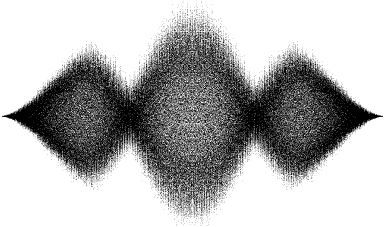
Cyclotomic Coefficients: Progress and Promise

Pieter Moree (MPIM, Bonn)



CIRM, Luminy
March 29, 2016

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Connections

-simplicial complexes (G. Musiker and V. Reiner, 2012)
-Kloosterman sums
-numerical semigroups
-cryptography

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-$\Phi_{2n}(X) = \Phi_n(-X)$, $n > 1$ odd

Some plots...

Courtesy of

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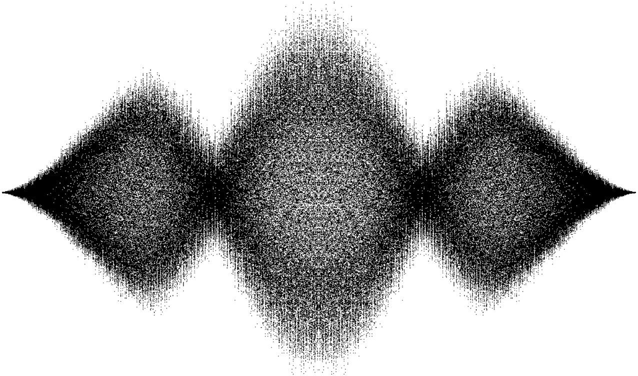

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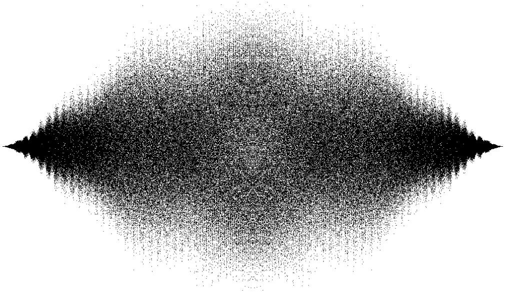

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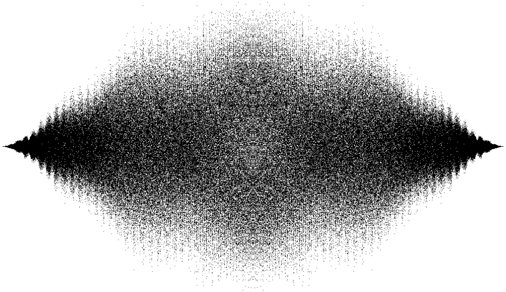

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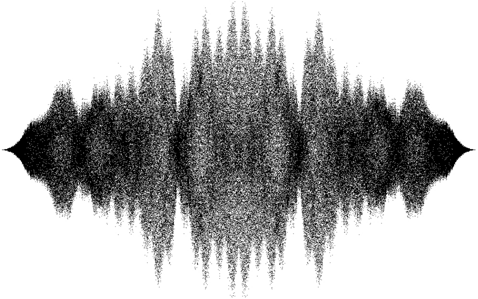

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height 2888582082 500892851

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This is still the state of the art in this direction.

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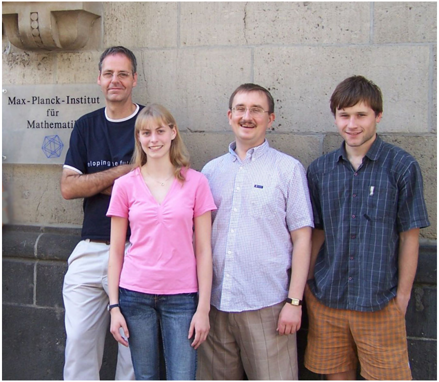

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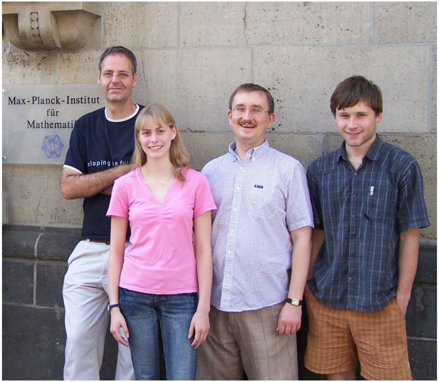

Defended her PhD in 2015 in Harvard...

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Thus the cyclotomic coefficient $a_{pq}(m)$ equals

$$\begin{cases} 1 & \text{if } m = ip + jq \text{ with } 0 \le i \le \rho - 1,\ 0 \le j \le \sigma - 1; \\ -1 & \text{if } m = ip + jq - pq \text{ with } \rho \le i \le q-1,\ \sigma \le j \le p-1; \\ 0 & \text{otherwise.} \end{cases}$$

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Example: Lehmer/Möller infinite family satisfies

$C(pqr) = [-\frac{p-1}{2}, \frac{p+1}{2}],$ (Bachman, 2004)

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-Camburu-Ciolan-Luca-M.-Shparlinski (2016): For infinitely many $n = pqr$ with pairwise distinct odd primes p, q and r, we have

$$J_n \ll n^{7/8+o(1)}.$$

Is $\Phi_n(X)$ a special divisor of $X^n - 1$?

If $\Phi_n(X) | X^{p^2q} - 1$, then

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Andreas Decker
PhD student
comp. analytic number th.
Bonn University

Φ_1	Φ_p	Φ_q	Φ_{pq}	Φ_{p^2}	Φ_{p^2q}	coefficients
1	0	0	0	1	0	$[-1, 1]$
0	1	0	0	1	0	$\{1\}$
1	1	0	0	1	0	$[-1, 1]$
0	0	1	0	1	0	$[\min([\frac{q}{p}], 1), \min([\frac{q-1}{p}] + 1, p)]$
1	0	1	0	1	0	$[-1, 1]$
0	1	1	0	1	0	$[1, \min(p^2, q)]$
1	1	1	0	1	0	$[-1, 1]$
0	0	0	1	1	0	$[-\min(p, q - p^*), \min(p, p^*)]$
1	0	0	1	1	0	$[-\gamma(p, q), \gamma(p, q)]$
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$\gamma(p.q) = \min(p, p^*) + \min(p, q - p^*)$.

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Consider

$$\Psi_n(X) = \frac{X^n - 1}{\Phi_n(X)} = \prod_{d|n,\ d<n} \Phi_d(X) = \sum_{k=0}^{\infty} c_n(k) X^k.$$

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$$B(pqr) = p - 1 \iff q \equiv r \equiv \pm 1 (\mathrm{mod}\ p) \text{ and } r < \frac{p-1}{p-2}(q-1)$$

Maximum gaps, I

- Maximum gap: Given $f(X) = c_1 X^{e_1} + \cdots + c_t X^{e_t} \in \mathbb{Z}[X]$, with $c_i \neq 0$ and $e_1 < \cdots < e_t$, we define the *maximum gap* of f as

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- More managable when turned into a problem involving the maximum gaps of inverse cyclotomic polynomials.

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Claim: $\mathcal{R}_3(x) = o(\mathcal{Q}_3(x))$.

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Thus the claim is true and in particular

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...where Kloosterman and Sister Beiter meet...

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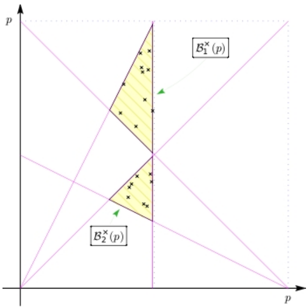

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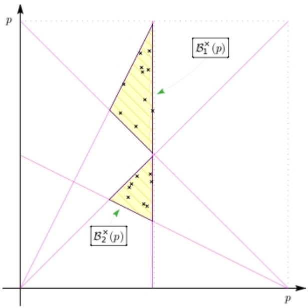

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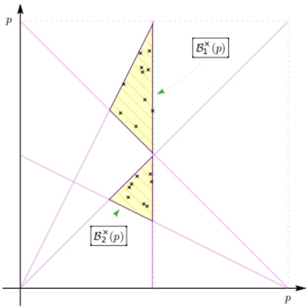

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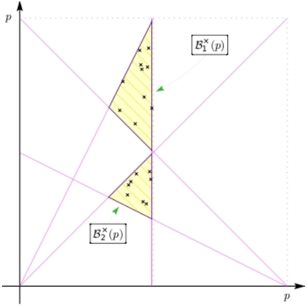

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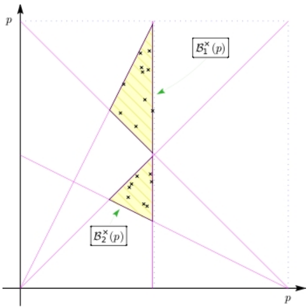

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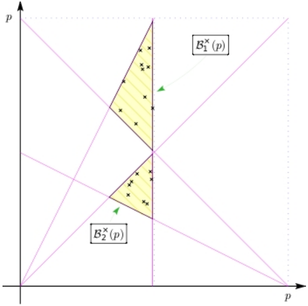

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For p large enough, $M(p) \geq (2/3 - \epsilon)p$.

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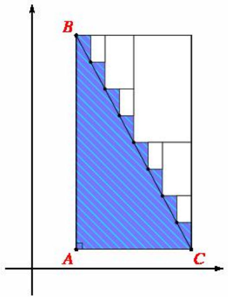

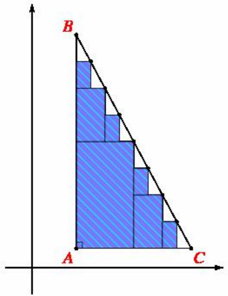

NUMERICAL SEMIGROUP APPROACH

Φ_{pq} and the Frobenius number

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In example: $\Phi_{15}(X) = 1 - X + X^3 - X^4 + X^5 - X^7 + X^8$

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S is symmetric if $S \cup (F(S) - S) = \mathbb{Z}$.

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Let $P_S(X) = a_0 + a_1 X + \cdots + a_k X^k$. Then

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-Let $S = \langle p, q \rangle$. Coefficients gaps are related to gapblocks and elementblocks of S.

Thank you for listening!

Non-zero binary coefficients

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Hence

$$\theta(35) = 2 \cdot 3 \cdot 3 - 1 = 17.$$

Sparse binary cyclotomic polynomials

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-Bombieri-Vinogradov theorem
-Two-dimensional sieve
-Linnik's famous theorem concerning the least prime in AP

The Roşu construction I

Question: Can one construct non-Beiter ternary Φ_n with an optimally large set of coefficients. That is given $l \geq 1$ can one construct Φ_{pqr} with $C(pqr) = [-(p-1)/2 + l, (p+1)/2 + l]$?

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Eugenia
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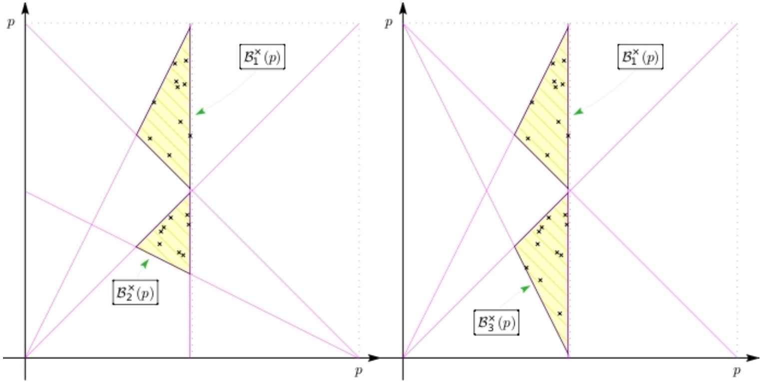
There are likely infinitely many primes p with $M_R(p) > M_{GM}(p)$:
$29, 37, 41, 83, 107, 109, 149, 179, 181, 223, 227, 233, 241, 269 \ldots$

Gallot/Moree versus Roşu $p = 241$



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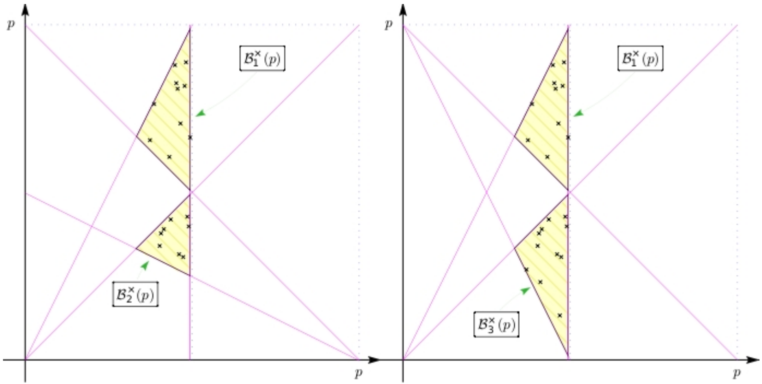

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$B_1(p) = \{x : 1 \le x \le \frac{p-3}{2},\ x + y \ge p,\ y \le 2x\}$

$B_2(p) = \{x : 1 \le x \le \frac{p-3}{2},\ x + 2y + 1 \ge p,\ x > y\}$

$B_3(p) = \{x : 1 \le x \le \frac{p-3}{2},\ 2x + y \ge p,\ x \ge y\}$

Gallot/Moree versus Roşu $p = 29$