Arithmetic Aspects of Piecewise Linear Dynamics

John A G Roberts

School of Mathematics and Statistics,
University of NSW, Sydney, Australia

CIRM Conference, April 2016

In 1969, Michel Hénon (Nice Observatory) constructed this toy model

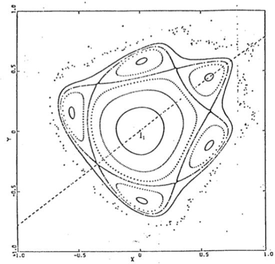
$$L^{RHen} : x' = y, \quad y' = -x + y^2 + a$$

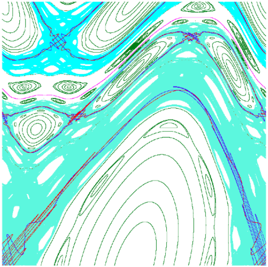
Intertwining order and chaos in Hamiltonian dynamics

In 1969, Michel Hénon (Nice Observatory) constructed this toy model

$$L^{RHen} : x' = y, \quad y' = -x + y^2 + a$$

Area-preserving, polynomial automorphism of $\mathbb{R}^2$ with *invariant (KAM) curves* intertwined to every magnification with *chaotic orbits*.



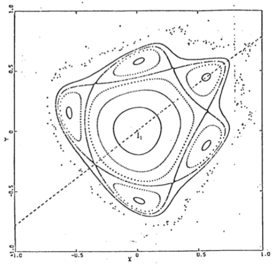


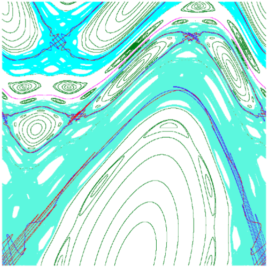
Intertwining order and chaos in Hamiltonian dynamics

In 1969, Michel Hénon (Nice Observatory) constructed this toy model

$$L^{RHen} : x' = y, \quad y' = -x + y^2 + a$$

Area-preserving, polynomial automorphism of $\mathbb{R}^2$ with *invariant (KAM) curves* intertwined to every magnification with *chaotic orbits*.





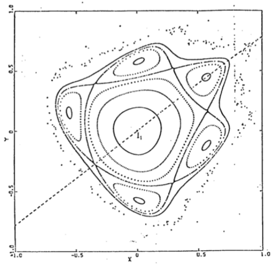
Open Problem: Rigorously show the measure of chaotic orbits is positive.

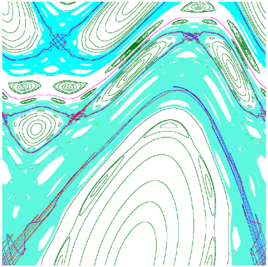
Intertwining order and chaos in Hamiltonian dynamics

In 1969, Michel Hénon (Nice Observatory) constructed this toy model

$$L^{RHen} : x' = y, \quad y' = -x + y^2 + a$$

Area-preserving, polynomial automorphism of $\mathbb{R}^2$ with *invariant (KAM) curves* intertwined to every magnification with *chaotic orbits*.





Open Problem: Rigorously show the measure of chaotic orbits is positive.

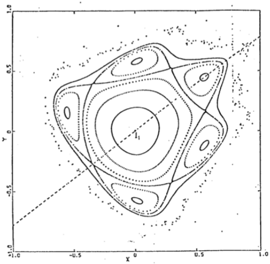
Happy April Fool's Day!

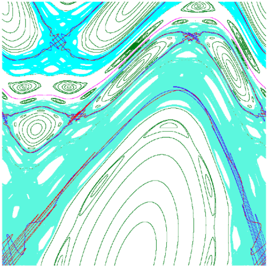
Intertwining order and chaos in Hamiltonian dynamics

In 1969, Michel Hénon (Nice Observatory) constructed this toy model

$$L^{RHen}: x' = y, \quad y' = -x + y^2 + a$$

Area-preserving, polynomial automorphism of $\mathbb{R}^2$ with *invariant (KAM) curves* intertwined to every magnification with *chaotic orbits*.



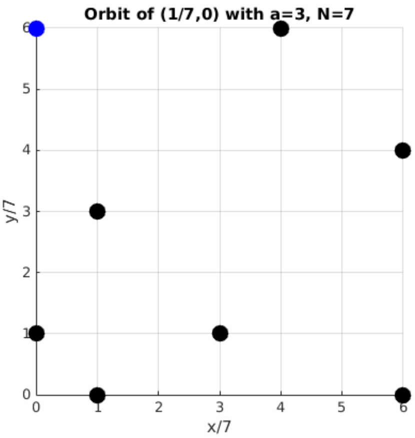


Open Problem: Rigorously show the measure of chaotic orbits is positive.

Happy April Fool's Day!

Happy Birthday Igor!

Plot of orbit over $\mathbb{F}_p^2$!

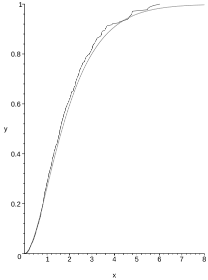


- Family of reversible polynomial automorphisms of $\mathbb{R}^2$
  $L^{RHen} : x' = y, \quad y' = -x + y^2 + a$

- Family of reversible polynomial automorphisms of $\mathbb{R}^2$
  $L^{RHen} : x' = y, \quad y' = -x + y^2 + a$
- Take x,y,$a \in \{0, 1, 2, \ldots, p-1\}$ and operate $\mod p$, p prime $\Rightarrow$ reduced L_p^{RHen}, reversible polynomial automorphism of $\mathbb{F}_p^2$

- Family of reversible polynomial automorphisms of $\mathbb{R}^2$
  $L^{RHen}: x' = y, \quad y' = -x + y^2 + a$
- Take x,y,$a \in \{0, 1, 2, \ldots, p-1\}$ and operate $\mod p$, p prime $\Rightarrow$ reduced L_p^{RHen}, reversible polynomial automorphism of $\mathbb{F}_p^2$
- Period $t(z)$ distribution function for the induced permutation:

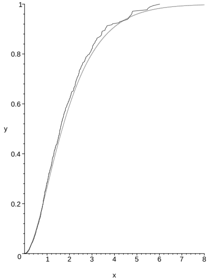
- Family of reversible polynomial automorphisms of $\mathbb{R}^2$
  $L^{RHen}: x' = y, \quad y' = -x + y^2 + a$
- Take x,y,$a \in \{0, 1, 2, \ldots, p-1\}$ and operate $\mod p$, p prime $\Rightarrow$ reduced L_p^{RHen}, reversible polynomial automorphism of $\mathbb{F}_p^2$
- Period $t(z)$ distribution function for the induced permutation:



$$\mathcal{R}_p(x) = \frac{\#\{z \in \mathbb{F}_p^2 : t(z) \leq \kappa x\}}{\#\mathbb{F}_p^2},$$

$$\kappa = \bar{T} = \frac{p^2}{\#\{\text{cycles}\}} \approx p$$

- Family of reversible polynomial automorphisms of $\mathbb{R}^2$
  $L^{RHen} : x' = y, \quad y' = -x + y^2 + a$
- Take x,y,$a \in \{0, 1, 2, \ldots, p-1\}$ and operate $\mod p$, p prime $\Rightarrow$ reduced L_p^{RHen}, reversible polynomial automorphism of $\mathbb{F}_p^2$
- Period $t(z)$ distribution function for the induced permutation:

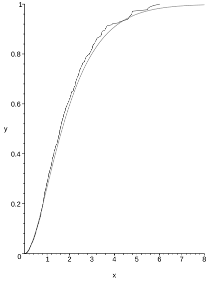


$$\mathcal{R}_p(x) = \frac{\#\{z \in \mathbb{F}_p^2 : t(z) \le \kappa x\}}{\#\mathbb{F}_p^2},$$

$$\kappa = \bar{T} = \frac{p^2}{\#\{\text{cycles}\}} \approx p$$

- Case p=997

- Family of reversible polynomial automorphisms of $\mathbb{R}^2$
  $L^{RHen} : x' = y, \quad y' = -x + y^2 + a$
- Take x,y,$a \in \{0, 1, 2, \ldots, p-1\}$ and operate mod p, p prime $\Rightarrow$ reduced L_p^{RHen}, reversible polynomial automorphism of $\mathbb{F}_p^2$
- Period $t(z)$ distribution function for the induced permutation:



$$\mathcal{R}_p(x) = \frac{\#\{z \in \mathbb{F}_p^2 : t(z) \leq \kappa x\}}{\#\mathbb{F}_p^2},$$

$$\kappa = \bar{T} = \frac{p^2}{\#\{\text{cycles}\}} \approx p$$

- Case p=997 versus $\lim_{p\to\infty} \mathcal{R}_p(x) = \mathcal{R}(x) = 1 - e^{-x}(1+x)$

Outline

1. Motivation
2. Arithmetic exponents in $\mathbb{Q}^2$ to separate order and chaos in PWL or PWA maps (cf. Lyapunov exponents for $\mathbb{R}^2$) [joint with Franco Vivaldi]

Outline

1. Motivation
2. Arithmetic exponents in $\mathbb{Q}^2$ to separate order and chaos in PWL or PWA maps (cf. Lyapunov exponents for $\mathbb{R}^2$) [joint with Franco Vivaldi]
3. A universal period distribution for Piece-wise Cat maps (gives a PWL map over $\mathbb{F}_p^2$) [joint with Tim Siu]

Outline

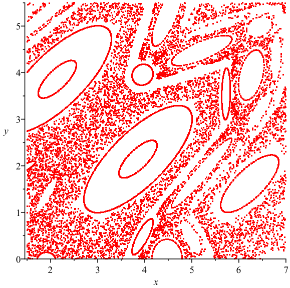
1. Motivation
2. Arithmetic exponents in $\mathbb{Q}^2$ to separate order and chaos in PWL or PWA maps (cf. Lyapunov exponents for $\mathbb{R}^2$) [joint with Franco Vivaldi]
3. A universal period distribution for Piece-wise Cat maps (gives a PWL map over $\mathbb{F}_p^2$) [joint with Tim Siu]
4. Complexity, Divisibility and Recurrence in PWL over $\mathbb{F}_p^2$ [joint with Franco Vivaldi]

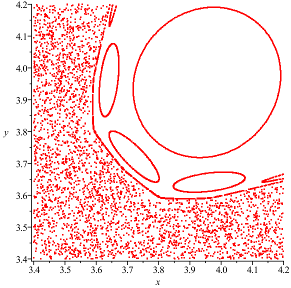
I. Arithmetic exponents in $\mathbb{Q}^2$ to separate order and chaos

$$F : \mathbb{R}^2 \to \mathbb{R}^2 \qquad (x, y) \mapsto (f(x) - y, dx) \tag{1}$$

with $d = 1$ (the map is area-preserving) and piecewise-affine function

$$f(x) = \begin{cases} \frac{3}{2}x + \frac{3}{2} & x < -1 \\ 0 & -1 \leqslant x \leqslant 1 \\ \frac{3}{2}x - \frac{3}{2} & x > 1 \end{cases} \tag{2}$$





When affine map parameters are in $\mathbb{Q}$, then $F : \mathbb{Q}^2 \to \mathbb{Q}^2$. We wish to monitor the arithmetic complexity of the orbits

- Height

$$H(m/n) = \max(|m|, |n|) \qquad \gcd(m, n) = 1. \tag{3}$$

Extend to two dimensions:

$$H(z) = \max(H(x), H(y)) \qquad z = (x, y).$$

and define arithmetic exponent of z

$$\lambda(z) = \lim_{t\to\infty} \frac{1}{t} \log H(F^t(z)) \tag{4}$$

- *p*-adic valuation.
For $m \in \mathbb{Z}$, define order $\nu_p(m)$ to be largest non-negative integer k such that p^k divides m, with $\nu(0) = \infty$ $[\nu_p(m) \leqslant \frac{\log m}{\log p}]$.

$$r = \frac{m}{n} \implies \nu_p(r) = \nu_p(m) - \nu_p(n)$$

and define *p*-adic (arithmetic) exponent of z

$$\lambda_p(z) = \lim_{t \to \infty} -\frac{1}{t} \nu_p(F^t(z)) \tag{5}$$

where $\nu_p(z) = \min(\nu_p(x), \nu_p(y))$.

These exponents relate to arithmetic entropy and canonical height – see *The arithmetic of dynamical systems* by J. Silverman and his recent *arXiv* articles.

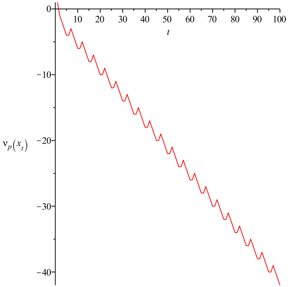


Figure 1: $\nu_p(F^t(z_0))$ versus t for rational $z_0 = (2, 0)$ in island.

.

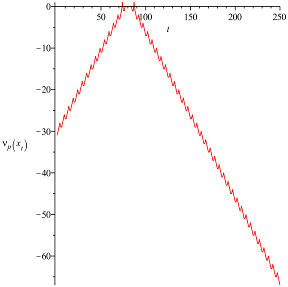


Figure 2: $\nu_p(F^t(z_1))$ for z_1 close to z_0 in island

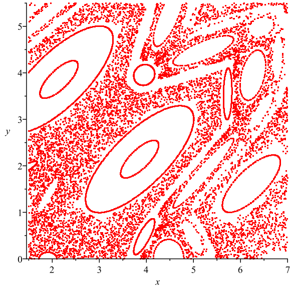


Figure 3: Phase Portrait of F with $d = 1$.

.

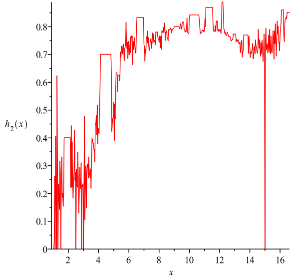


Figure 4: Behaviour of $\lambda_2(z_0)$ with initial conditions $z_0 = (x, 0)$.

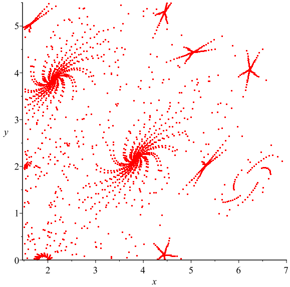


Figure 5: Phase Portrait of F with $d = 497/499$.

.

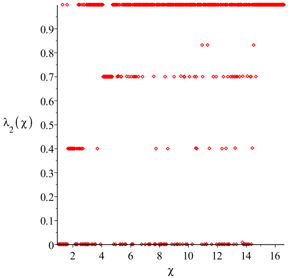


Figure 6: Behaviour of $\lambda_2(z_0)$ with initial conditions $z_0 = (x, 0)$.

Theorem (R+Vivaldi 15)

Almost all points of a rational island have the same exponents λ and λ_p for all primes p. The latter are rational numbers.

Islands in the piecewise affine map correspond to invariant regions of a suitable affine map so need to do p-adic linear dynamics.

Theorem (R+Vivaldi 15)

Let $F : \mathbb{Q}^2 \to \mathbb{Q}^2$ with $z = (x, y) \mapsto \mathrm{M}\, z + s$ and $\mathrm{M} \in (2, \mathbb{Q})$, $T = \mathrm{trace}(\mathrm{M})$ and $D = \det(\mathrm{M})$. If $s = (0, 0)$, then for almost all $z \in \mathbb{Q}^2$ we have:

i) if $\nu_p(D) > 2\nu_p(T)$ then $\lambda_p(z) = -\nu_p(T)$;

ii) if $\nu_p(D) \leqslant 2\nu_p(T)$ then $\lambda_p(z) = -\nu_p(D)/2$.

If $s \neq (0, 0)$ then the above expressions for λ_p must be replaced by $\max(-\nu_p(T), 0)$ and $\max(-\nu_p(D)/2, 0)$, respectively.

Explicit formulae

In many cases, one can describe $\nu_p(F^t(z))$ explicitly as a piecewise affine function of t:

$$\mathrm{M}^t \;=\; U_t\,\mathrm{M} - D\,U_{t-1}\,, \tag{6}$$

$U_t = U_t(T, D)$ obey the Lucas sequence of the first kind:

$$U_0 = 0, \quad U_1 = 1, \qquad U_{t+1}(T,D) \;=\; T\,U_t(T,D) - D\,U_{t-1}(T,D), \quad t \geqslant 1. \tag{7}$$

with solution

$$U_t(T,D) = \sum_{k=0}^{\lfloor (t-1)/2 \rfloor} c_k^{(t)}\; T^{t-2k-1}\,(-D)^k \tag{8}$$

where

$$c_k^{(t)} = \binom{t-k-1}{k}.$$

$$z_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = U_t(T,D) \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} - D\, U_{t-1}(T,D) \begin{pmatrix} x_0' \\ y_0' \end{pmatrix} + \begin{pmatrix} x^* \\ y^* \end{pmatrix} \qquad (9)$$

Use the ultrametric inequality :

$$\nu_p(x_t) \geqslant \min(\nu_p(\mathcal{T}_t^{(1)}), \nu_p(\mathcal{T}_t^{(0)}), \nu_p(x^*)). \qquad (10)$$

For example, if $\nu_p(D) > 2\nu_p(T)$:

$$\nu_p(x_t) \geqslant \min\{\nu_p(x_1')+(t-1)\nu_p(T), \nu_p(x_0')+(t-2)\,\nu_p(T)+\nu_p(D), \nu_p(x^*)\}. \qquad (11)$$

If $\nu_p(T) \geqslant 0$, linear terms are increasing, and we have two possibilities. If $x^* \neq 0$, then eventually $\nu_p(x_t) = \nu_p(x^*)$. If, $x^* = 0$, then eventually, under the non-degeneracy condition

$$\nu_p(x_1') + \nu_p(T) \neq \nu_p(x_0') + \nu_p(D) \qquad (12)$$

a unique minimum emerges in (11), and $\nu_p(x_t)$ becomes affine.

- Remark that arbitrarily close to each rational z_0 in an island with generic behaviour of $\nu_p(F^t(z_0))$ can be found a rational initial condition that initially grows in the wrong direction!
- $\lambda(z) = \lim_{t\to\infty} \frac{1}{t} \log H(F^t(z))$ can also be proved to be constant in rational islands of piecewise affine planar maps.
- We conjecture that the arithmetic exponents of a piecewise affine map of $\mathbb{Q}^2$ exist for almost all points with bounded orbit. We are exploring how $\nu_p(F^t(z_0))$ behaves for z_0 in a chaotic region of the map to see how the arithmetic exponents "see" the border of regular and chaotic behaviour.

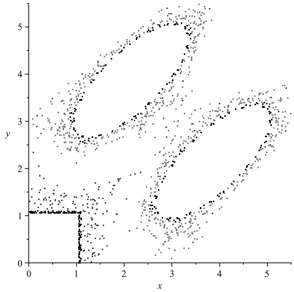


Figure 7: Chaotic orbit of z_0 of F with $d = 1$.

.

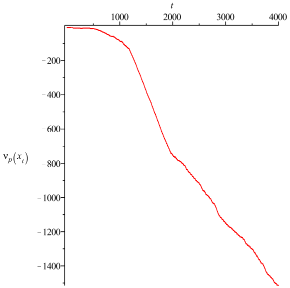


Figure 8: $\nu_p(F^t(z_0))$ versus t

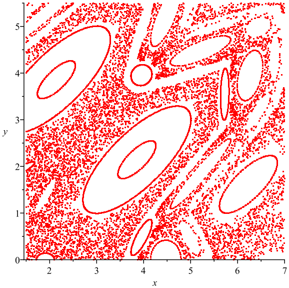


Figure 9: Phase Portrait of F with $d = 1$.

.

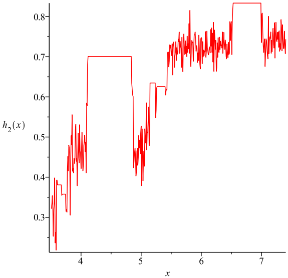


Figure 10: Behaviour of $\lambda_2(z_0)$ with initial conditions $z_0 = (x, 0)$.

Conjecture. *Let f be an area-preserving piecewise affine map of $\mathbb{Q}^2$. Then any arithmetic exponent has a (non-strict) local maximum at almost all points $z \in \mathbb{Q}^2$ for which the Lyapunov exponent is zero.*

II. Universal period distribution for PWL over $\mathbb{F}_p^2$

Consider the matrix

$$A = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{where } a \in \mathbb{Z}. \tag{13}$$

Consider the matrix

$$A = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{where } a \in \mathbb{Z}. \tag{13}$$

We induce the dynamical system S on the 2-torus $\mathbb{T}^2$

$$S(x, y) = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1 = \begin{pmatrix} ax - y \\ x \end{pmatrix} \mod 1. \tag{14}$$

Consider the matrix

$$A = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{where } a \in \mathbb{Z}. \tag{13}$$

We induce the dynamical system S on the 2-torus $\mathbb{T}^2$

$$S(x, y) = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1 = \begin{pmatrix} ax - y \\ x \end{pmatrix} \mod 1. \tag{14}$$

When $|a| > 2$, such cat maps are chaotic mappings significant in theory and applications.

Periodic orbits

- Rational points of the form $(\frac{x}{N}, \frac{y}{N})$ will get sent to rational points $(\frac{x'}{N}, \frac{y'}{N})$ with $x', y' < N$.

Periodic orbits

- Rational points of the form $(\frac{x}{N}, \frac{y}{N})$ will get sent to rational points $(\frac{x'}{N}, \frac{y'}{N})$ with $x', y' < N$.
- There are a N^2 rational points of such form.

Periodic orbits

- Rational points of the form $(\frac{x}{N}, \frac{y}{N})$ will get sent to rational points $(\frac{x'}{N}, \frac{y'}{N})$ with $x', y' < N$.
- There are a N^2 rational points of such form.
- All rational points are periodic (finite number of points in its orbit).

Periodic orbits

- Rational points of the form $(\frac{x}{N}, \frac{y}{N})$ will get sent to rational points $(\frac{x'}{N}, \frac{y'}{N})$ with $x', y' < N$.
- There are a N^2 rational points of such form.
- All rational points are periodic (finite number of points in its orbit).
- For $|a| > 2$ all periodic points are rational.

Periodic orbits

- Rational points of the form $(\frac{x}{N}, \frac{y}{N})$ will get sent to rational points $(\frac{x'}{N}, \frac{y'}{N})$ with $x', y' < N$.
- There are a N^2 rational points of such form.
- All rational points are periodic (finite number of points in its orbit).
- For $|a| > 2$ all periodic points are rational.
- Then to study the periodic orbits we consider only the rational points.

Prime lattice

We discretise our mapping and focus on rational points with prime denominators $(\frac{x}{p}, \frac{y}{p})$.

Prime lattice

We discretise our mapping and focus on rational points with prime denominators $(\frac{x}{p}, \frac{y}{p})$. Then for a fixed p we obtain a reduced dynamical system S_p on the rational lattice $\Lambda_p = \{(\frac{x}{p}, \frac{y}{p}) \mid 0 \leq x, y < p\}$

$$S_p(x, y) = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1 \quad \equiv \quad S_p : \mathbb{F}_p^2 \to \mathbb{F}_p^2 : \begin{pmatrix} ax - y \\ x \end{pmatrix}$$

Prime lattice

We discretise our mapping and focus on rational points with prime denominators $(\frac{x}{p}, \frac{y}{p})$. Then for a fixed p we obtain a reduced dynamical system S_p on the rational lattice $\Lambda_p = \{(\frac{x}{p}, \frac{y}{p}) \mid 0 \leq x, y < p\}$

$$S_p(x,y) = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1 \quad \equiv \quad S_p : \mathbb{F}_p^2 \to \mathbb{F}_p^2 : \begin{pmatrix} ax - y \\ x \end{pmatrix}$$

- We can find the orbit and period of every point (by explicit computation).

Prime lattice

We discretise our mapping and focus on rational points with prime denominators $(\frac{x}{p}, \frac{y}{p})$. Then for a fixed p we obtain a reduced dynamical system S_p on the rational lattice $\Lambda_p = \{(\frac{x}{p}, \frac{y}{p}) \mid 0 \leq x, y < p\}$

$$S_p(x, y) = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1 \quad \equiv \quad S_p : \mathbb{F}_p^2 \to \mathbb{F}_p^2 : \begin{pmatrix} ax - y \\ x \end{pmatrix}$$

- We can find the orbit and period of every point (by explicit computation).
- What is the distribution of orbit lengths?

Prime lattice

We discretise our mapping and focus on rational points with prime denominators $(\frac{x}{p}, \frac{y}{p})$. Then for a fixed p we obtain a reduced dynamical system S_p on the rational lattice $\Lambda_p = \{(\frac{x}{p}, \frac{y}{p}) \mid 0 \leq x, y < p\}$

$$S_p(x,y) = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1 \quad \equiv \quad S_p : \mathbb{F}_p^2 \to \mathbb{F}_p^2 : \begin{pmatrix} ax - y \\ x \end{pmatrix}$$

- We can find the orbit and period of every point (by explicit computation).
- What is the distribution of orbit lengths?
- For any prime p and $a \neq \pm 2$ all orbits (except the origin) have the same period (e.g. via ideal theory of Percival+Vivaldi '87).

Prime lattice

We discretise our mapping and focus on rational points with prime denominators $(\frac{x}{p}, \frac{y}{p})$. Then for a fixed p we obtain a reduced dynamical system S_p on the rational lattice $\Lambda_p = \{(\frac{x}{p}, \frac{y}{p}) \mid 0 \leq x, y < p\}$

$$S_p(x,y) = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1 \quad \equiv \quad S_p : \mathbb{F}_p^2 \to \mathbb{F}_p^2 : \begin{pmatrix} ax - y \\ x \end{pmatrix}$$

- We can find the orbit and period of every point (by explicit computation).
- What is the distribution of orbit lengths?
- For any prime p and $a \neq \pm 2$ all orbits (except the origin) have the same period (e.g. via ideal theory of Percival+Vivaldi '87).

Example

For $a = 3, p = 7$ we have 6 orbits each with period 8 and 1 fixed point (the origin).

Distribution function

Define the distribution function

$$\mathcal{D}_p(x) = \frac{\#\{z \in \mathbb{F}_p^2 : t(z) \leq \kappa x\}}{\#\mathbb{F}_p^2} \qquad (15)$$

where $\kappa = \frac{p^2}{\#orbits}$ = mean period length.

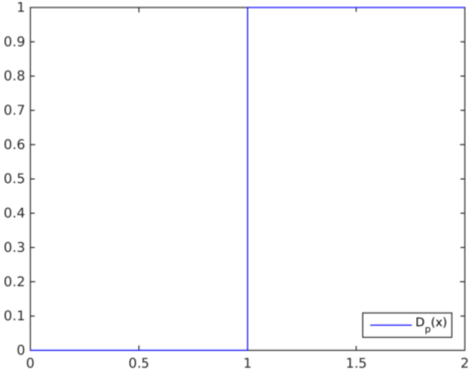


Figure 11: An example of the period distribution function $D_p(x)$ for S_p

Cat map

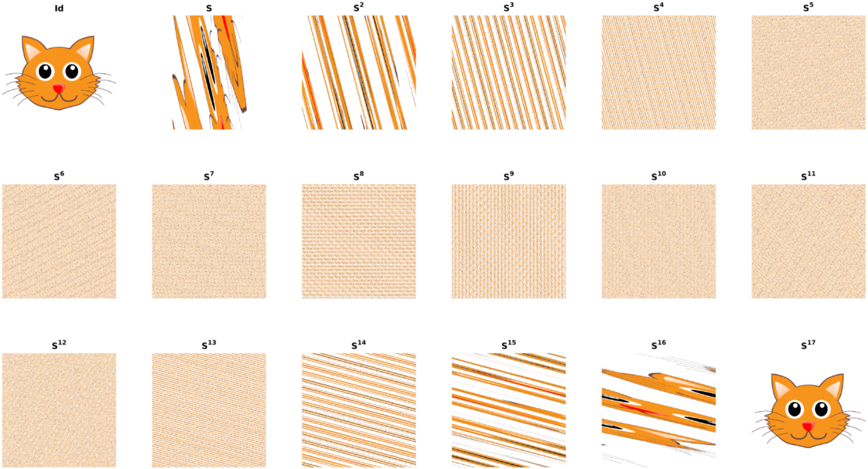


Figure 12: We apply the mapping to the whole phase space one iteration at a time with $p = 509, a = 4$. We come back after 17 iterations.

Piecewise Cat Map

Piecewise Cat Map

We introduce a family of piecewise Cat maps $T_{ab} : \Lambda_p \to \Lambda_p$:

$$T_{ab}(x, y) = \begin{cases} (bx - y, x) & \text{mod } 1 \quad \text{if } 0 \le x < s \\ (ax - y, x) & \text{mod } 1 \quad \text{if } s \le x < 1 \end{cases}$$

with parameters $a, b \in \mathbb{Z}$.

Alternate form

This map can also be written as $T_{ab} : \mathbb{F}_p^2 \to \mathbb{F}_p^2$

$$T_{ab}(x, y) = \begin{bmatrix} F_{ab}(x) & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$F_{ab}(x) = \begin{cases} b \in \mathbb{F}_p & \text{if } x \in \{0, 1, \ldots, \lfloor s\,p \rfloor\} \\ a \in \mathbb{F}_p & \text{if } x \in \{\lfloor s\,p \rfloor + 1, \ldots, p - 1\} \end{cases}$$

R-reversibility

Definition

- An involution is a map G which is equal to its inverse,

$$G = G^{-1}.$$

- A map acting on a space of p^2 points is said to be *R-reversible* if it is the composition of two involutions G and H,

$$L = H \circ G$$

  and G and H have p fixed points.

R-reversibility

Definition

- An involution is a map G which is equal to its inverse,

$$G = G^{-1}.$$

- A map acting on a space of p^2 points is said to be *R-reversible* if it is the composition of two involutions G and H,

$$L = H \circ G$$

and G and H have p fixed points.

Involutions for Piecewise Cat map

$$G : x' = x, \qquad y' = F_{ab}(x)x - y \qquad\qquad H : x' = y, \qquad y' = x.$$

$$\mathrm{Fix}(G) = \{(x, y) : 2y = F_{ab}(x)x\}, \qquad \mathrm{Fix}(H) = \{(x, x)\}.$$

Roberts and Vivaldi 2005

Definition

Below we define our empirical period distribution function:

$$\mathcal{D}_p(x) = \frac{\#\{z \in \mathbb{F}_p^2 : t(z) \leq \kappa x\}}{\#\mathbb{F}_p^2}$$

Conjecture

Let L be an R-reversible map (with a single family of reversing symmetries) acting on a space of p^2 points. Then

$$\lim_{p\to\infty} \mathcal{D}_p(x) = \mathcal{R}(x) := 1 - e^{-x}(1+x), \qquad x \geq 0$$

where the normalisation constant κ is the mean period of the orbits,

$$\kappa = \frac{p^2}{\# orbits}.$$

Conjecture

For our Piecewise Cat Map T_{ab} and any fixed switch $0 < s < 1$, then for almost all parameters $(a, b) \in \mathbb{F}_p^2$, we have

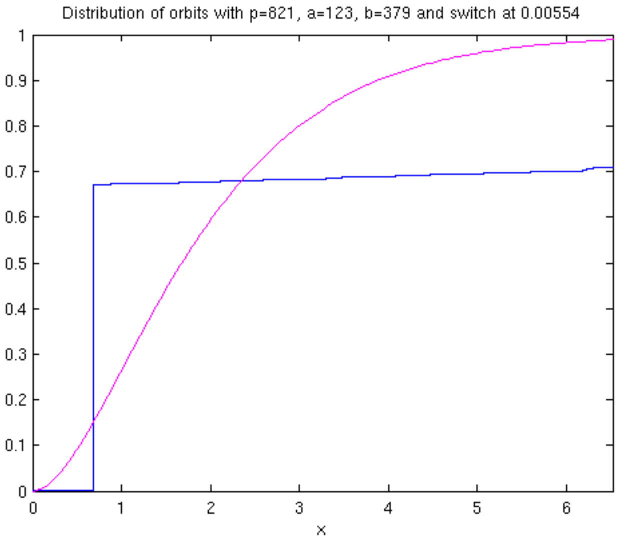
$$\lim_{p\to\infty} \mathcal{D}_p(x) = \mathcal{R}(x) := 1 - e^{-x}(1 + x), \qquad x \geq 0,$$

to be compared to the singular distribution when $s = 0$ (i.e. for the single cat map).

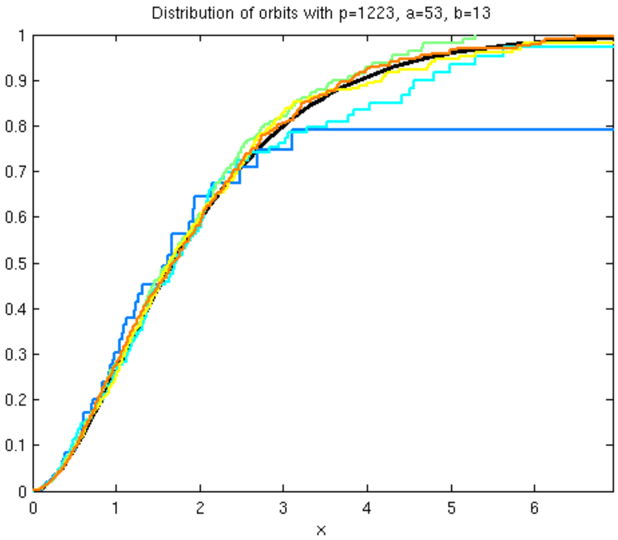
Previous work confirming $\mathcal{R}(x)$

- Polynomial automorphisms on finite fields (e.g. Hénon map) - Roberts and Vivaldi 2005
- Casati-Prosen map (piece-wise constant map on the torus) - Neumarker, Roberts and Vivaldi 2012
- Wehler K3 Surfaces over Finite Fields - Faria and Hutz (2015)

Plot of distribution with fixed prime and changing switch



Plot of distribution with increasing prime



Quantifying convergence

Quantifying convergence

In order to quantify convergence, we define a function to represent the distance (in 1-norm) between the exponential distribution function $\mathcal{R}$ and the empirical distribution function $\mathcal{D}_p$. This is written as

$$\mathcal{E}_p(a, b) = \int_0^\infty |\mathcal{D}_{p,a,b}(x) - \mathcal{R}(x)| dx.$$

Quantifying convergence

In order to quantify convergence, we define a function to represent the distance (in 1-norm) between the exponential distribution function $\mathcal{R}$ and the empirical distribution function $\mathcal{D}_p$. This is written as

$$\mathcal{E}_p(a, b) = \int_0^{\infty} |\mathcal{D}_{p,a,b}(x) - \mathcal{R}(x)| dx.$$

Example

In the previous figure, with $a = 53$, $b = 13$ we have

$$\begin{aligned}
\mathcal{E}_{71}(a, b) &= 0.755843573324966, \\
\mathcal{E}_{281}(a, b) &= 0.217055546096392, \\
\mathcal{E}_{523}(a, b) &= 0.127933982374603, \\
\mathcal{E}_{809}(a, b) &= 0.080157583604951, \\
\mathcal{E}_{1223}(a, b) &= 0.053647256826975.
\end{aligned}$$

Plot of $\mathcal{E}_p(a, b)$

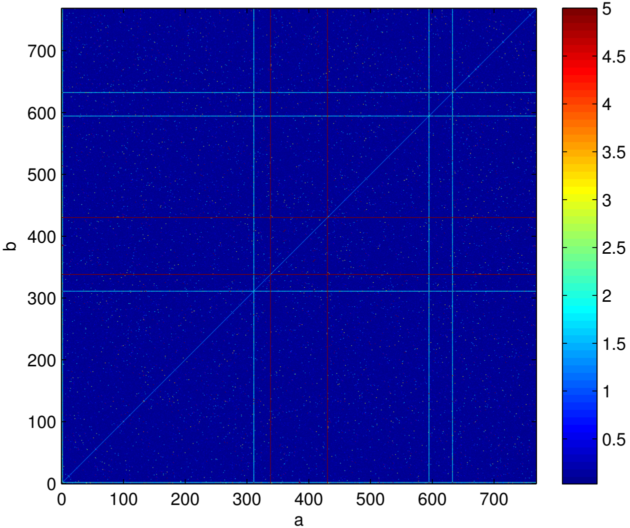


Figure 13: Plot of error $\mathcal{E}_p(a, b)$ capped at 5 with $p = 769$ for parameter space a, b

Anomalous distributions

Vertical lines

These occur when the underlying linear map contains orbits with small period and is related to the spatial distribution of these orbits.

Example

With $p = 769$ and $a = 311$ we find that M_a has 84480 orbits of period 7. We also find that 665 of those orbits have $1/2 \leq x < 1$ for each point. This transfers directly to the map T_{ab} explaining the vertical line for $a = 311$.

Impossible? to prove for a particular case but probabilistic approach over $\mathbb{F}_p$

- $E(N)$ a subset of permutations of symmetric group S_N and expected period distribution $\langle \mathcal{D}_N(x) \rangle = \sum_{t=1}^{\lfloor x\kappa \rfloor} \langle P_t \rangle, \; x \geq 0,$

$$P_t = \frac{1}{N} \#\{x \in \Omega \, : \, x \text{ has minimal period } t\}, t = 1, 2, \ldots, N.$$

- $E(N)$ a subset of permutations of symmetric group S_N and expected period distribution $\langle \mathcal{D}_N(x)\rangle = \sum_{t=1}^{\lfloor x\kappa \rfloor} \langle P_t\rangle,\ x \geq 0,$

$$P_t = \frac{1}{N}\#\{x \in \Omega\ :\ x \text{ has minimal period } t\}, t = 1, 2, \ldots, N.$$

- $E(N) = S_N$, $\langle P_t\rangle = \frac{1}{N} \Rightarrow \langle \mathcal{D}_N(x)\rangle = \frac{\lfloor x\kappa \rfloor}{N} = x$ with $\kappa = N$.

- $E(N)$ a subset of permutations of symmetric group S_N and expected period distribution $\langle \mathcal{D}_N(x) \rangle = \sum_{t=1}^{\lfloor x\kappa \rfloor} \langle P_t \rangle, \ x \geq 0,$

$$P_t = \frac{1}{N} \#\{x \in \Omega \ : \ x \text{ has minimal period } t\}, t = 1, 2, \ldots, N.$$

- $E(N) = S_N$, $\langle P_t \rangle = \frac{1}{N} \Rightarrow \langle \mathcal{D}_N(x) \rangle = \frac{\lfloor x\kappa \rfloor}{N} = x$ with $\kappa = N$.

Theorem (Roberts+Vivaldi 09)

$E(N) = (H, G)$: *all pairs of random involutions of S_N with $g = \#\text{Fix } G$ and $h = \#\text{Fix } H$ satisfying*

$$\lim_{N\to\infty} g(N) + h(N) = \infty \qquad \lim_{N\to\infty} \frac{g(N) + h(N)}{N} = 0.$$

Then, as $N \to \infty$, $\langle \mathcal{D}_N(x) \rangle \to \mathcal{R}(x)$ for $\kappa = 2N/(g(N) + h(N))$. Moreover, almost all points in Ω belong to symmetric cycles.

$$F_{ab} : \mathbb{R}^2 \to \mathbb{R}^2 \qquad (x, y) \mapsto (f_{ab}(x) - y, x)$$

Now on the plane, area-preserving and piecewise-linear function:

$$f_{ab}(x) = a \ (x \geq 0) \text{ and } b \ (x < 0).$$

Lagarias Rains (2005): invariant curves? Proved existence for some $a, b \in \mathbb{R}$ built piecewise from arcs of conic. Experiments suggested other where $a, b \in \mathbb{Q}$.

$\Rightarrow$ foliations of (closed) invariant curves for $F_{ab} : \mathbb{Q}^2 \to \mathbb{Q}^2$ (backed up by theory of Herman), hence bounded rational orbits.

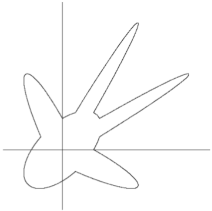
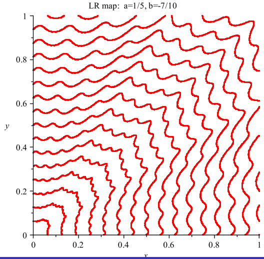


Figure 4.1: Invariant circle for $a = \sqrt[4]{2}$, $b = -\sqrt[4]{2}$ ("Bugs Bunny").



Observations:

(1) The arithmetic exponents introduced in Part I are found to be constant and independent of initial condition.

(2) The complexity of the orbit symbolic dynamics is near Sturmian e.g. $C_{w_t}(n) = n + 7$ for 2 symbols L and R for large t. So orbit is

$$\mathbf{x}_t = w_t(\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix})\, \mathbf{x}_0.$$

Note w_t is a matrix word over $\mathbb{Q}$. Only one matrix in word and mod $p \Rightarrow$ special case of Lucas sequence: $x_{t+1} = a\, x_t - x_{t-1}$.

Questions about the orbit complexity:

(1) What's the divisibility mod p of an orbit, e.g. if and when does $p \mid \text{numerator}(x_t)$?

(2) Does the reduction mod p of an orbit to $\mathbb{F}_p^2$ reflect the ordered motion in $\mathbb{Q}^2$ [Note: unlike Part II, we *cannot* reduce from $\mathbb{Q}^2$ the rule for the PWL dynamics].

Consider the lattice $L_p := \{(k,\ell) \mid 0 \le k,\ell < p\}$ with p prime and

$$\mathbf{x} \sim \mathbf{y} \quad \text{if} \quad \mathbf{x} = j\,\mathbf{y} \mod p, \quad \text{for some } j \in \mathbb{F}_p^{\times}.$$

Partitions L_p: $\mathbf{0}$ and the $p+1$ equivalence classes (rays):

$$\mathcal{R}_p := \{(1,y) : y \in \{0,1,2,\ldots,p-1\} \cup (0,1)\} \simeq P(\mathbb{F}_p).$$

$$\begin{pmatrix} T & -1 \\ 1 & 0 \end{pmatrix} \mod p \;\Rightarrow\; m_T : P(\mathbb{F}_p) \to P(\mathbb{F}_p) \quad y \mapsto \frac{1}{T-y} \mod p.$$

- $p \mid \text{numerator}(x_t) \Leftrightarrow$ hit ray p i.e ∞ at time t. Recover recurrence result for Lucas sequence in terms of Legendre symbol.
- For PWL problem, we study complexity of driving word $w_t(m_{T_1}, m_{T_2})$ in 2 symbols versus complexity of response word in the $p+1$ symbols of R_p and density of each ray in the response word.

Thank you

Thanks for your attention!