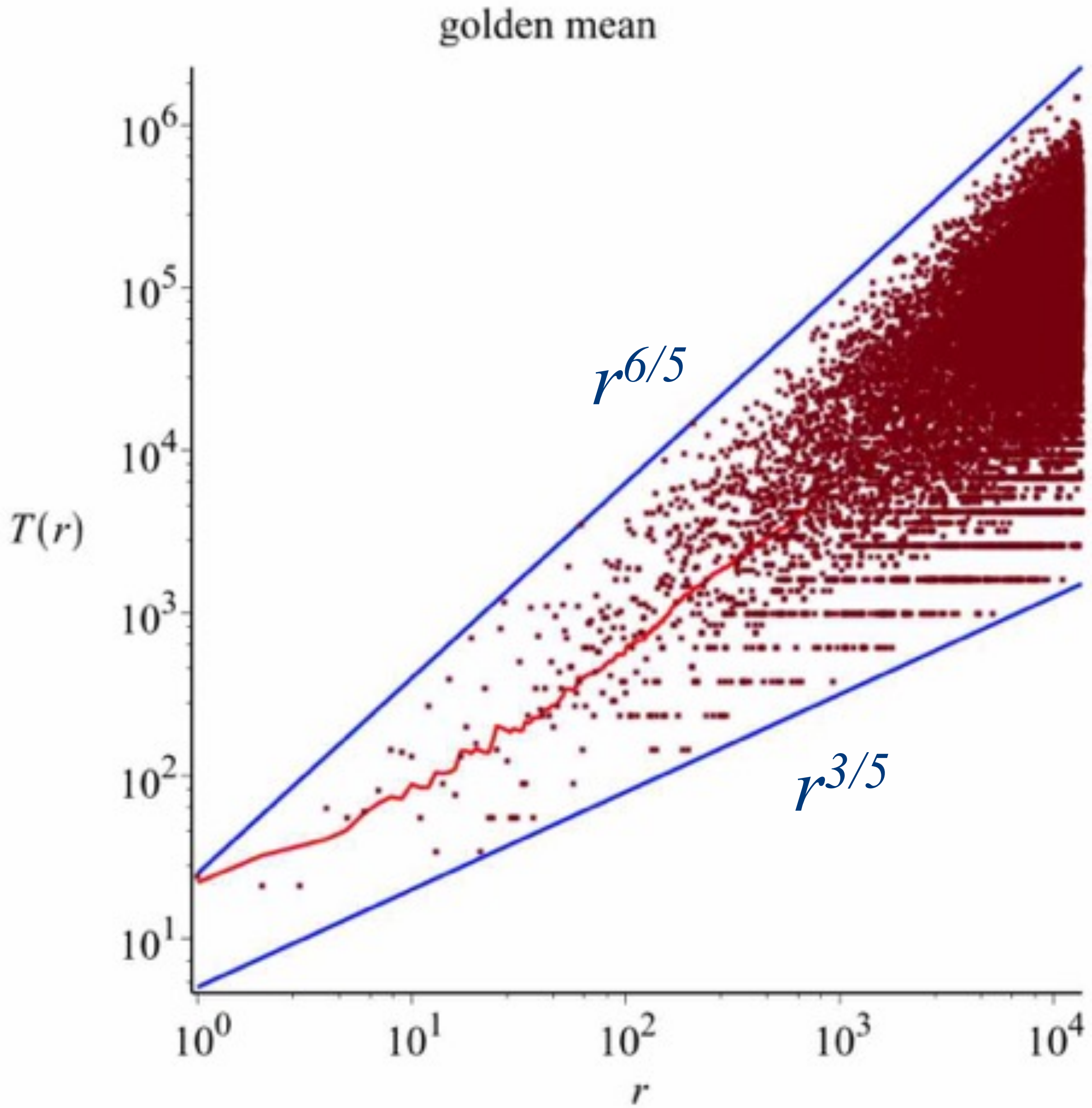

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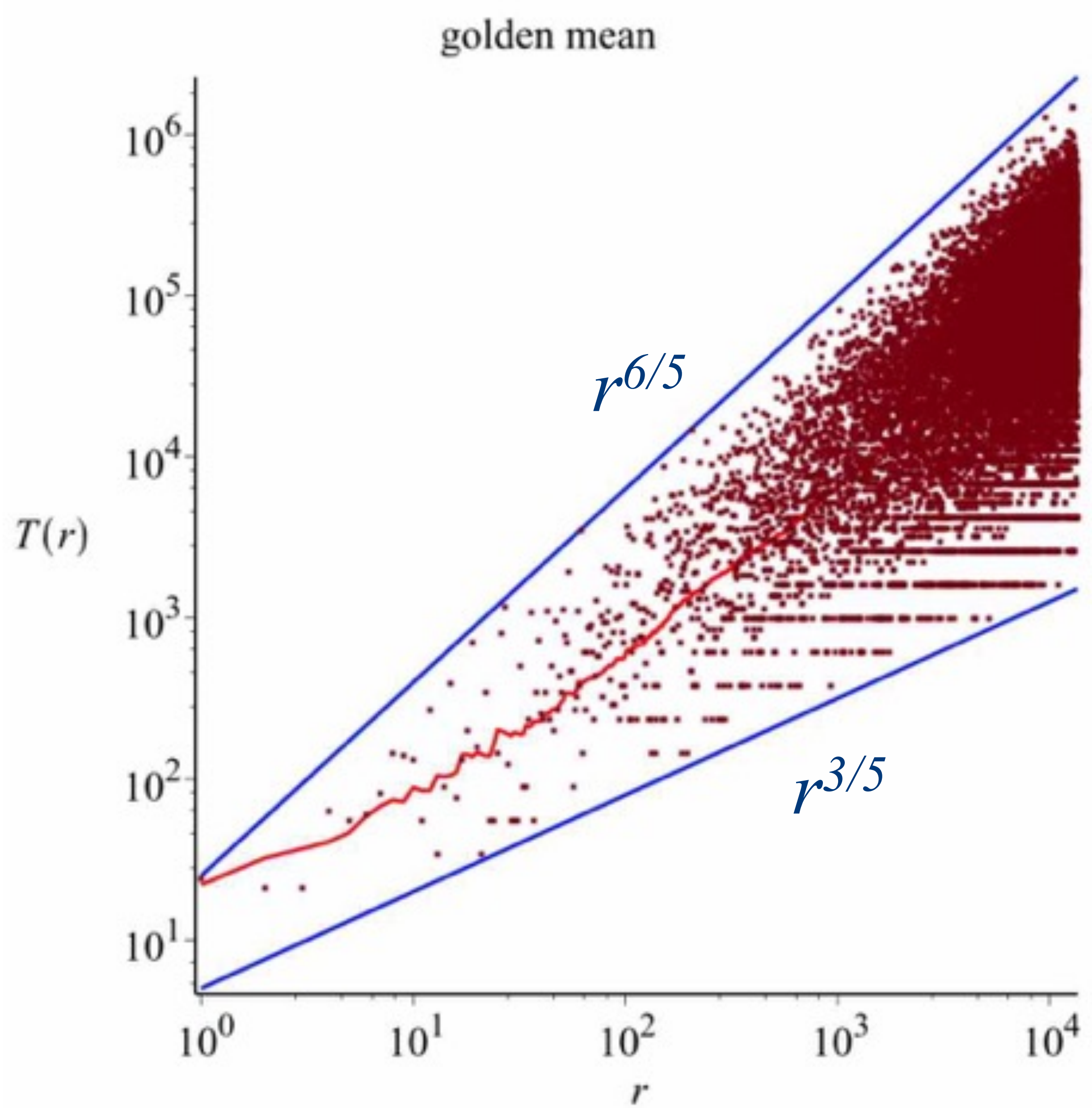

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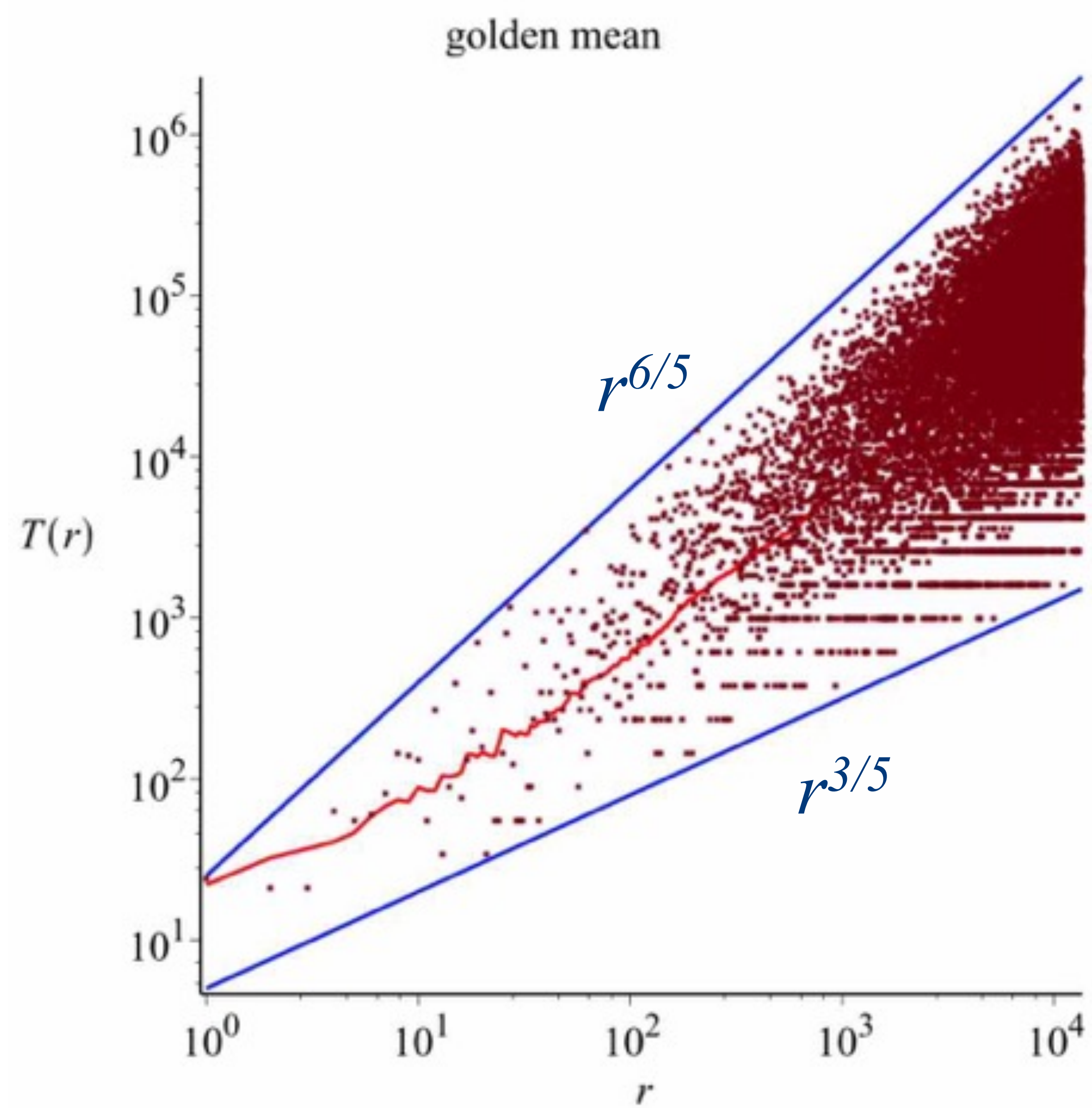

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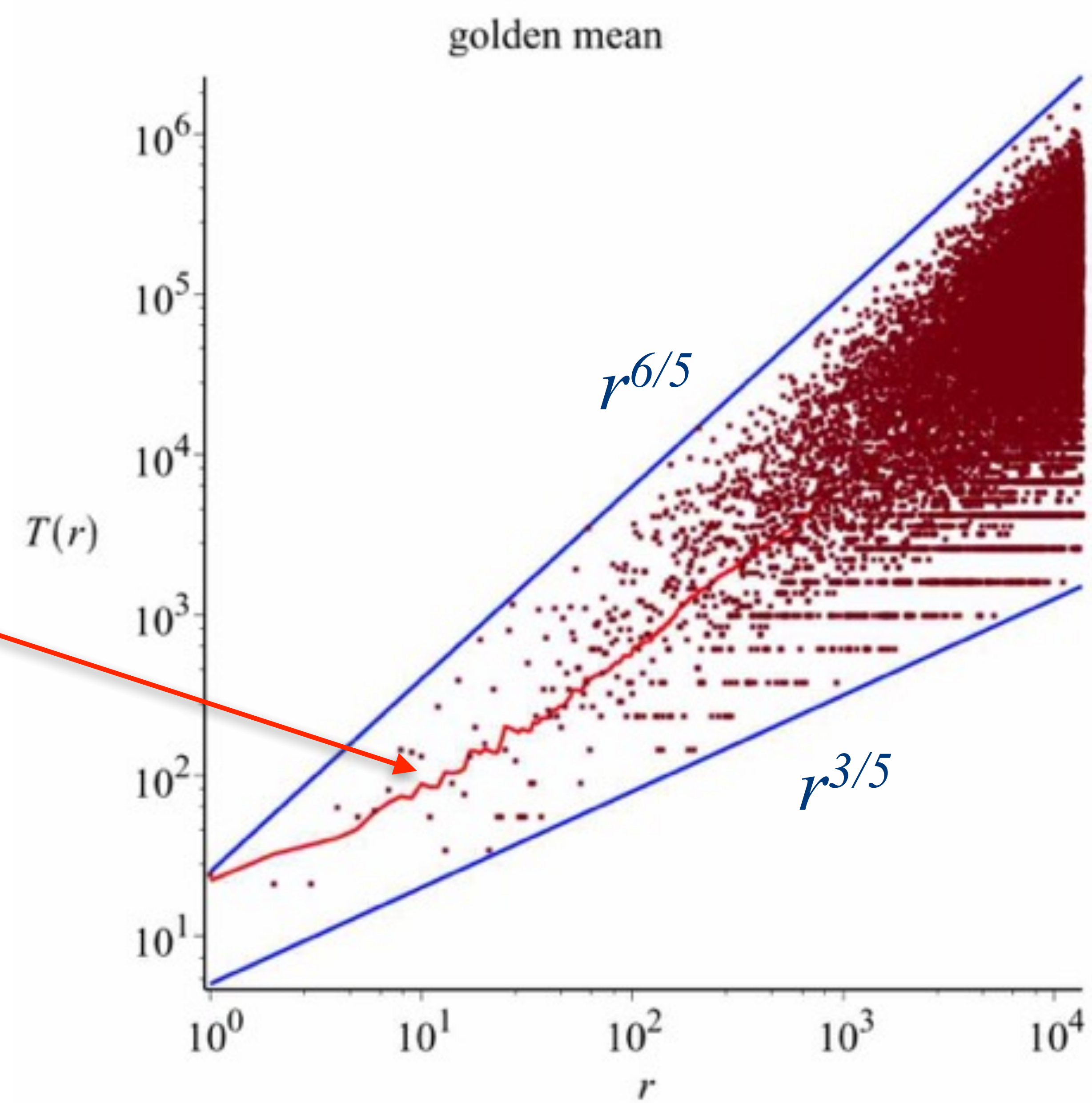

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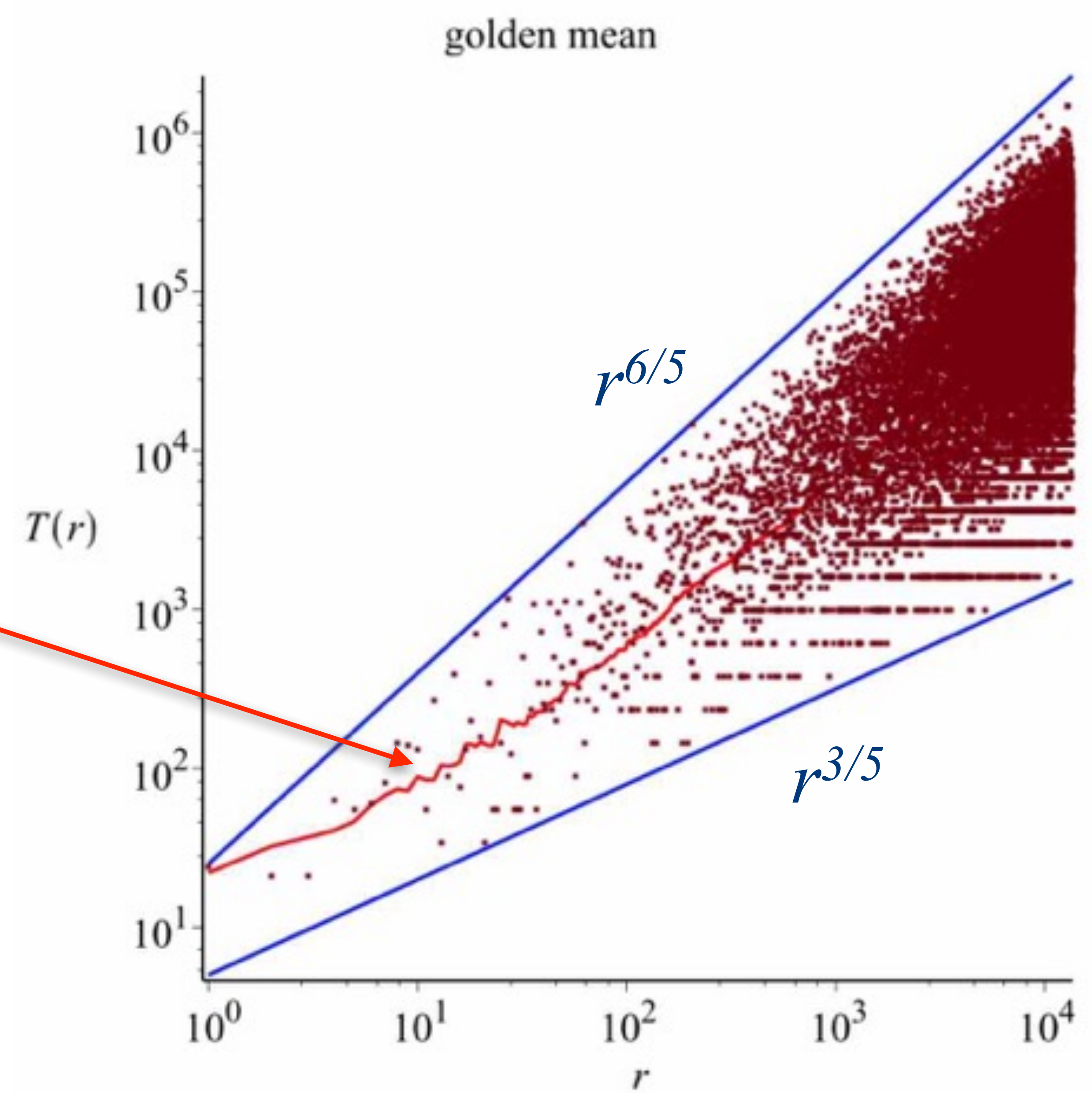

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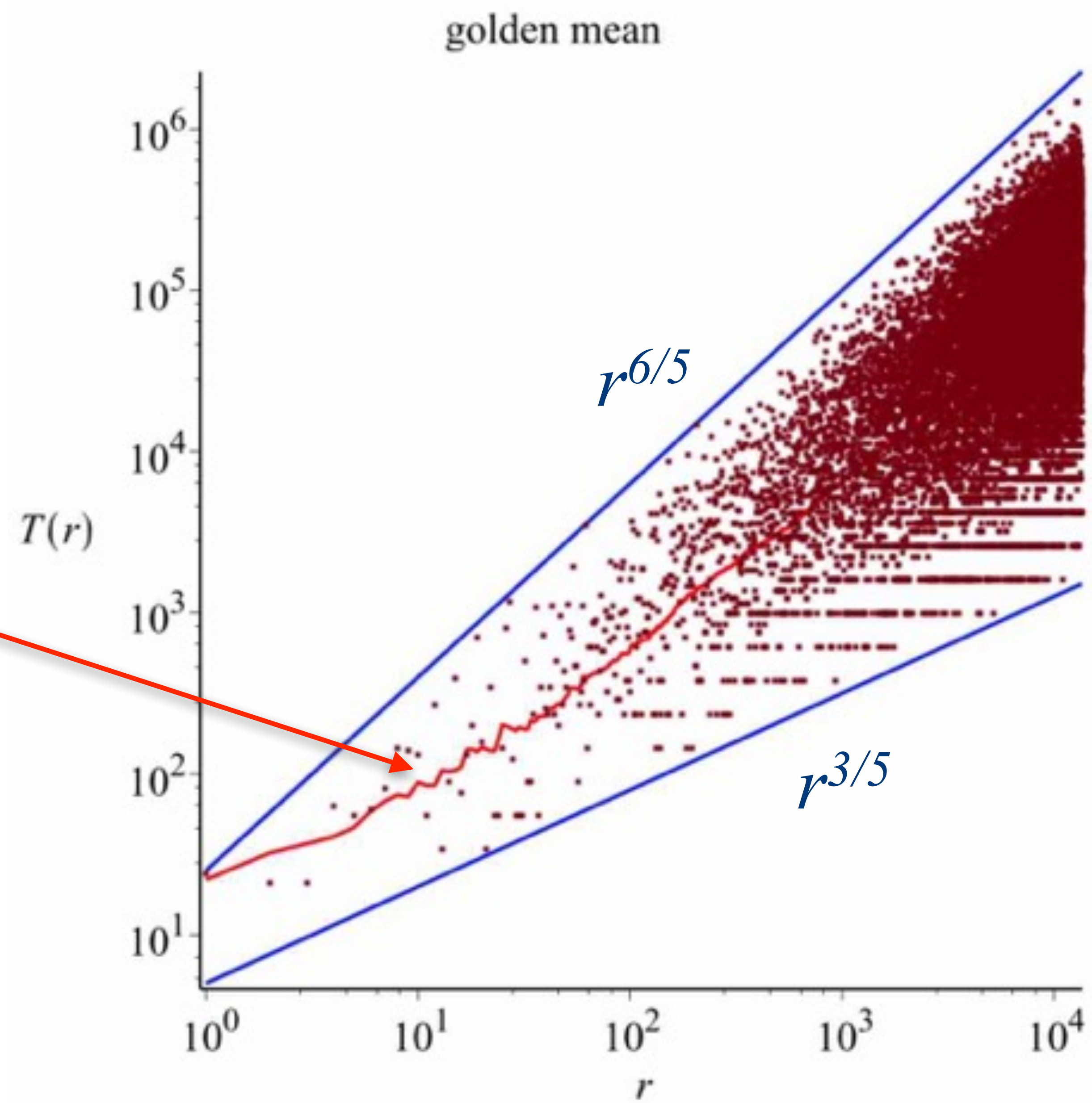

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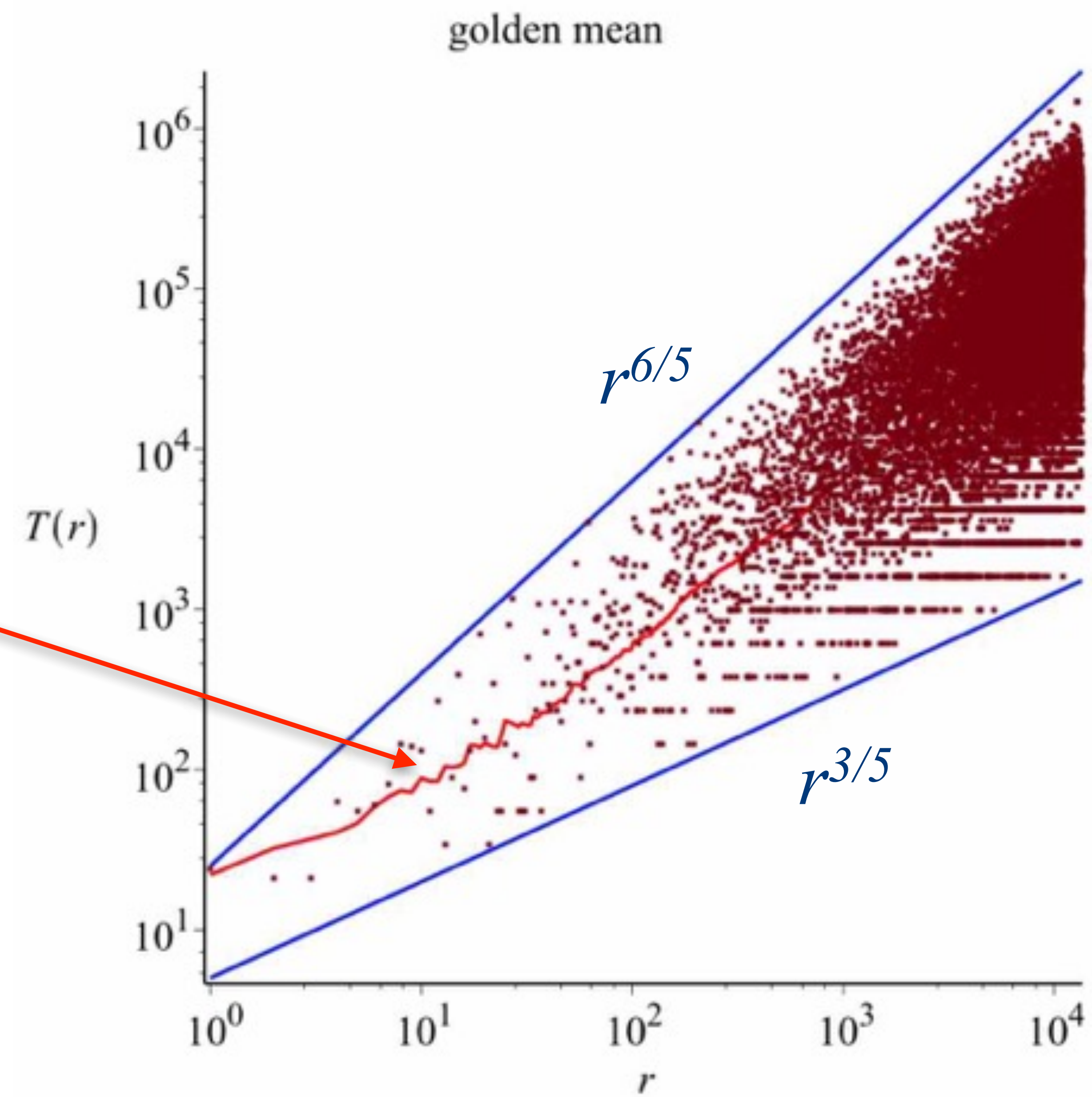

Results:

- The departure of round-off orbits from exact orbits obeys a central limit theorem. [Vladimirov, fv]
- The period function is defined at infinitely many points on the symmetry axis. [Akiyama]

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- Look for different discrete structures, for which bounding invariant sets exist.
- Look for systems with more manageable (zero entropy) fluctuations.

Geometric discretization: strip maps

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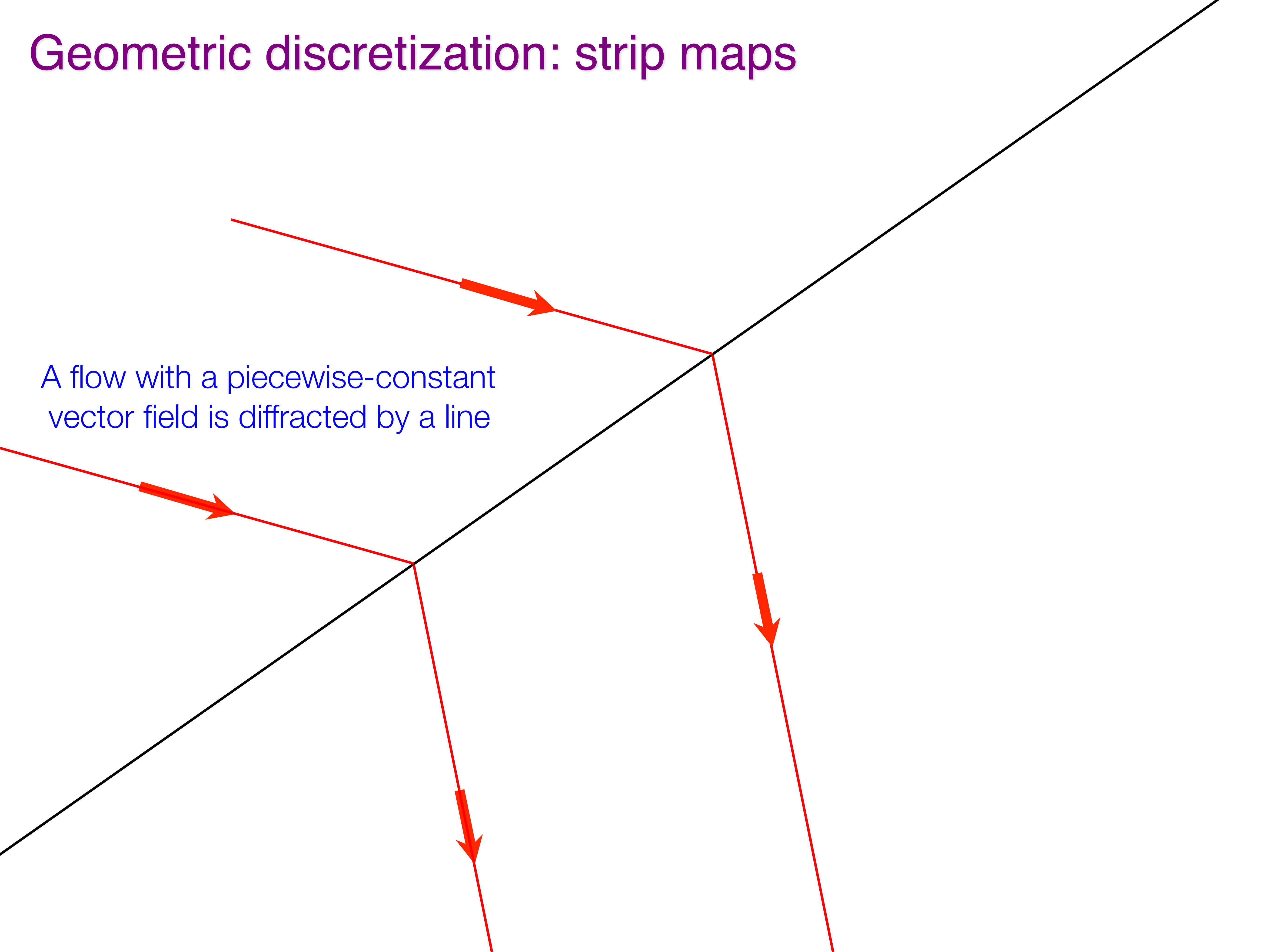

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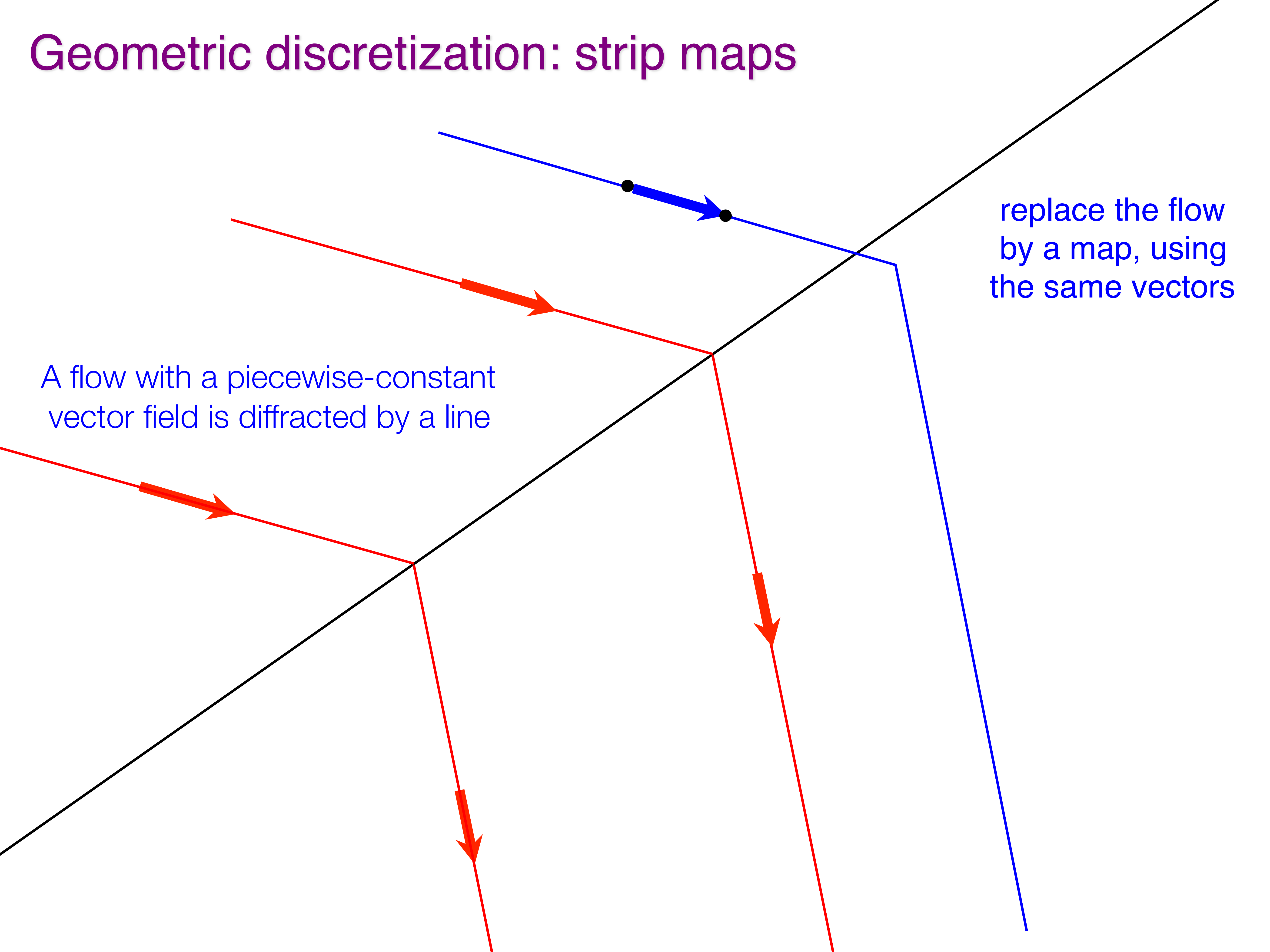

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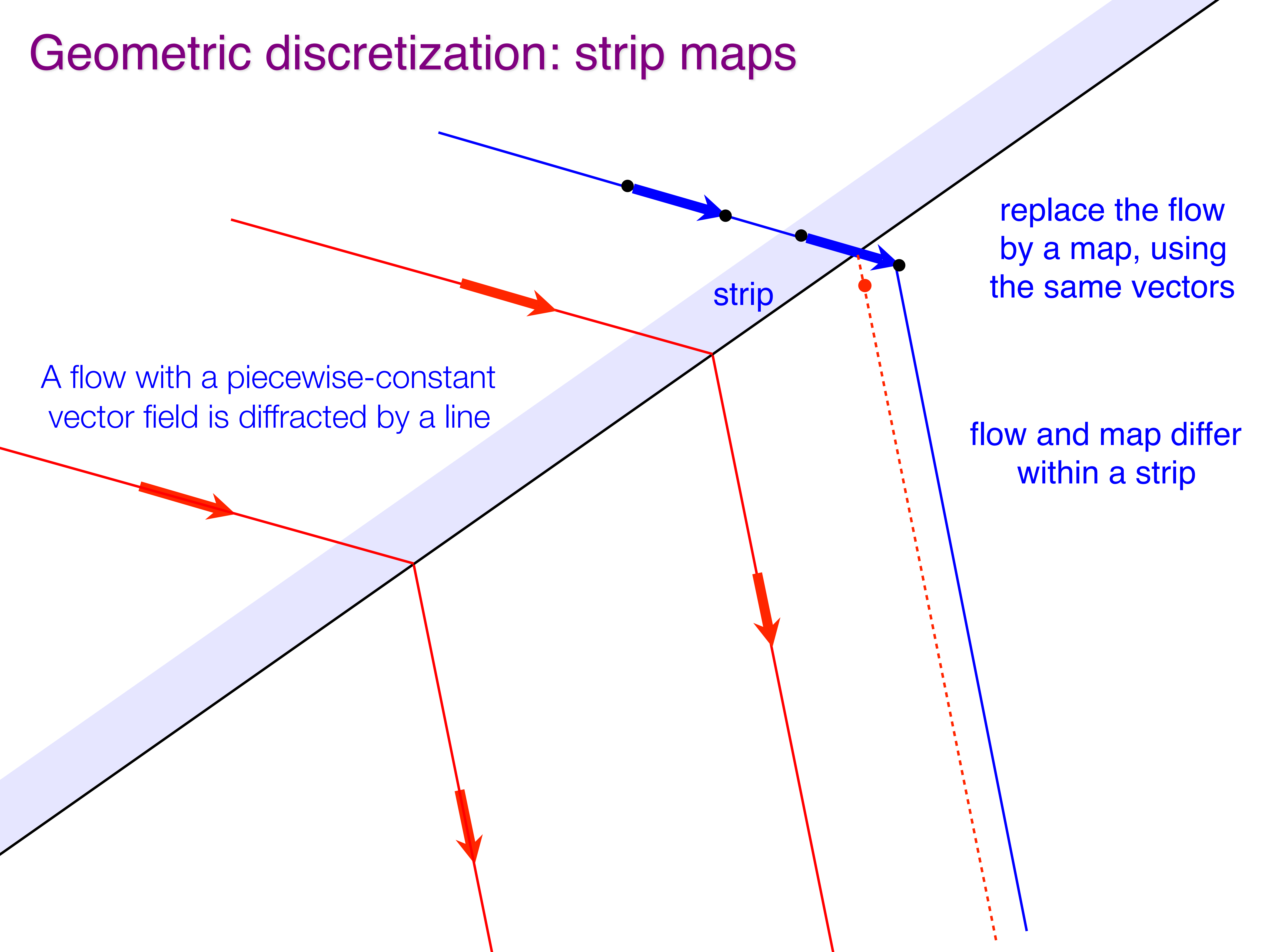

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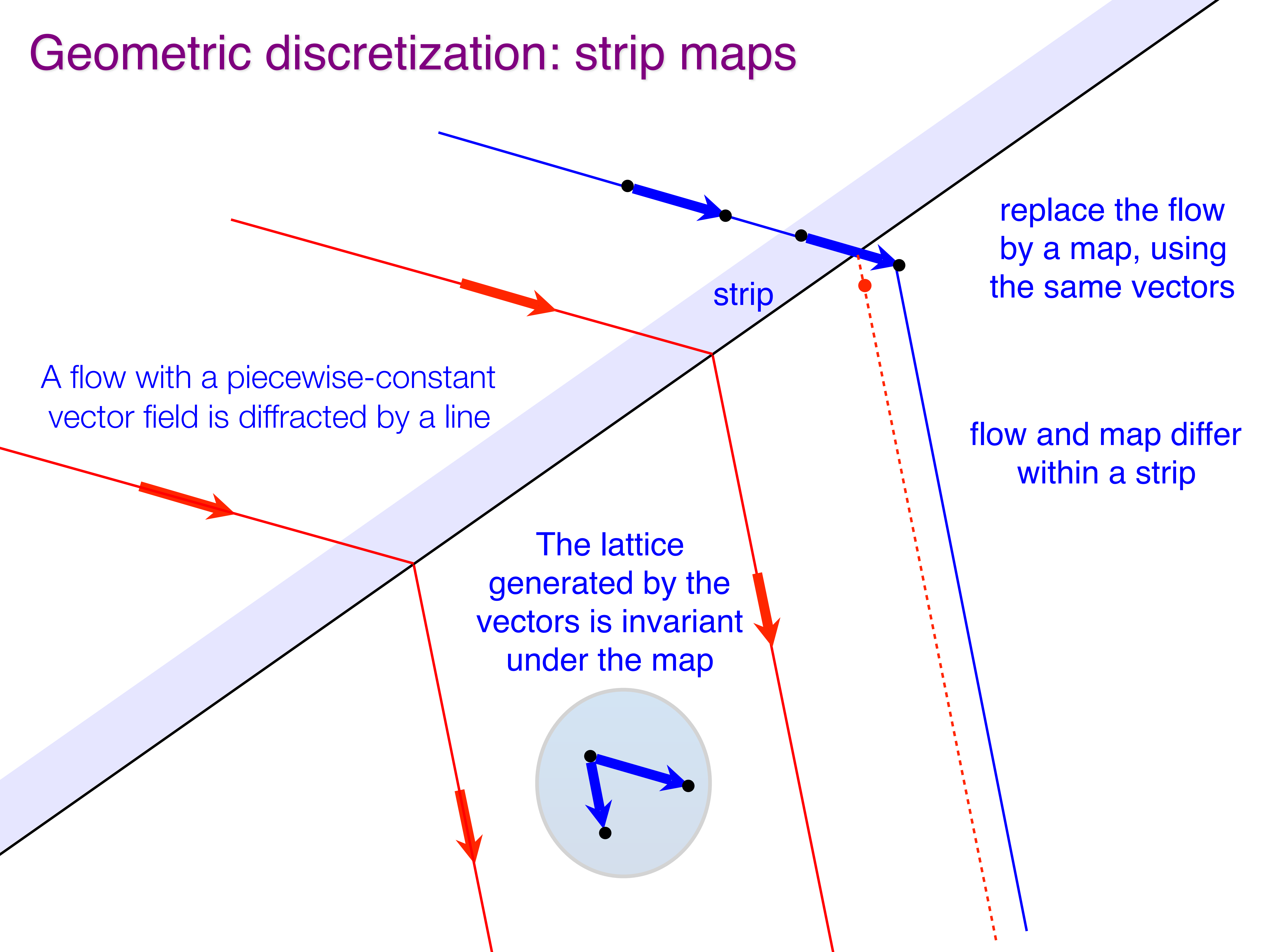
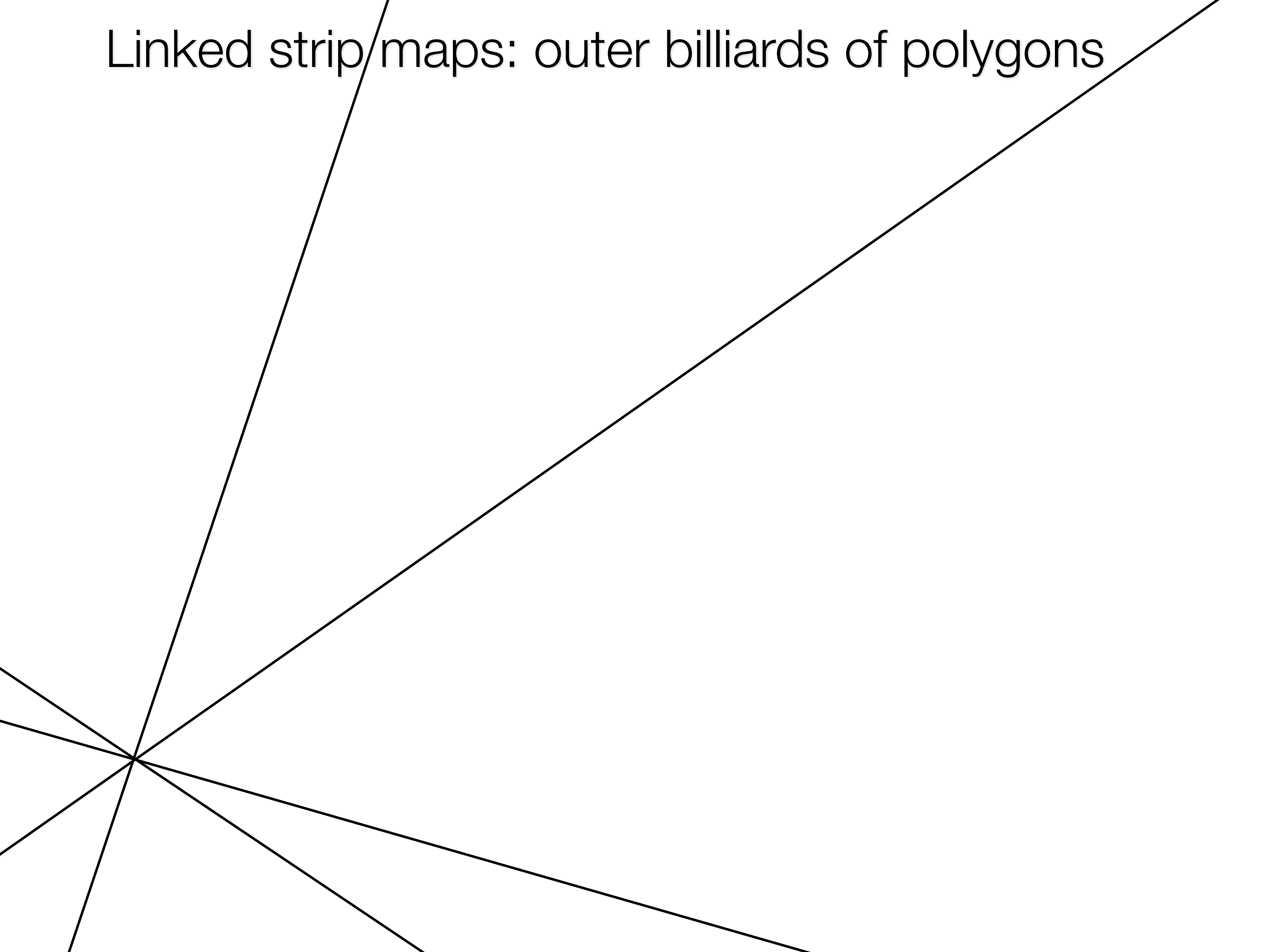


Linked strip maps: outer billiards of polygons



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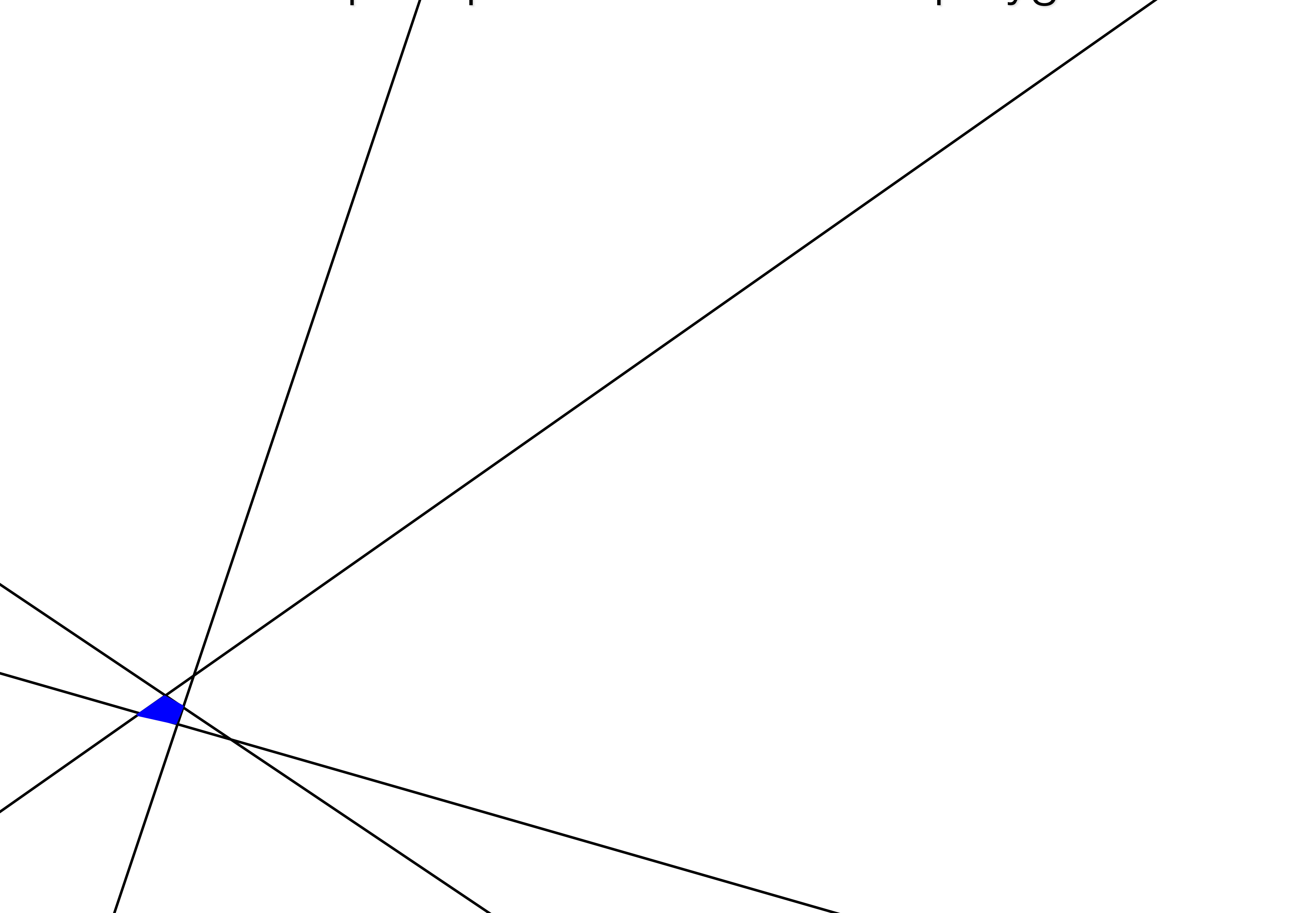

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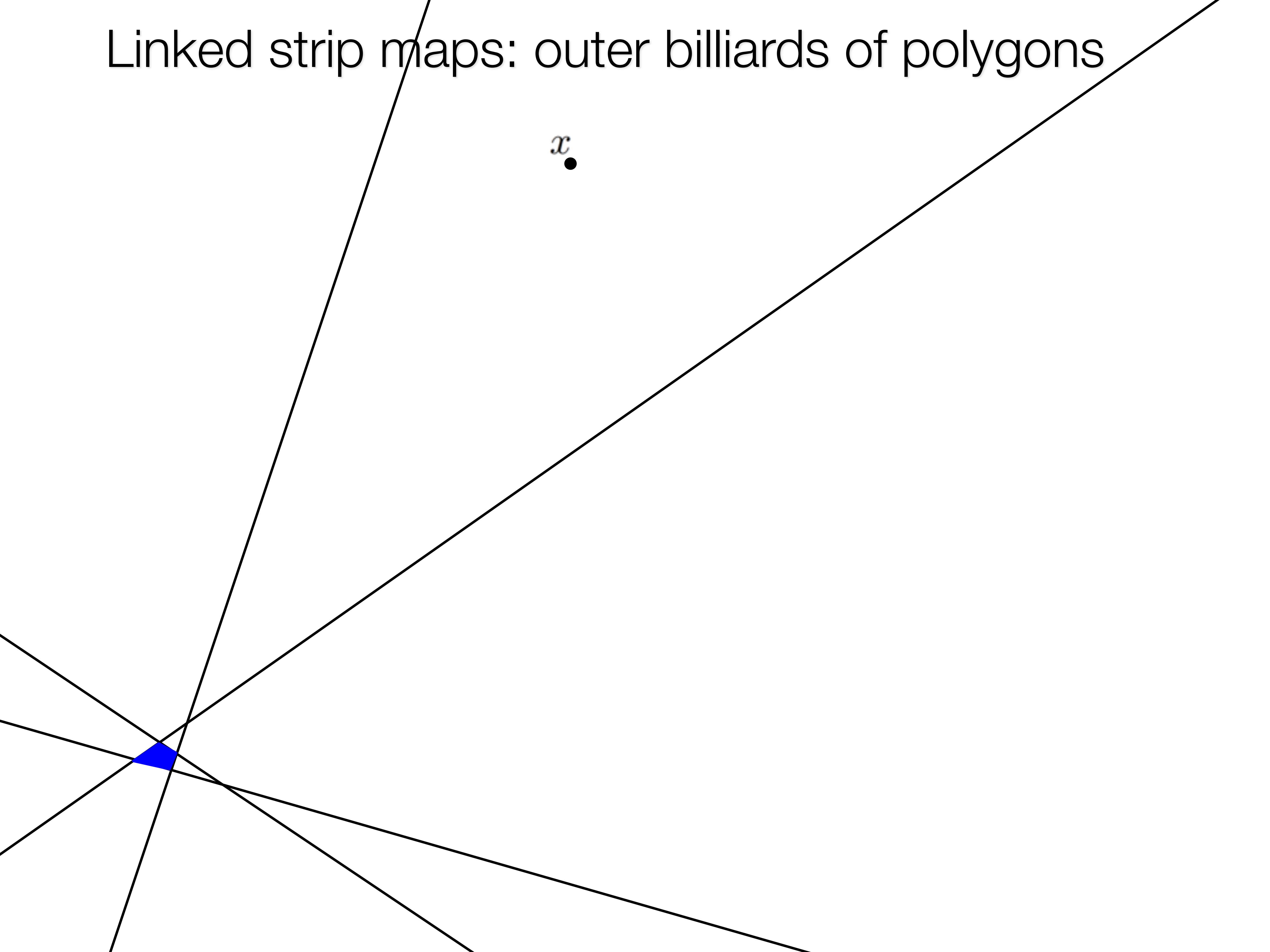

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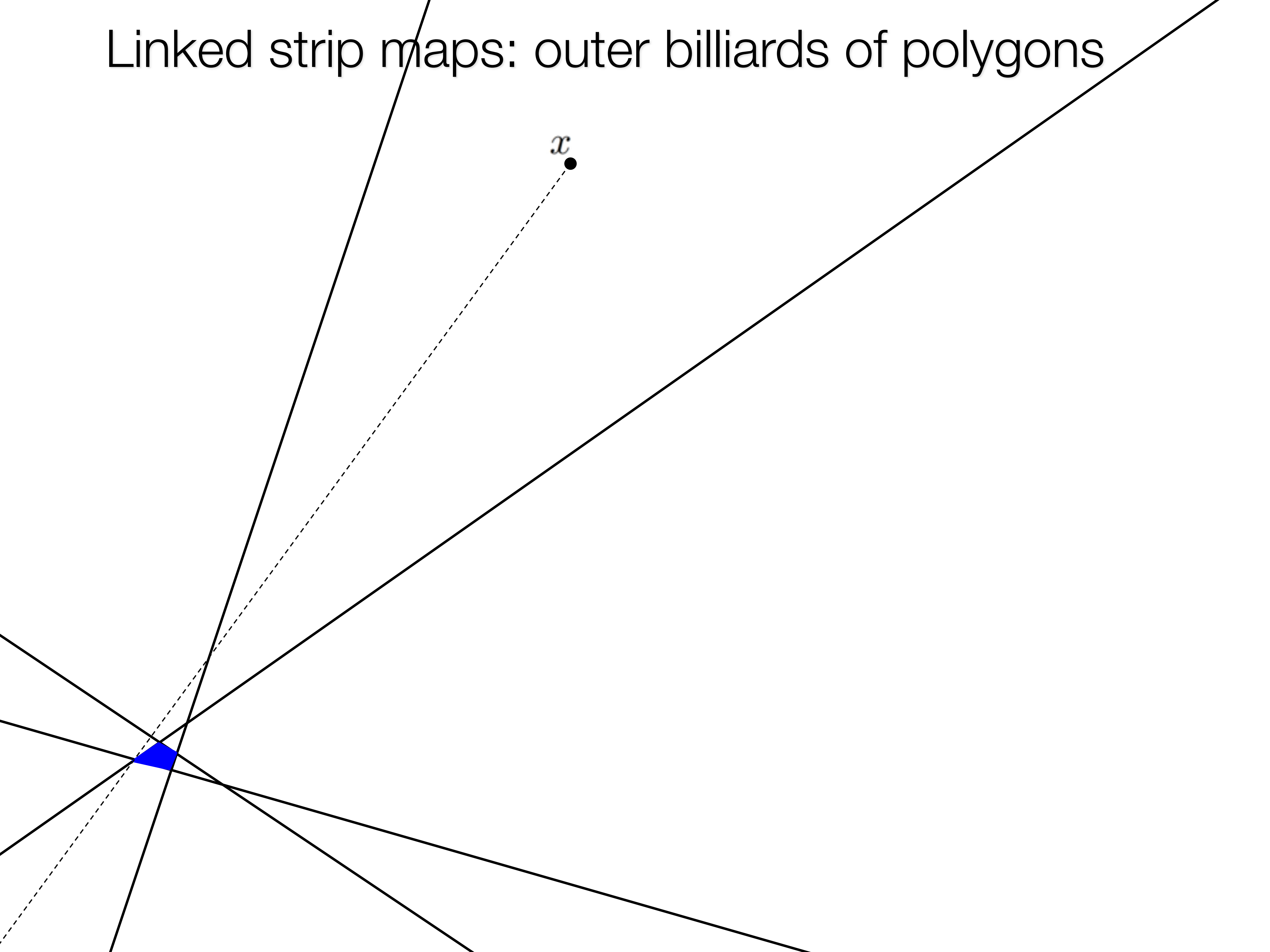

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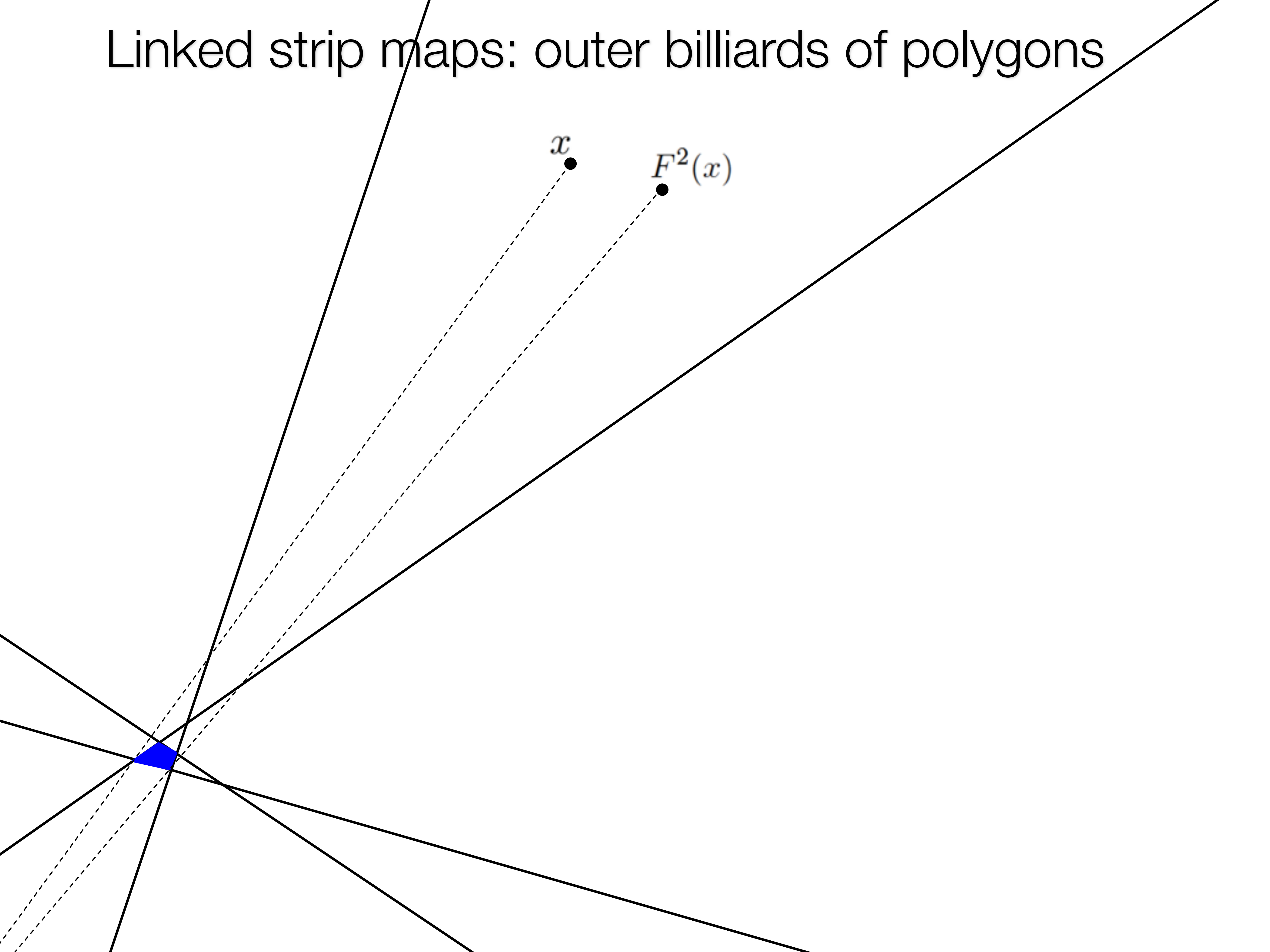

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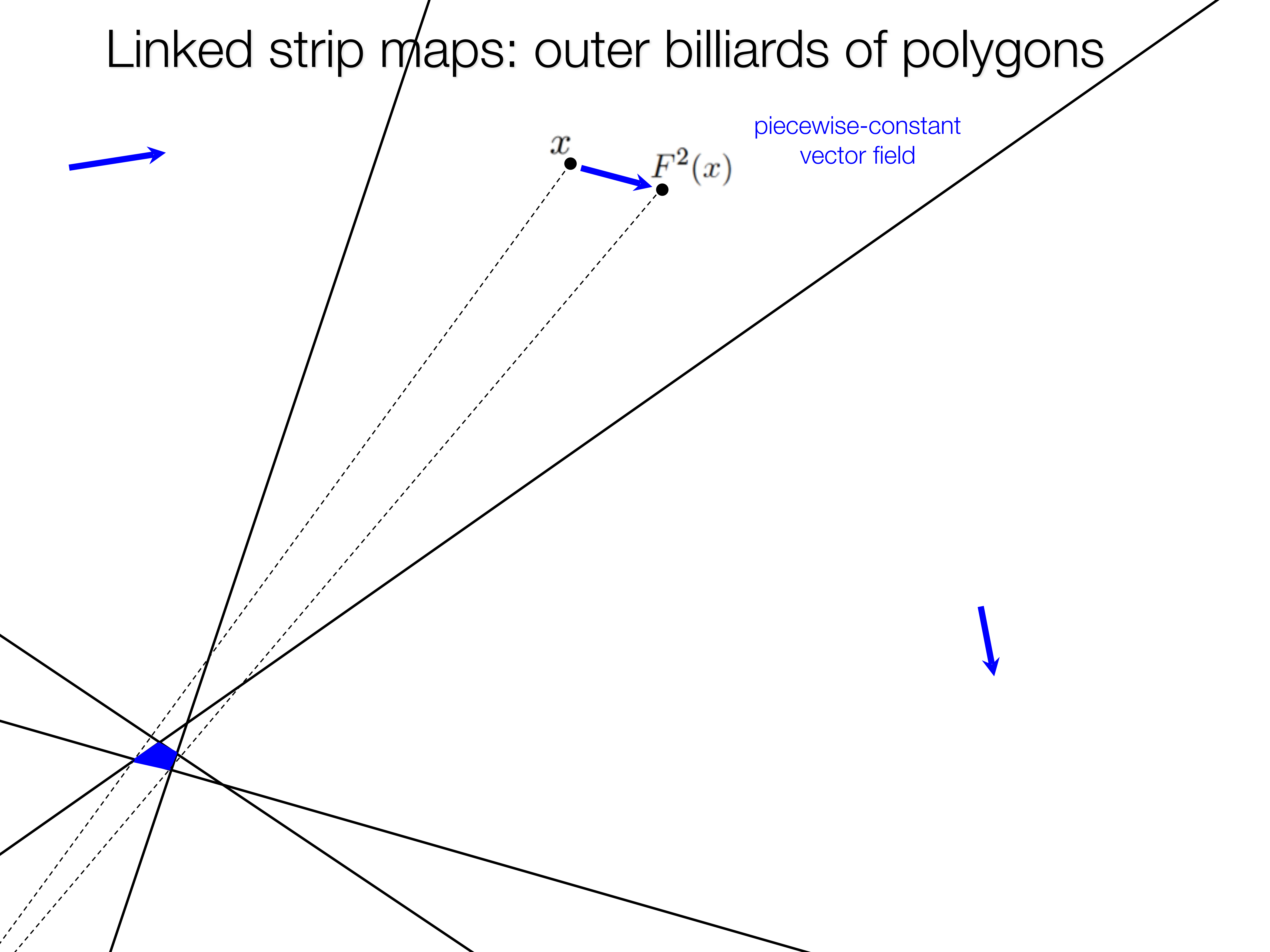

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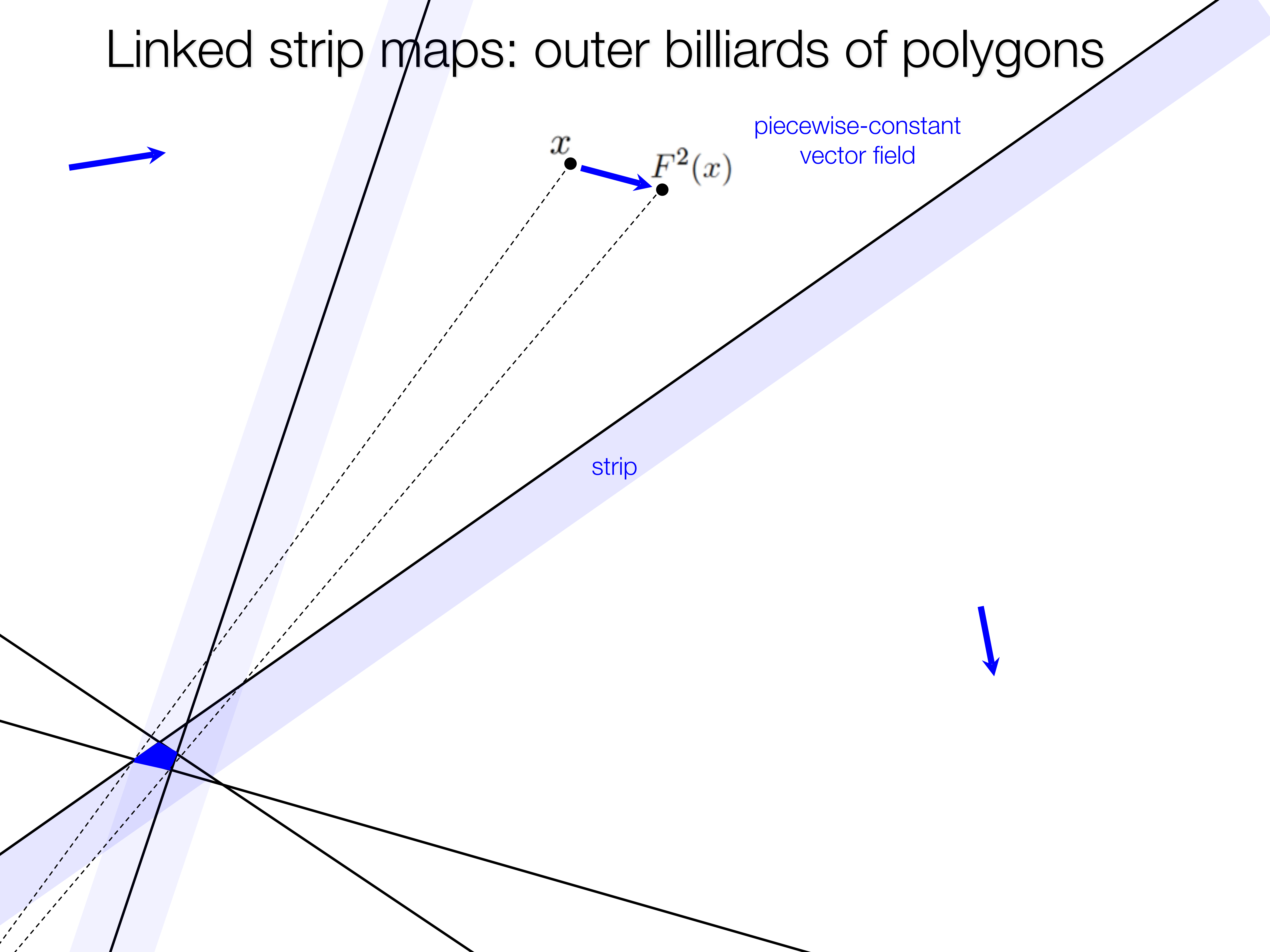

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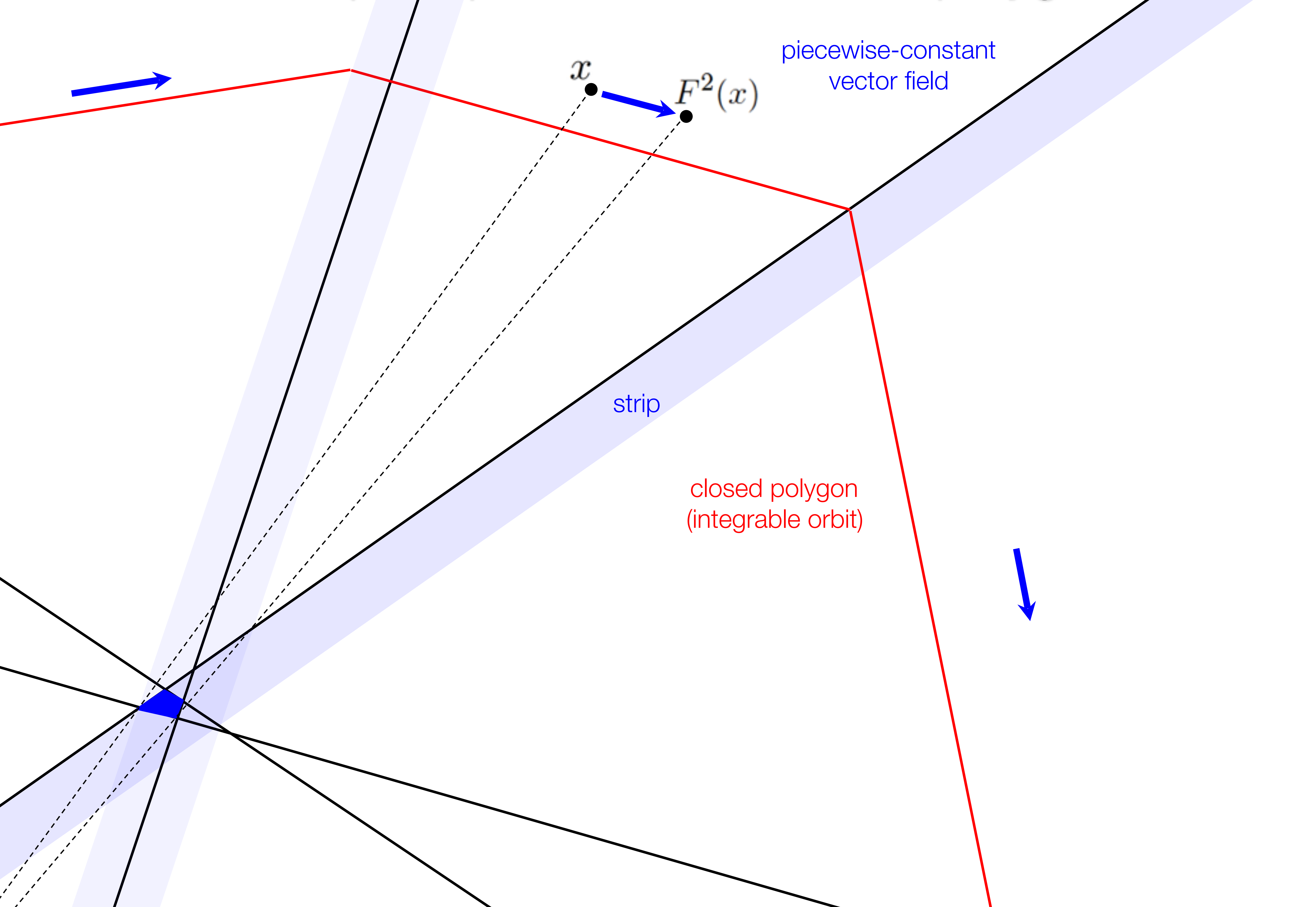

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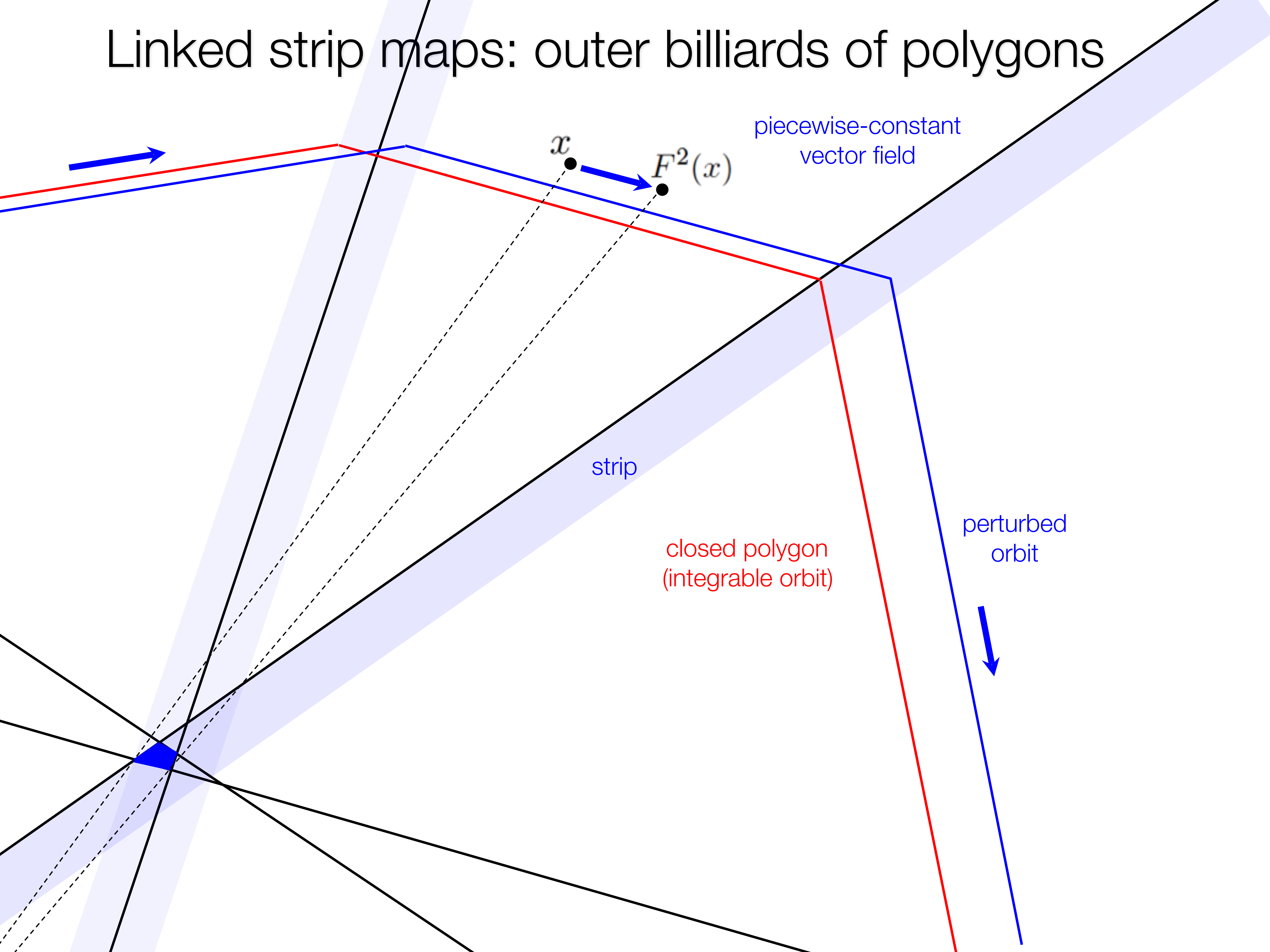
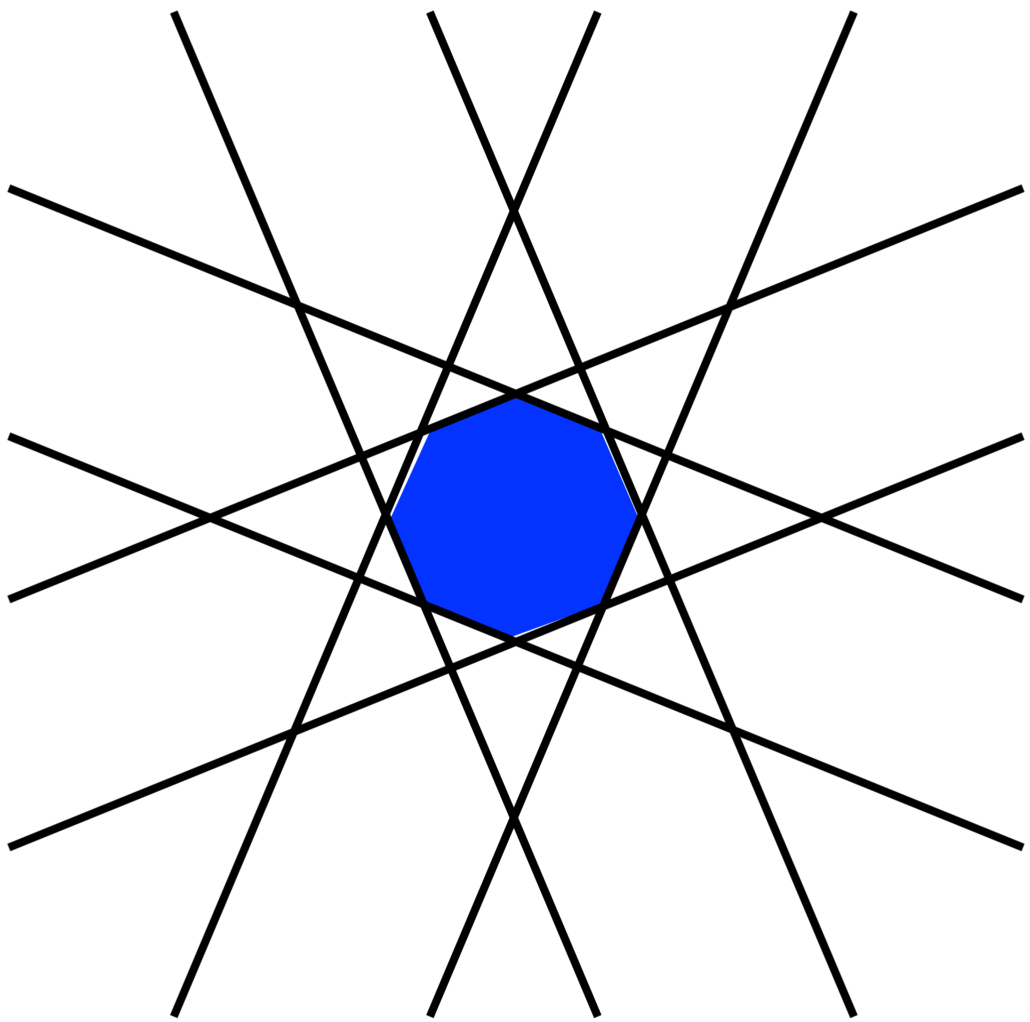
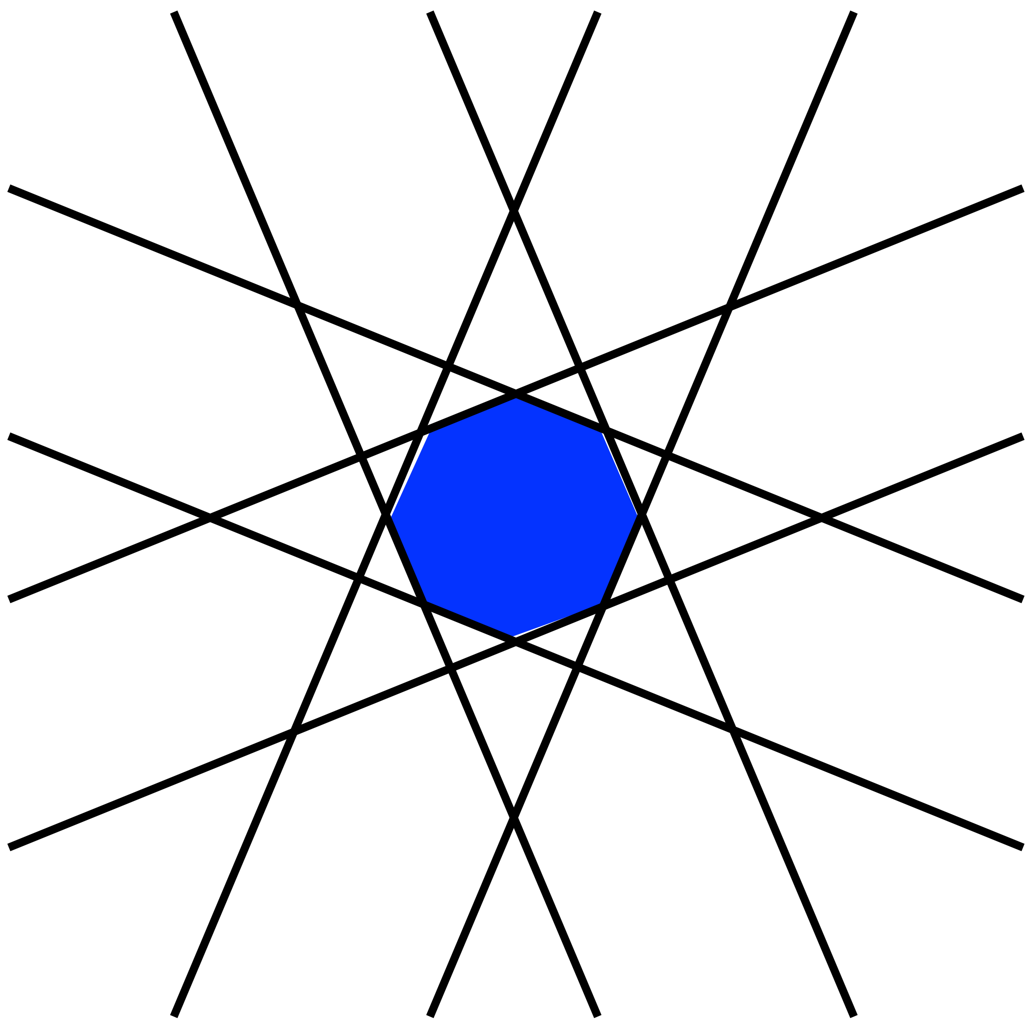




The images of the lines tangent to the polygon constitute the discontinuity set of the system, which partitions the plane into convex polygons.



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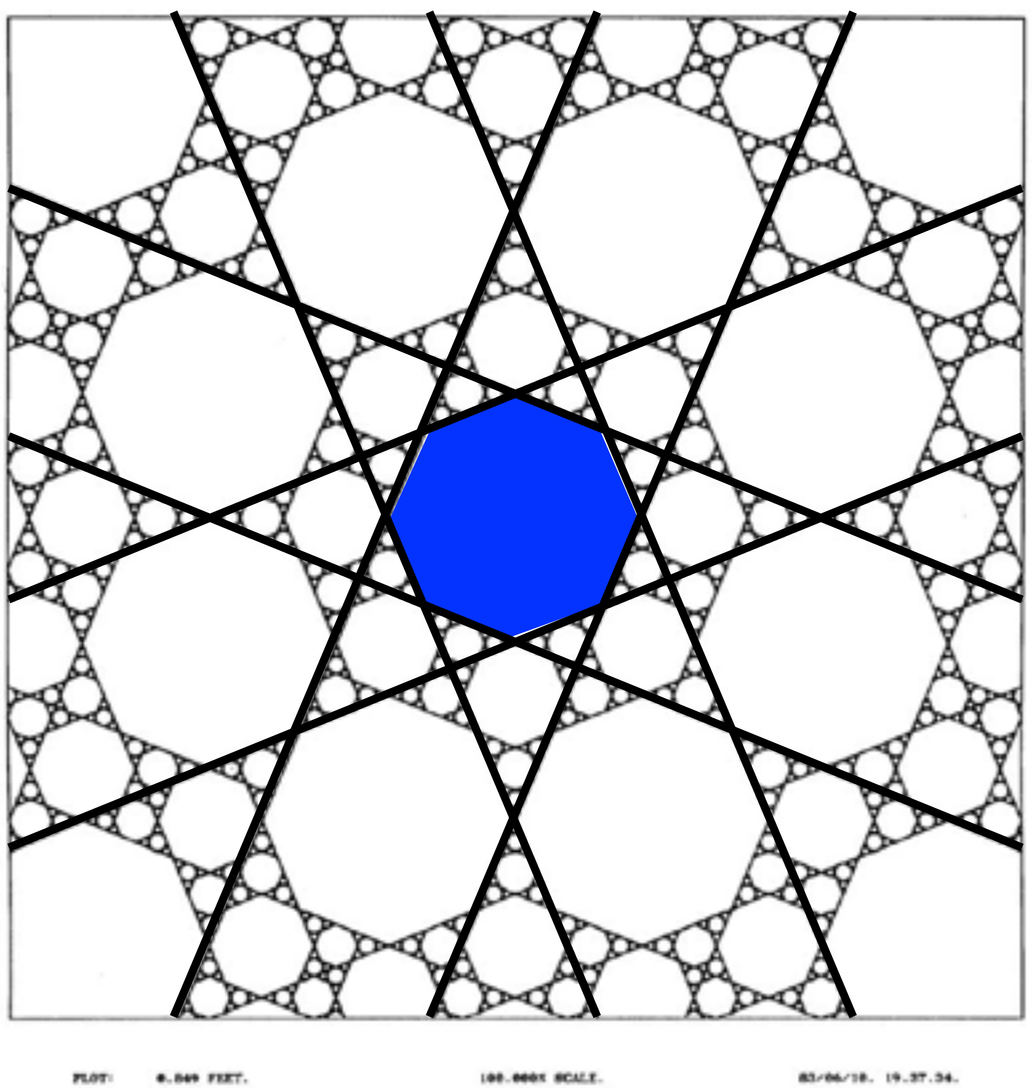

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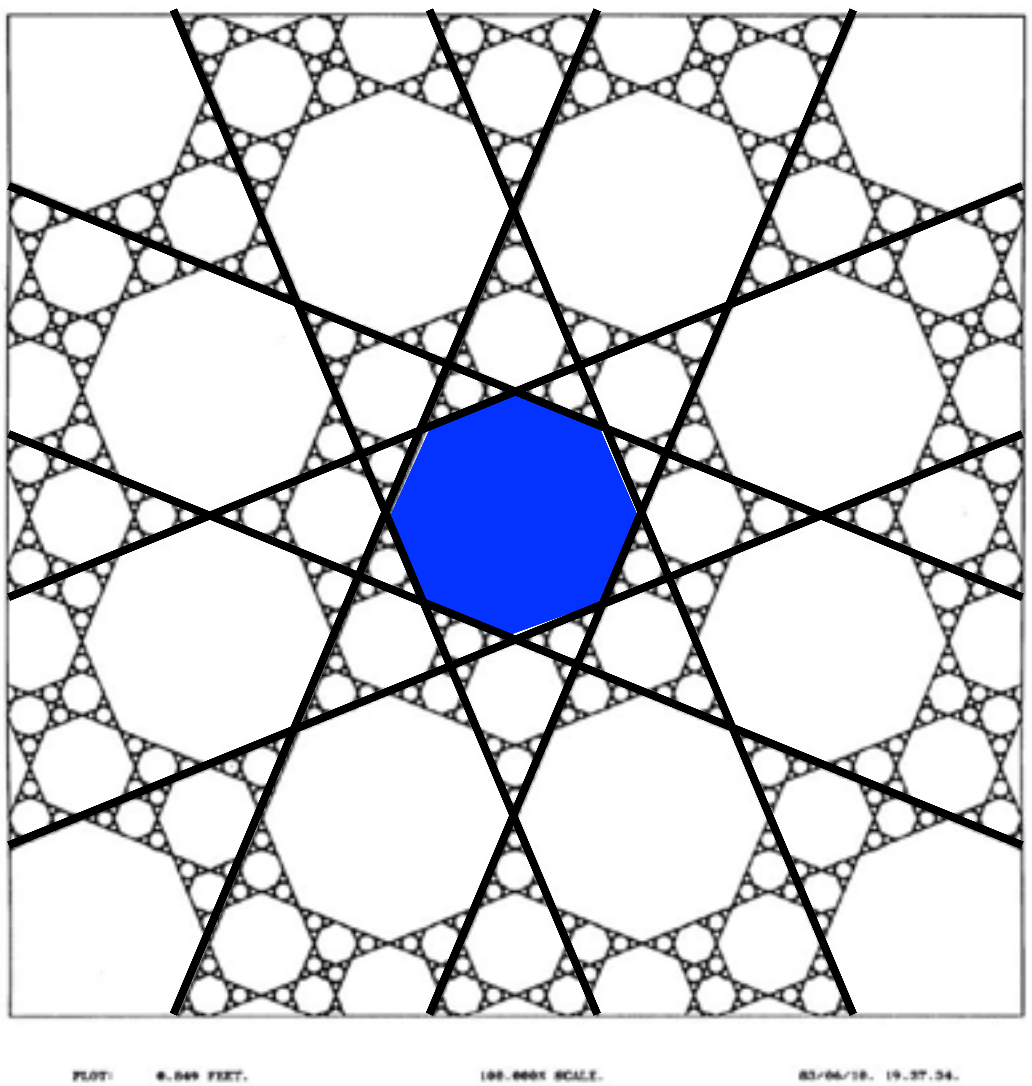

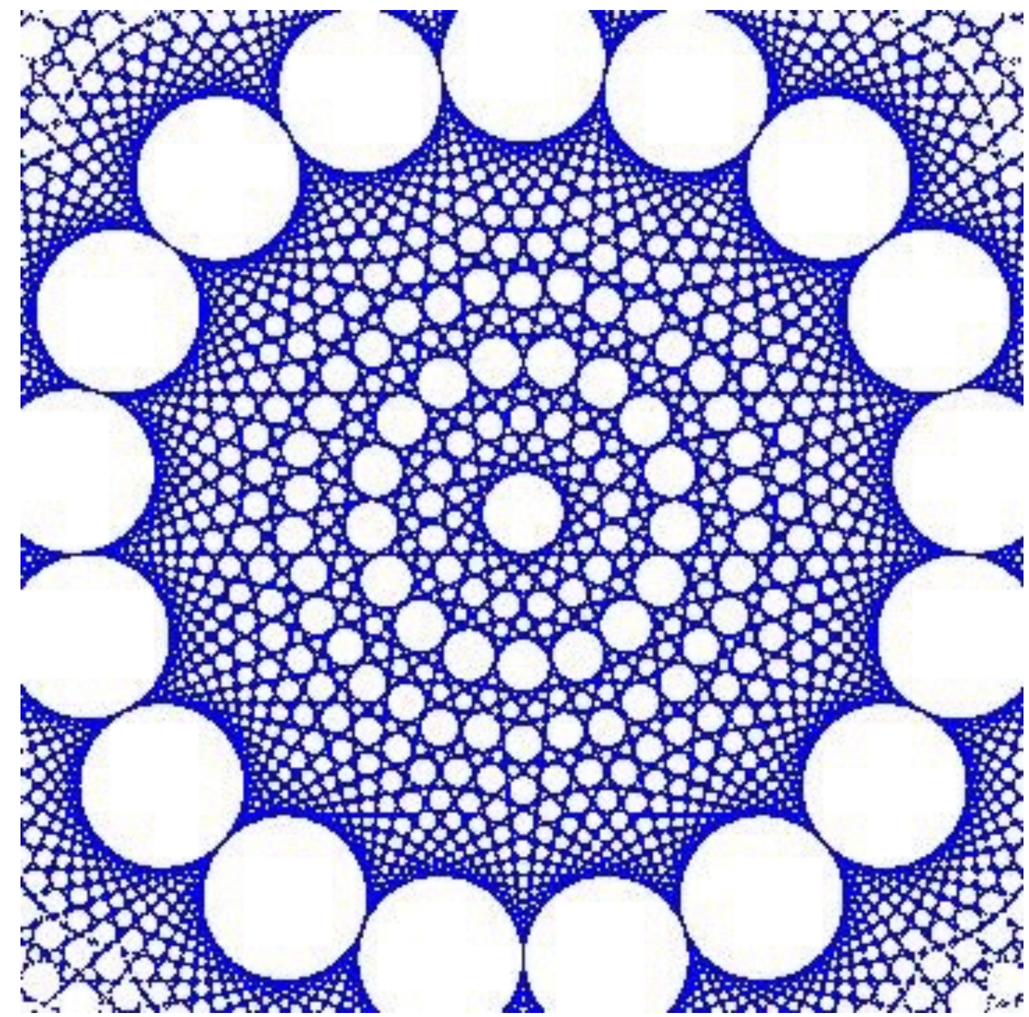

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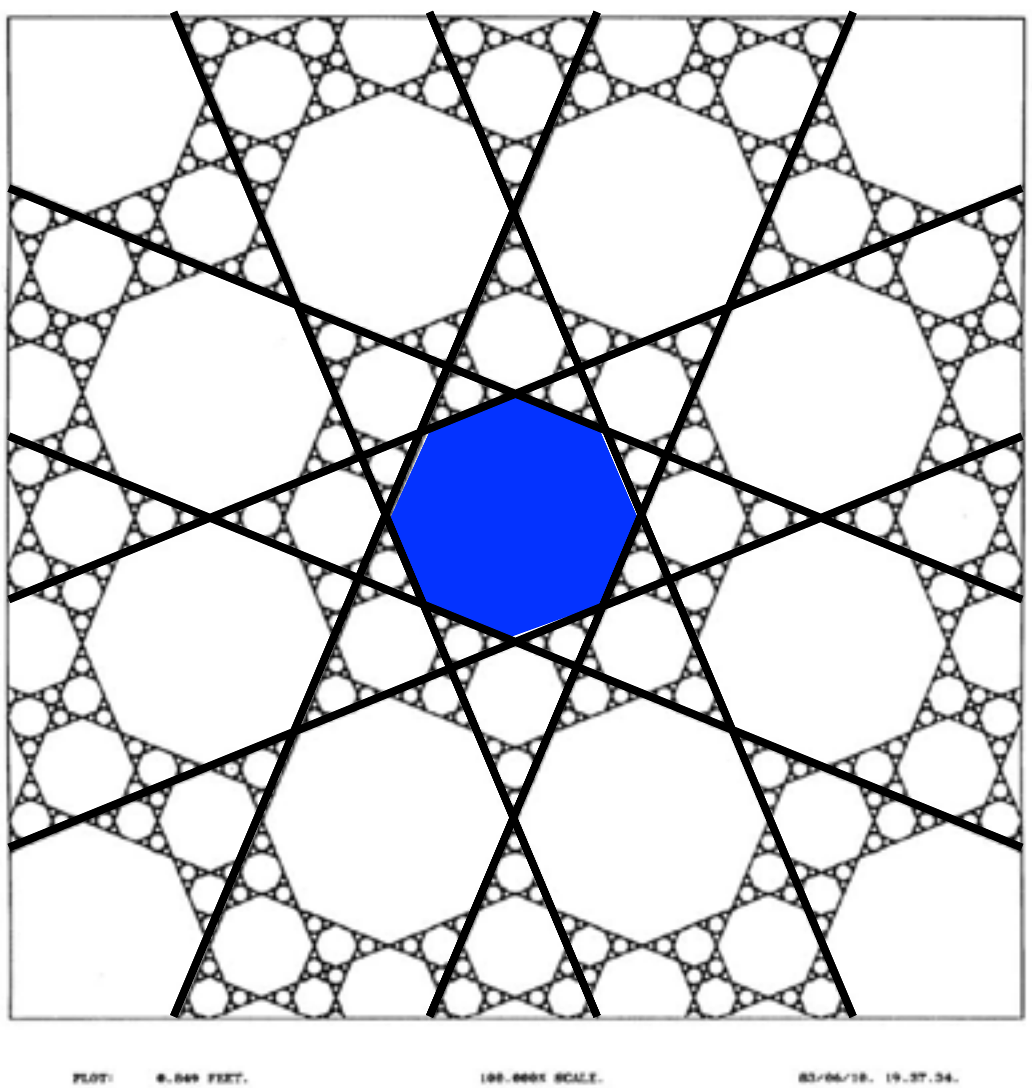

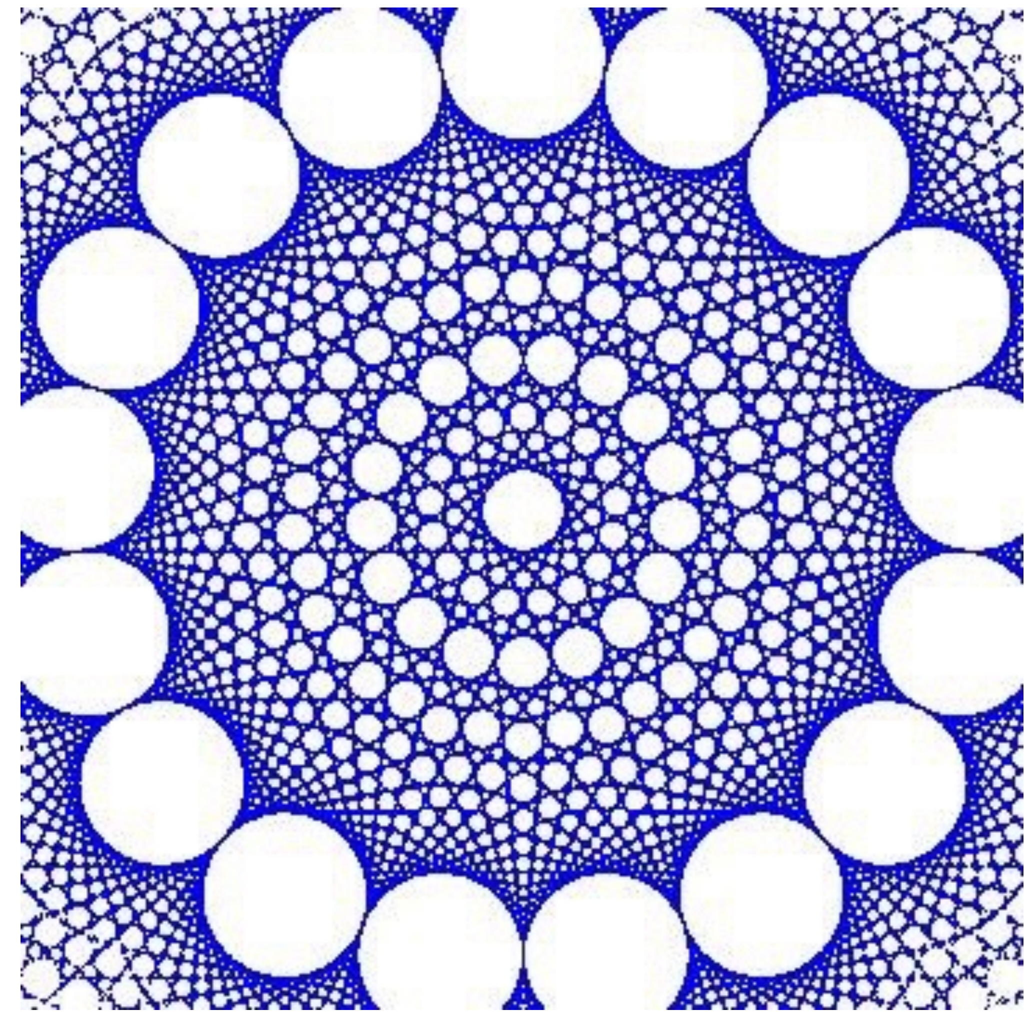

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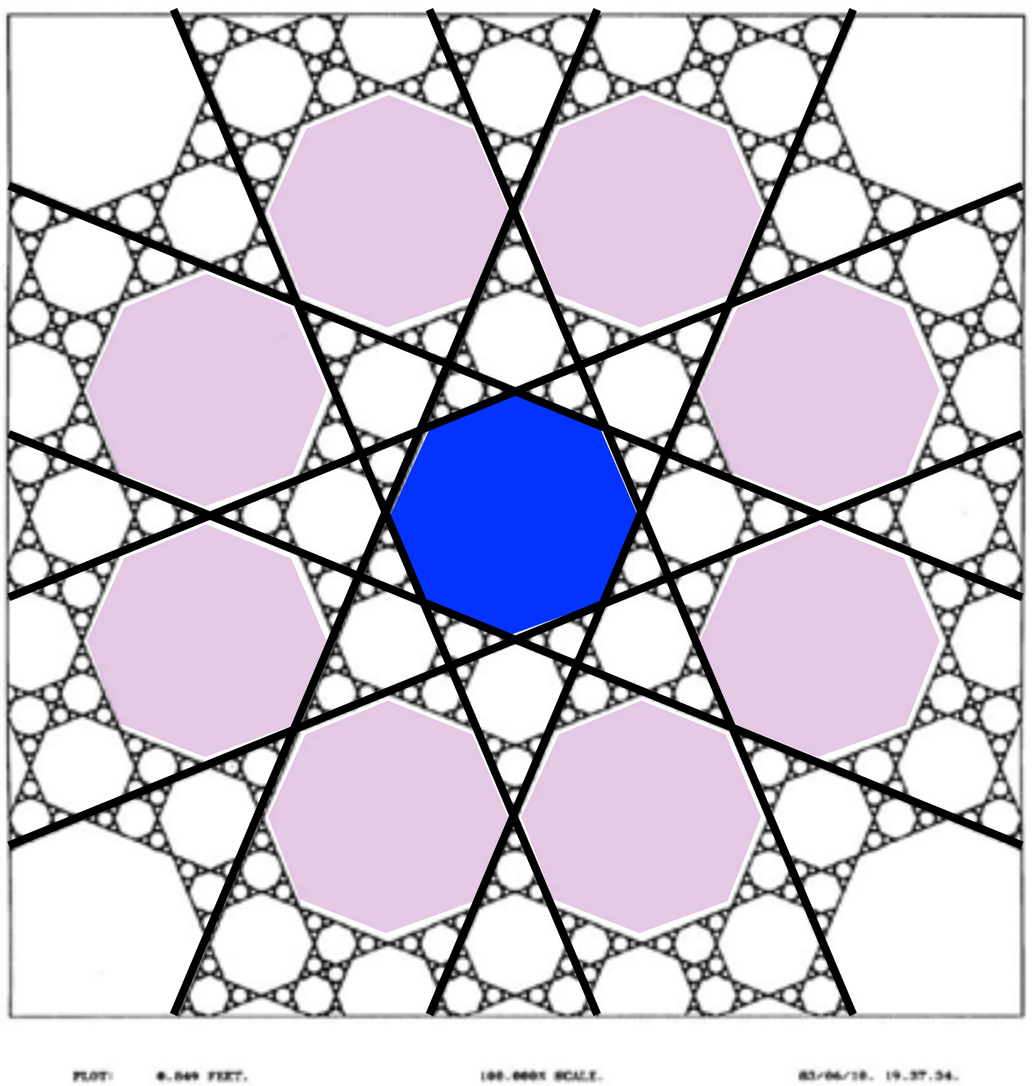

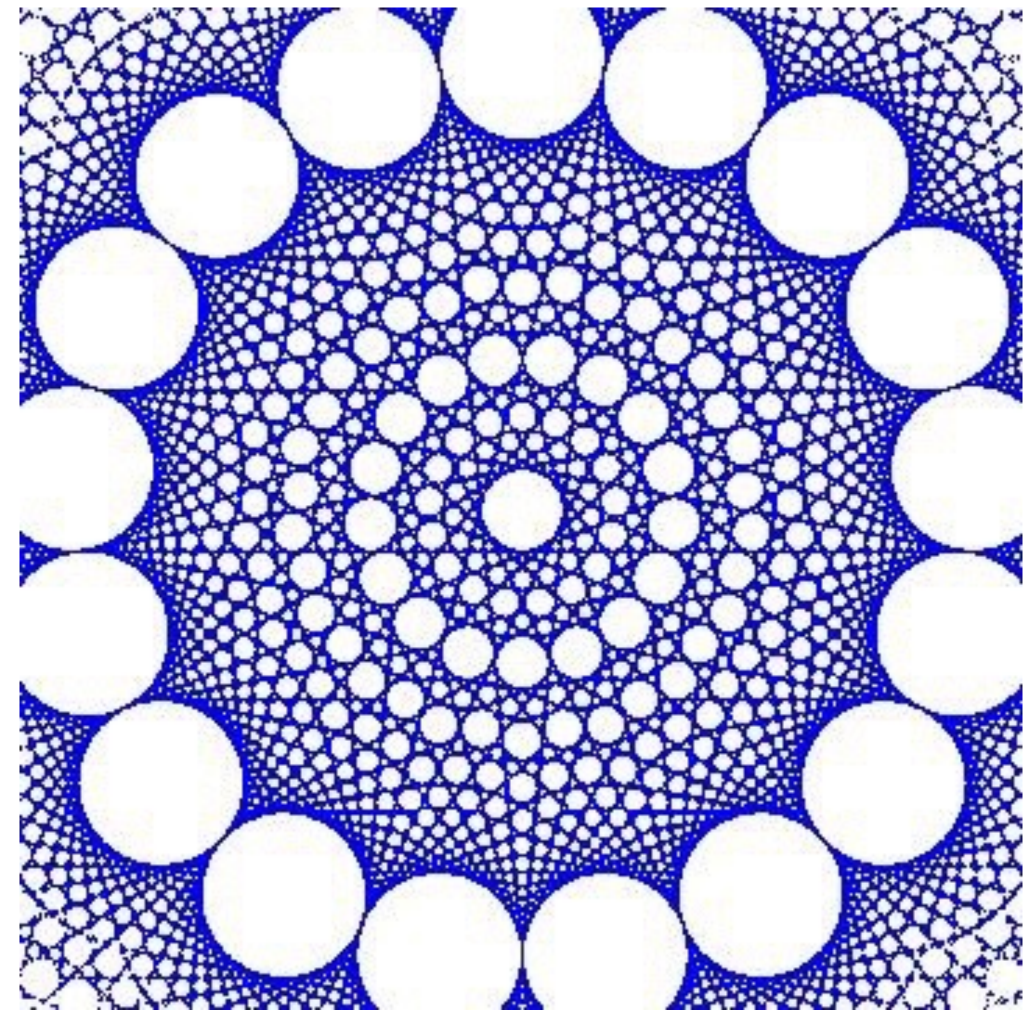

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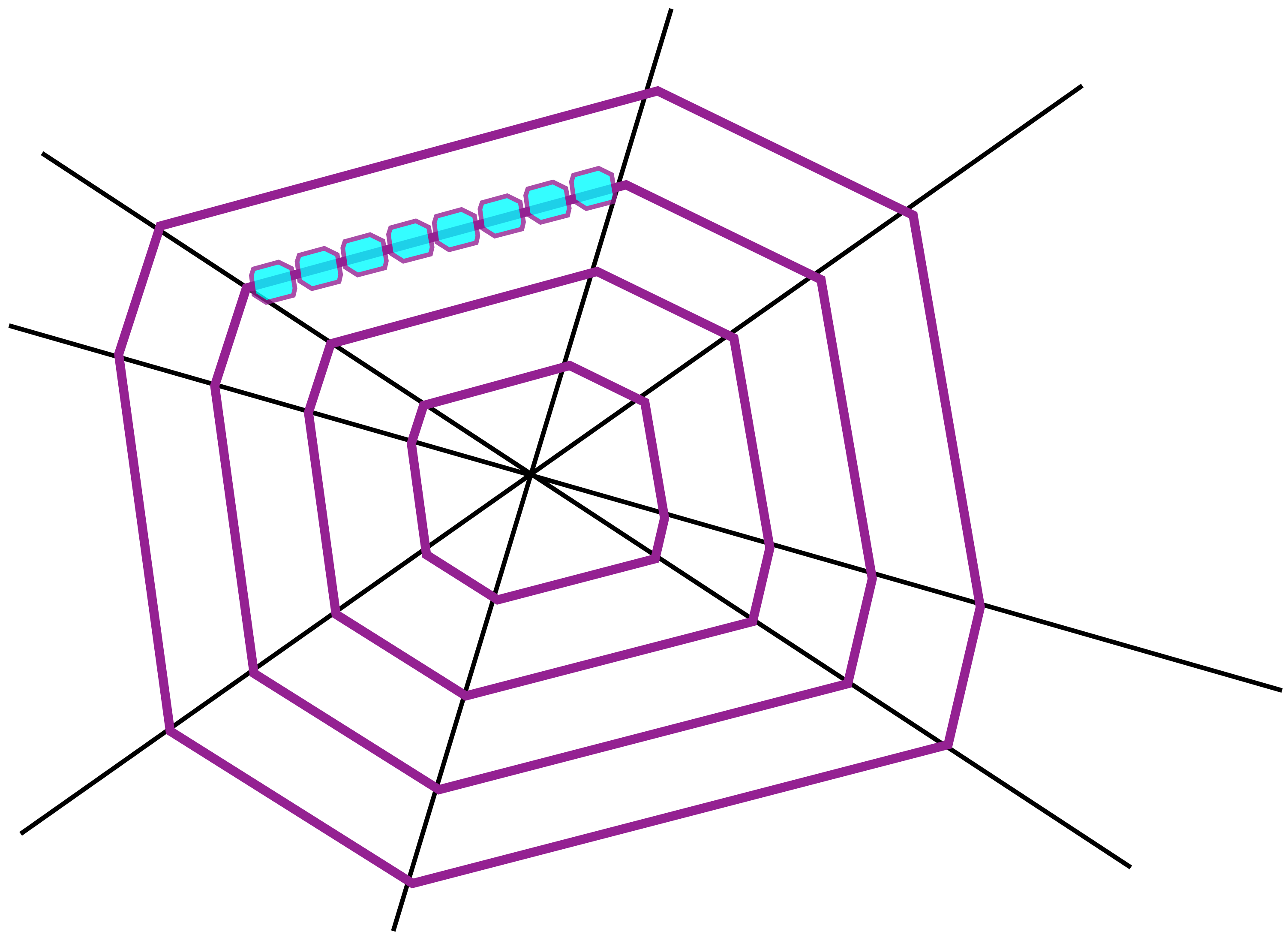

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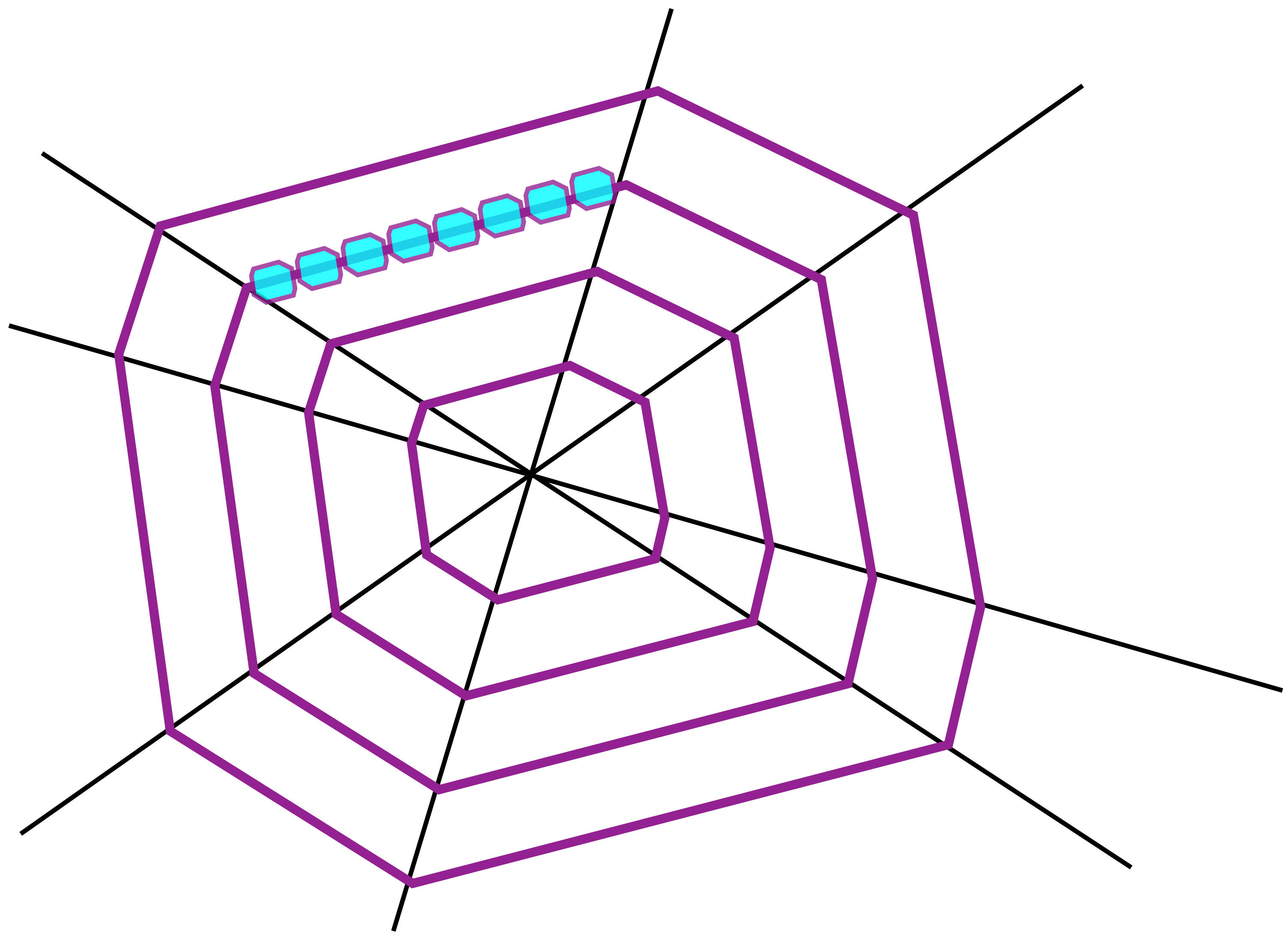

If the rationality conditions are not satisfied, then there are unbounded orbits (Schwartz, 2007).

A discrete twist map of the torus

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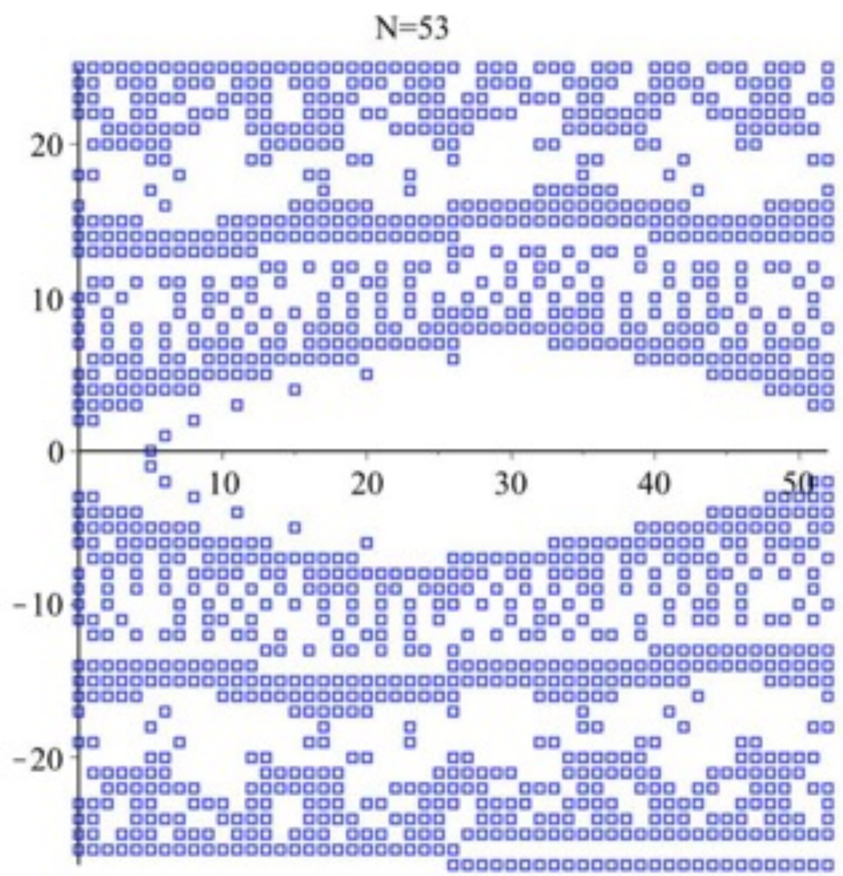

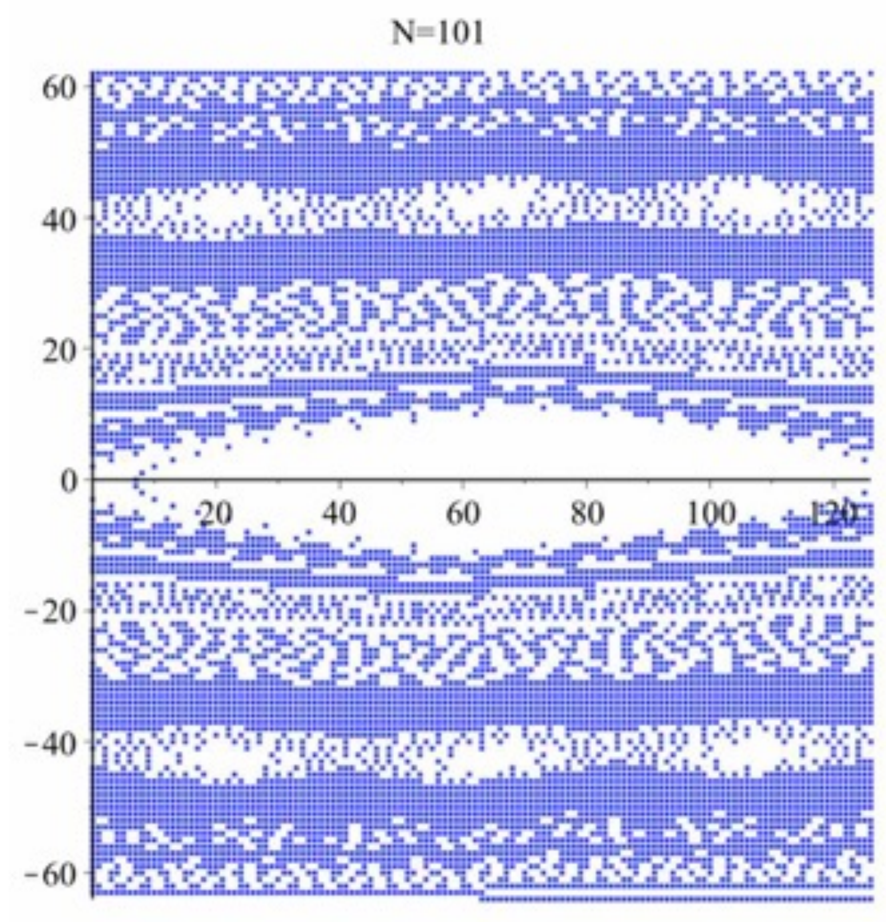

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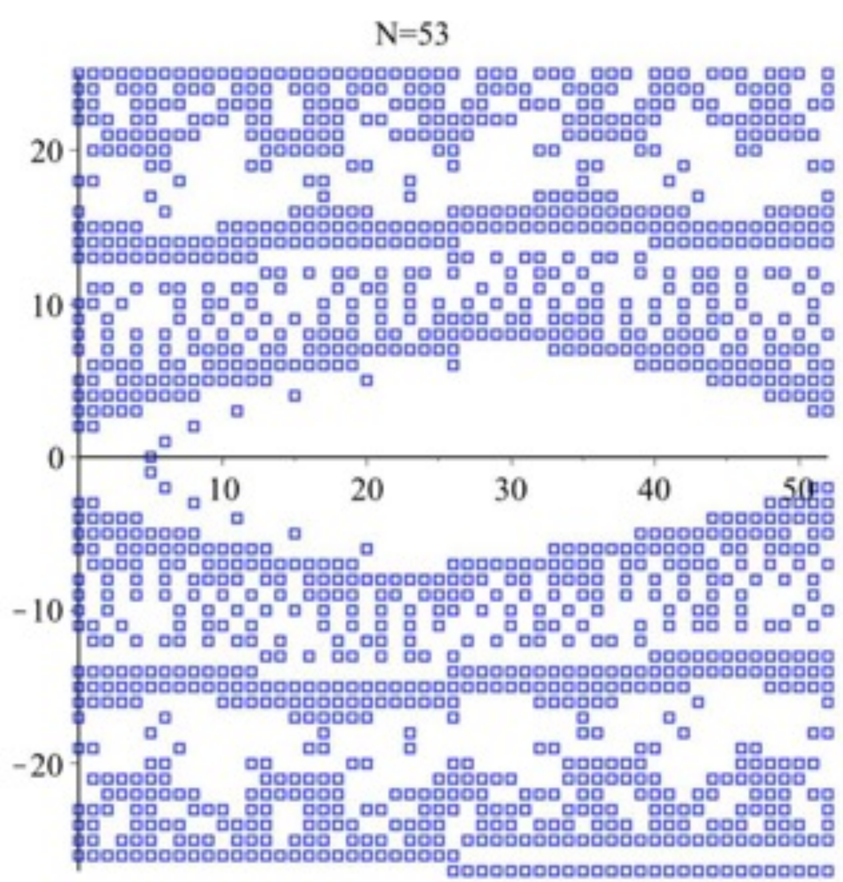

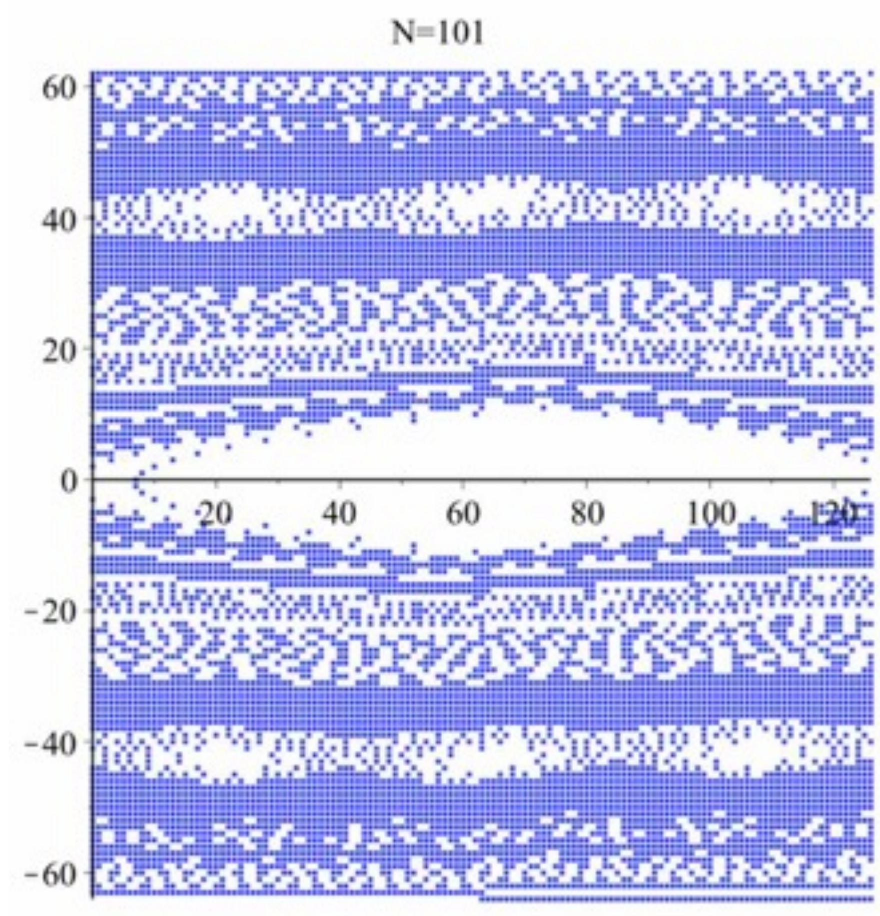

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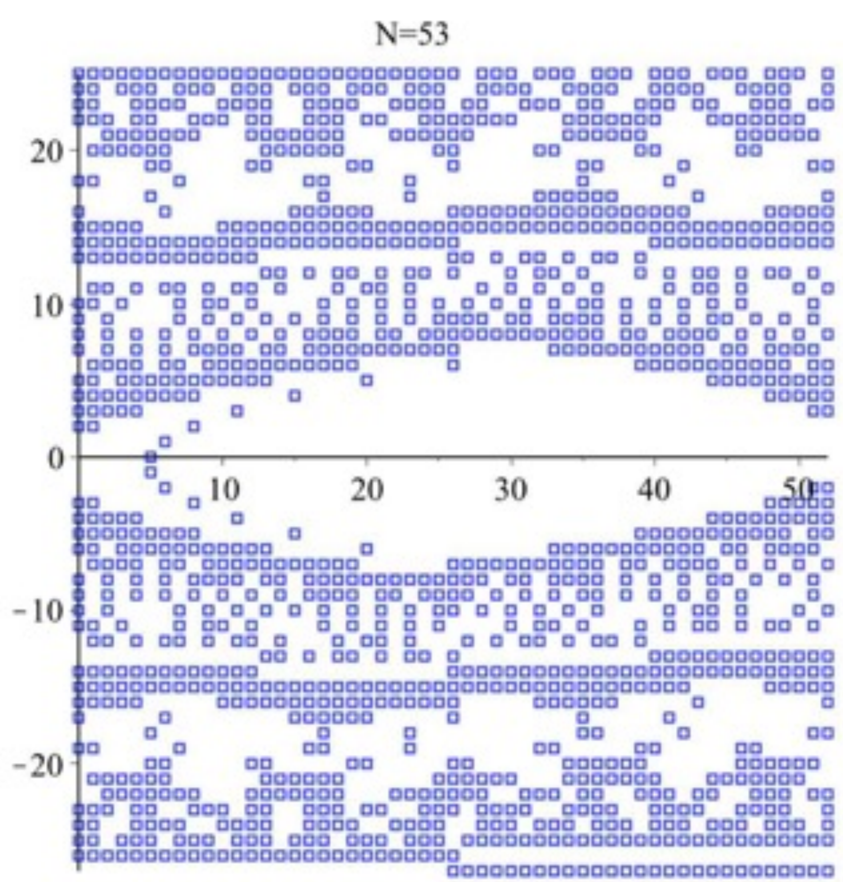

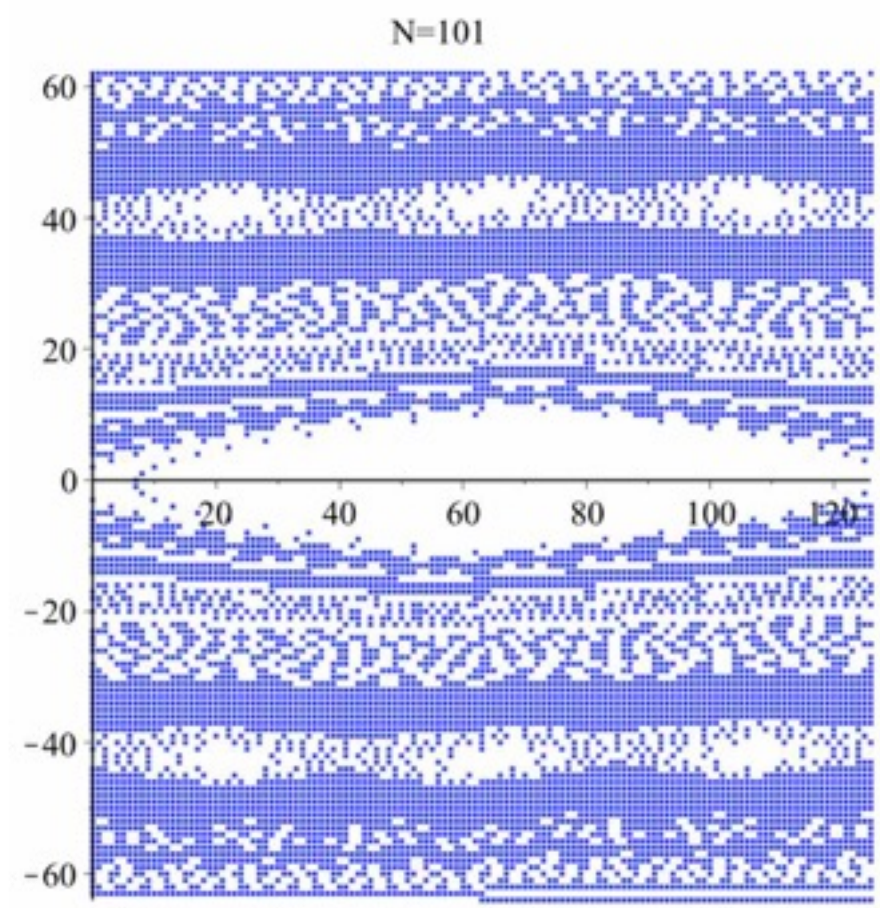

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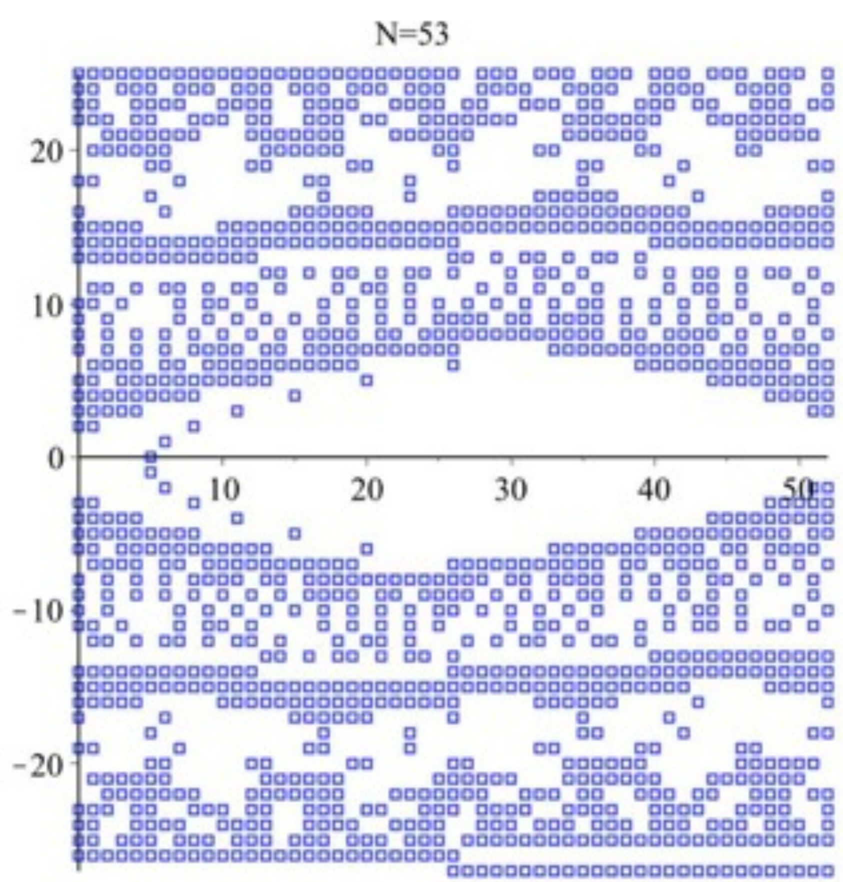

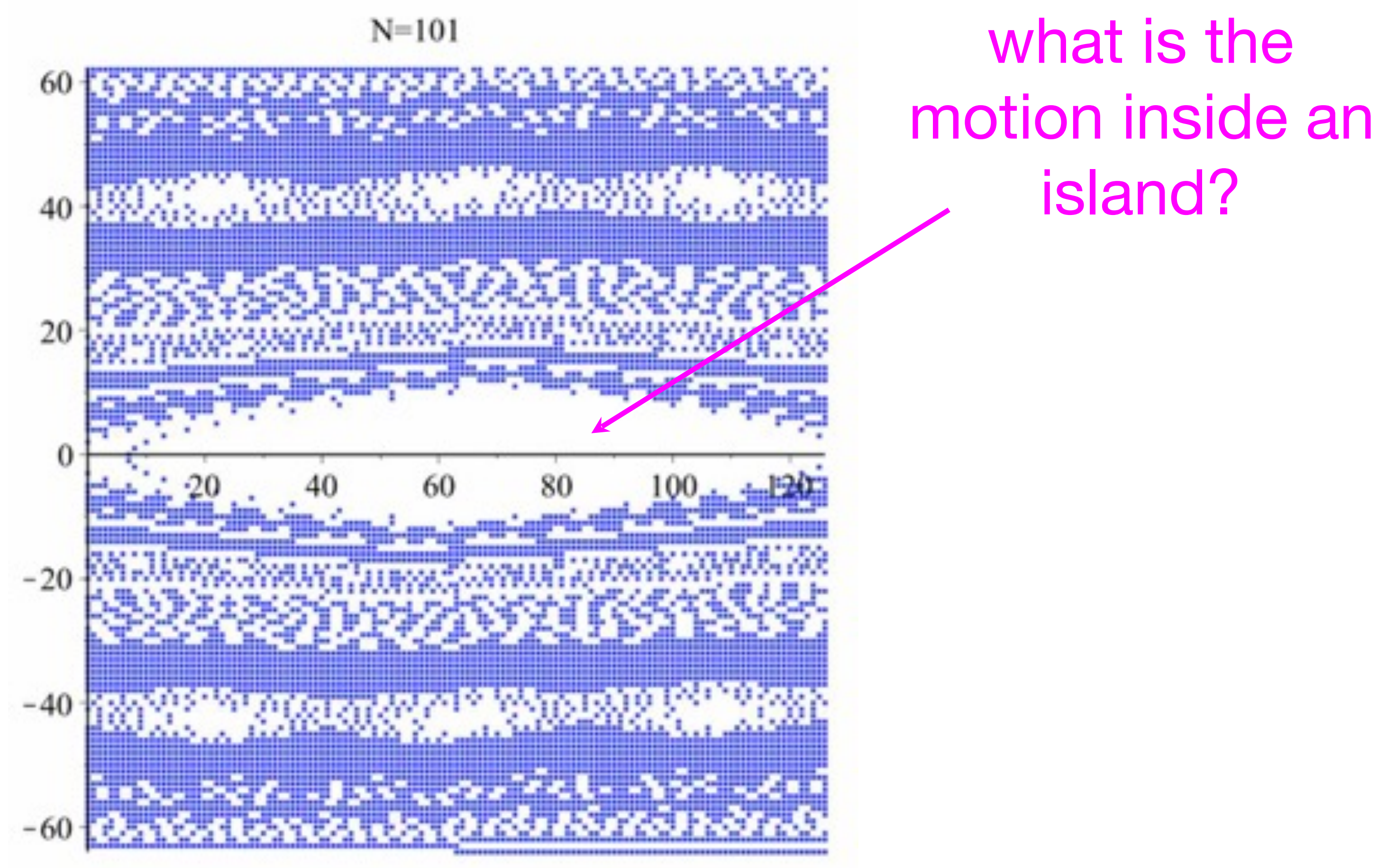

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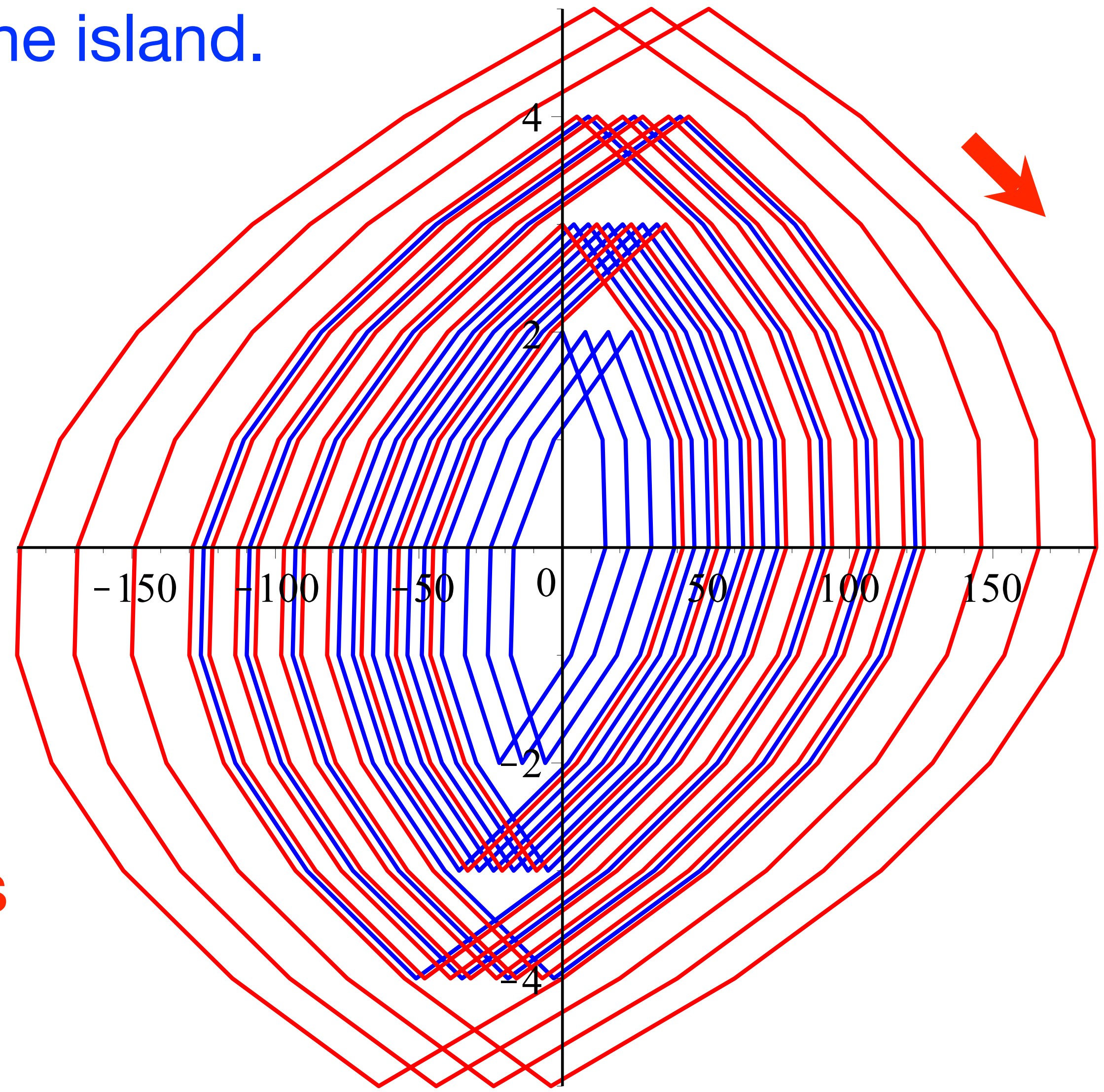
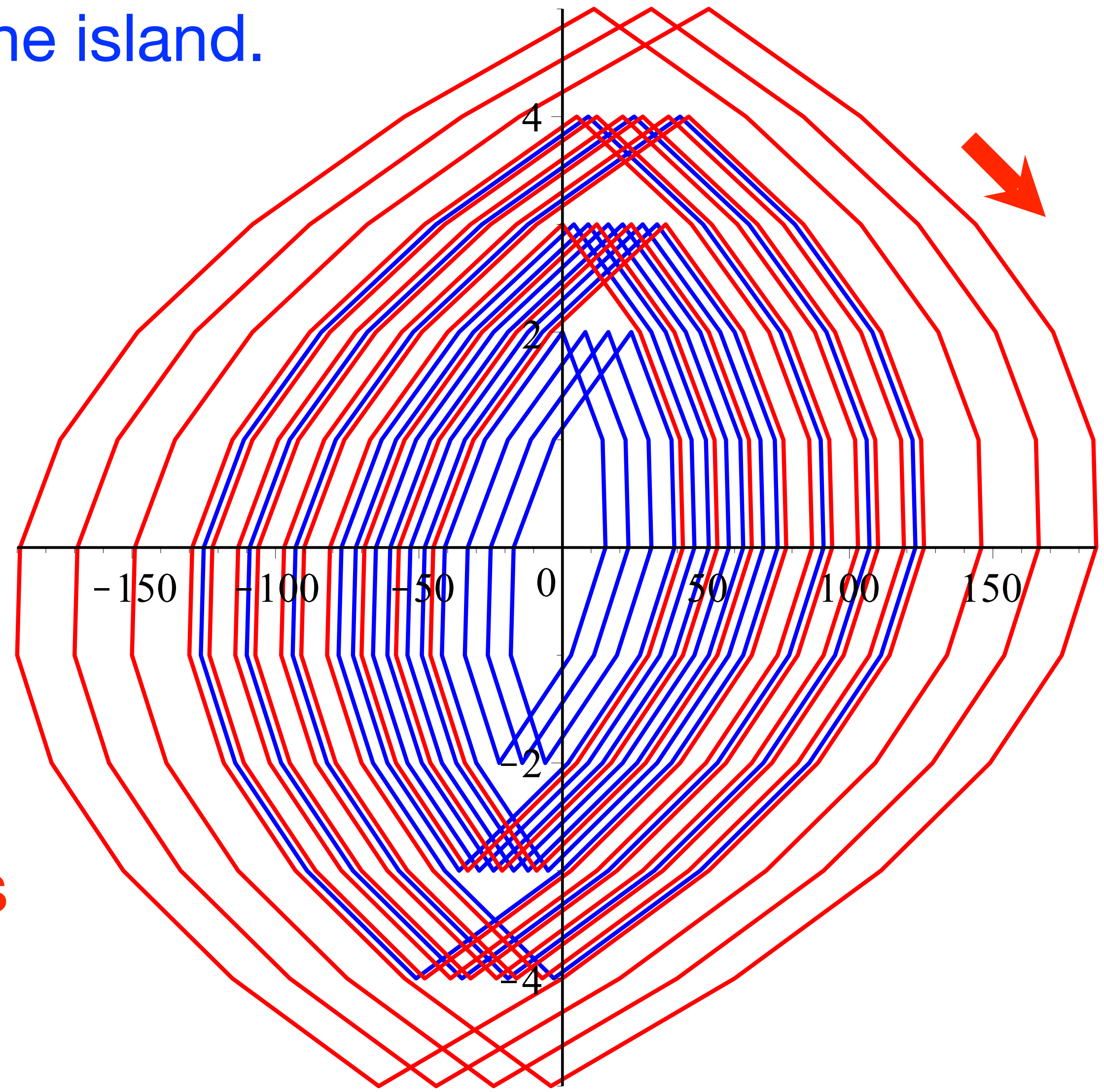


Overlapping orbits

The local map for an island

$$\mathscr{F} : \mathbb{Z}^2 \to \mathbb{Z}^2 \qquad \begin{aligned} y_{t+1} &= y_t - \operatorname{sign}(x_t) \\ x_{t+1} &= x_t + \alpha y_{t+1} + \beta \end{aligned} \qquad 0 \leqslant \beta < \alpha$$

The parameters α and β depend on the rotation number of the island.



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Conjecture (Zhang, fv). If $\gcd(\alpha, 2\beta)$ is odd, then all orbits are periodic, and their period, sufficiently far from the origin, eventually becomes constant. Otherwise all orbits are unbounded.

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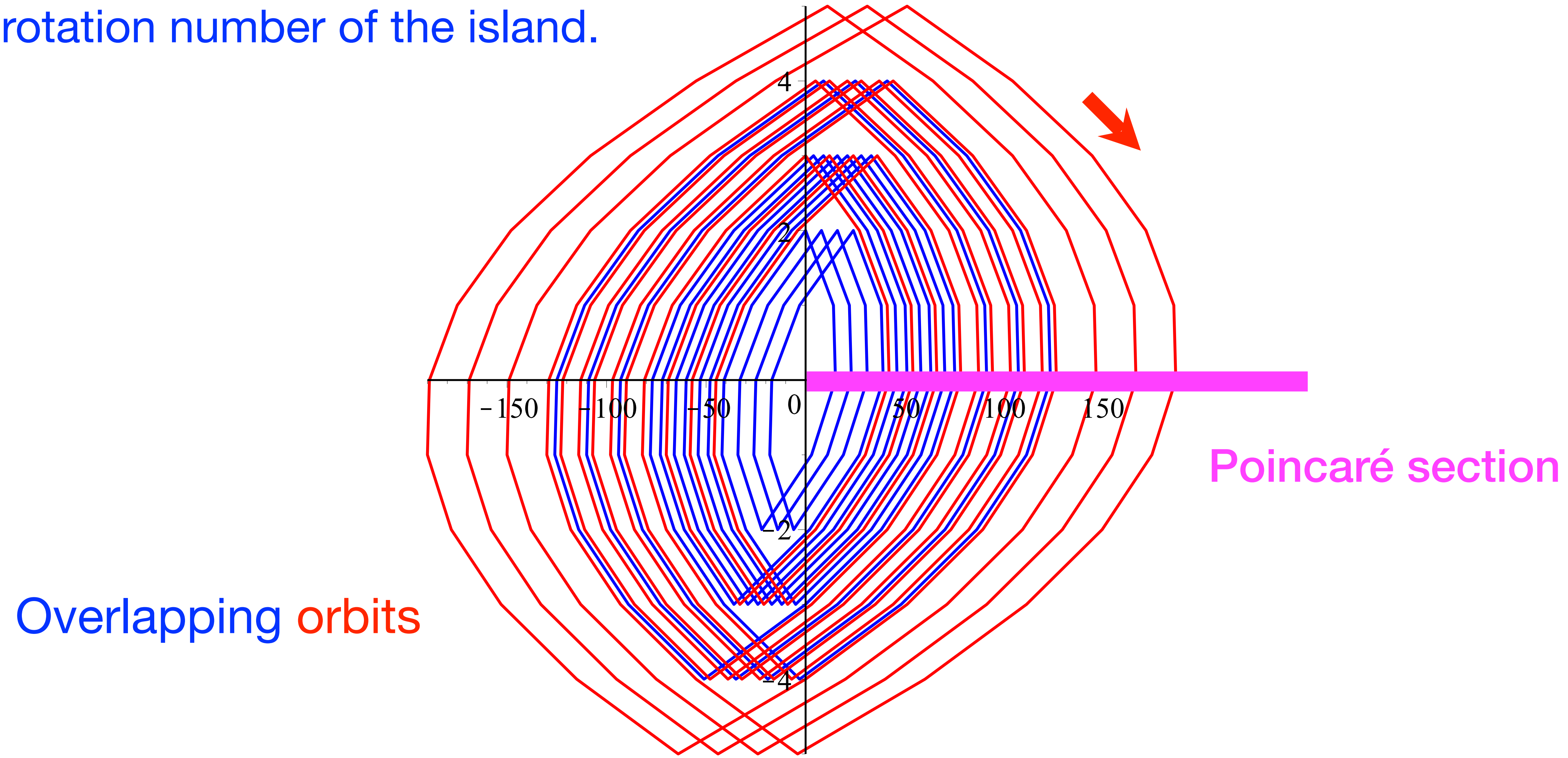

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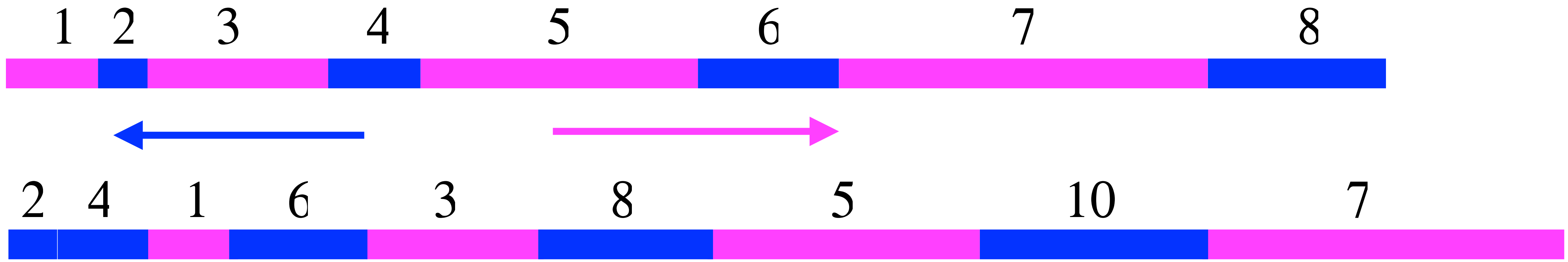

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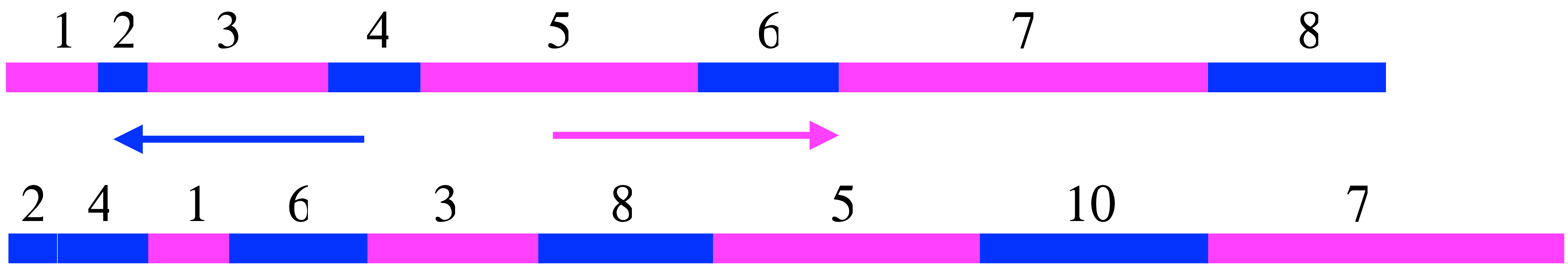

Asymptotically, after scaling, this structure becomes spatially periodic, with the same combinatorics.

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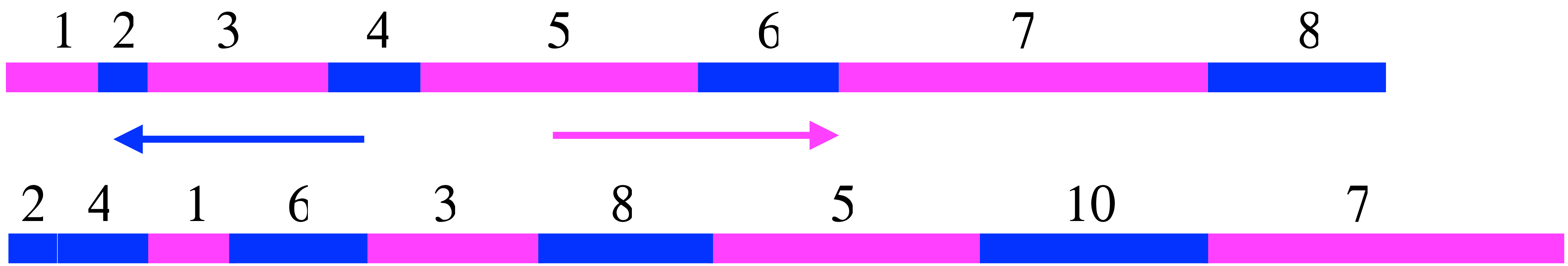

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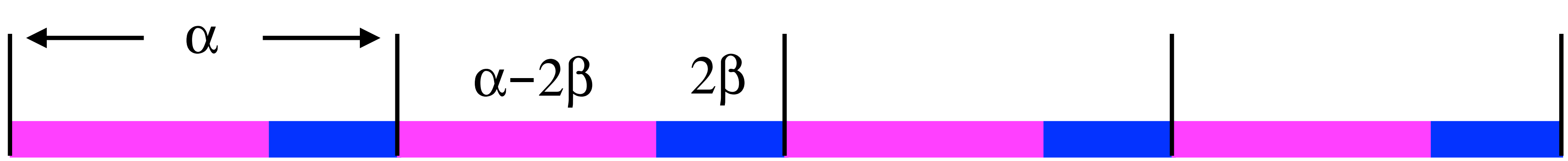

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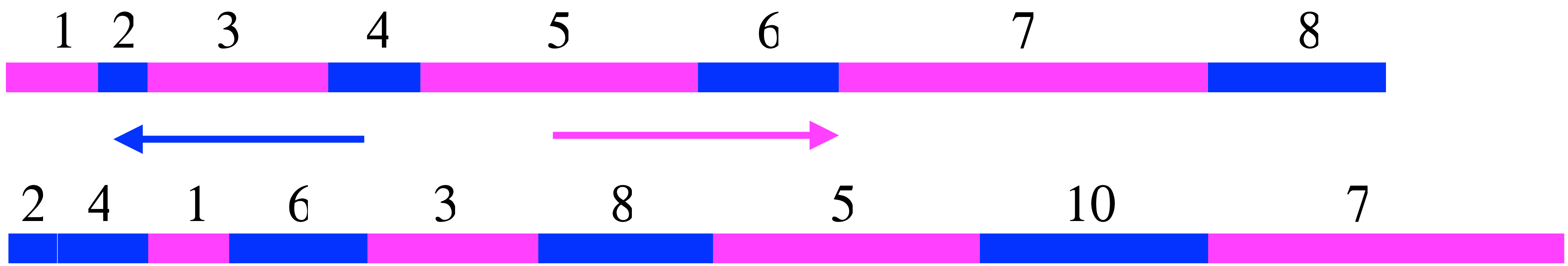

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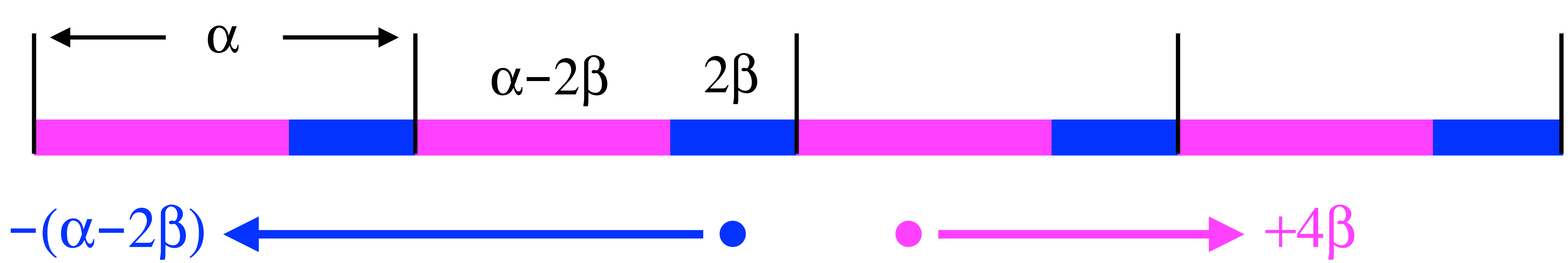

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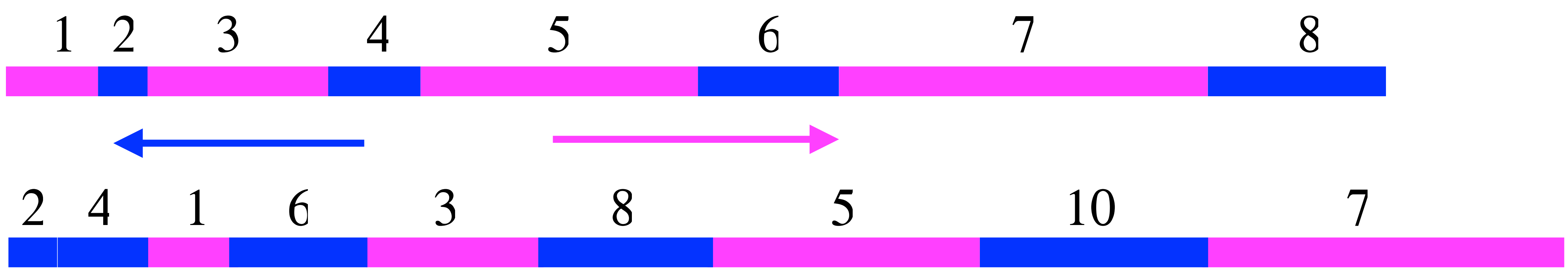

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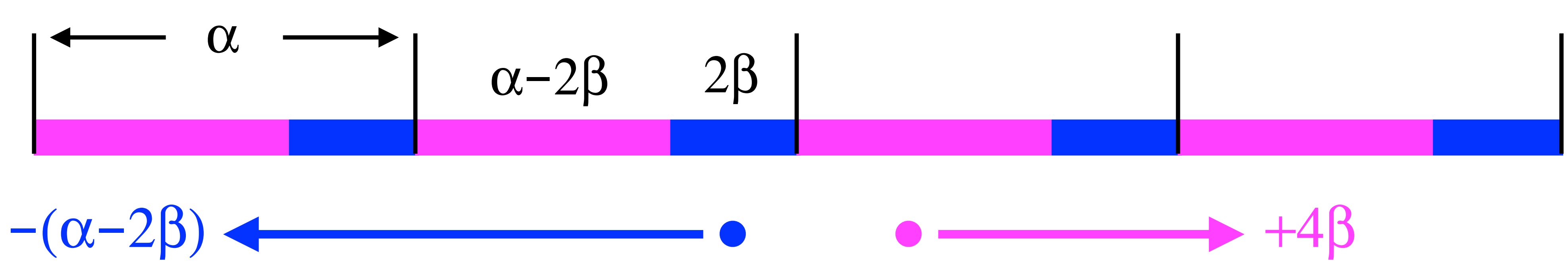
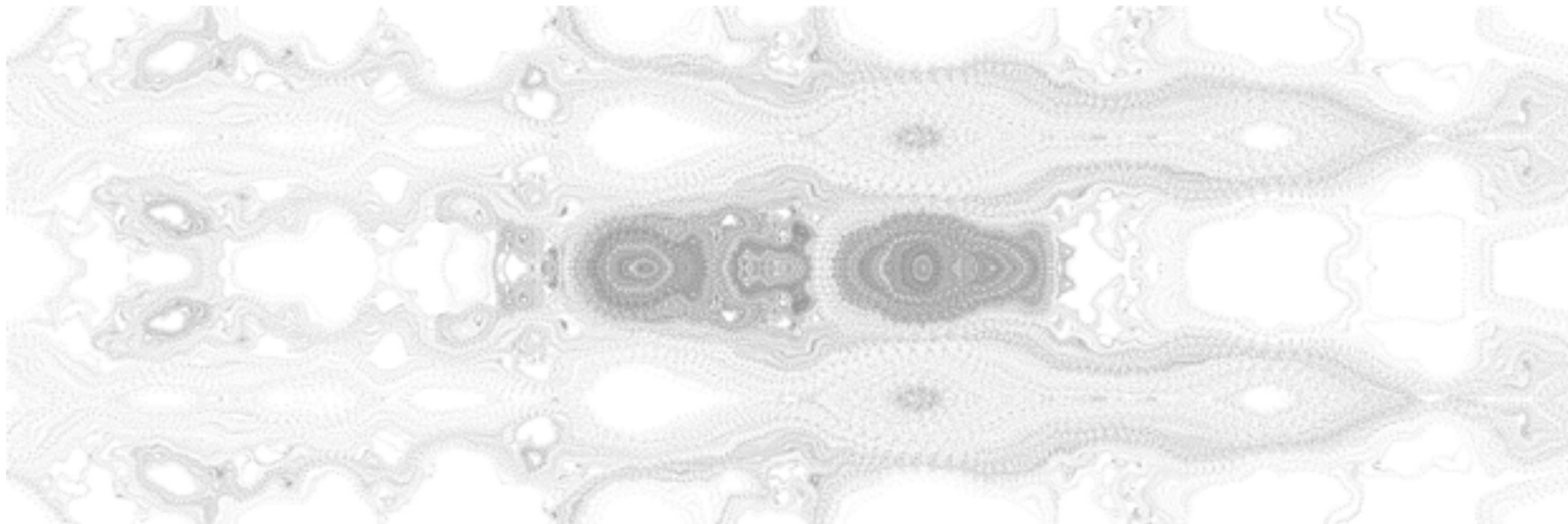
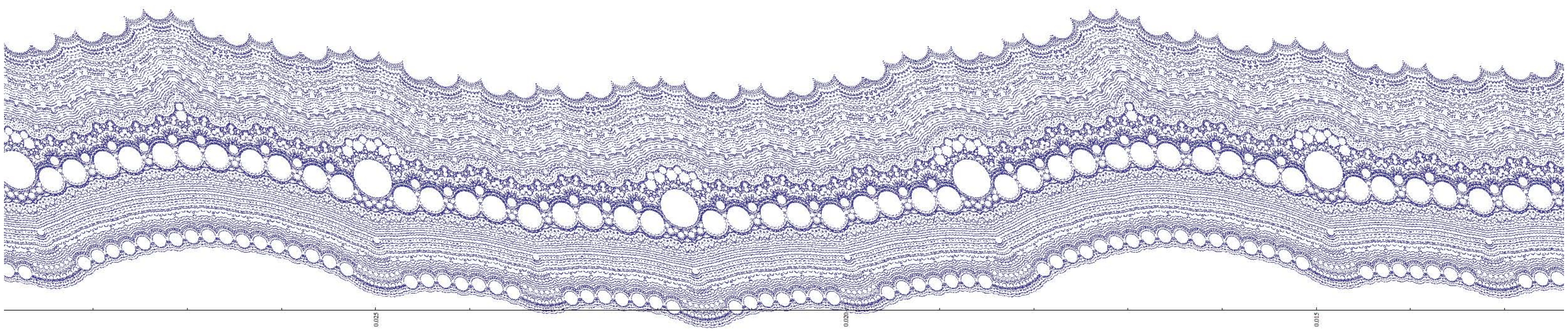
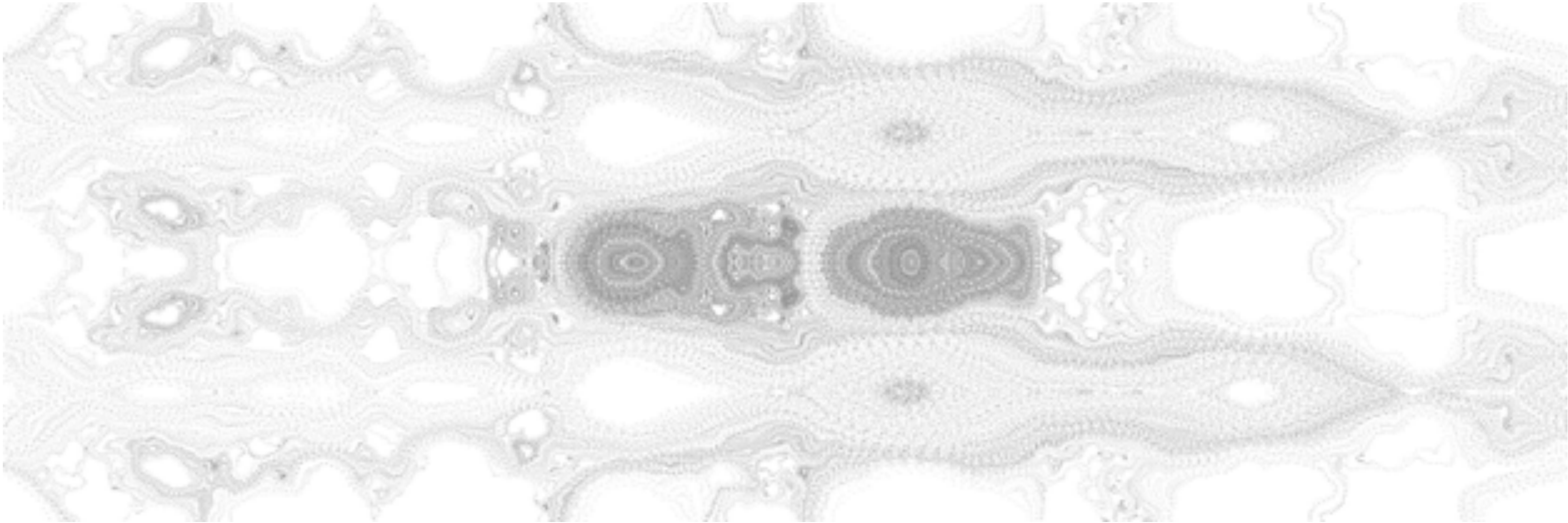


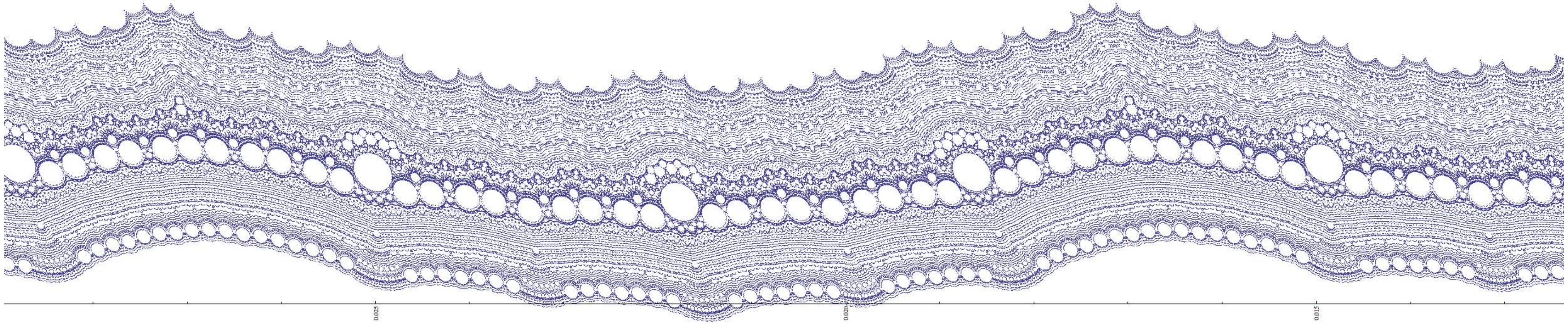
- The conjecture is true for the scaled system (from symmetry and uniform distribution modulo α).





0.025
0.020
0.015





Thank you for your attention