An efficient splitting Fourier pseudospectral method for Vlasov-Poisson-Fokker-Planck system

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- Plasma, in physics, is fully ionized gas of low density, containing approximately equal numbers of positive and negative ions.
- Plasma is the most abundant form of ordinary matter in the Universe, most of which is in stars, including the Sun. Both lightning and electric sparks are everyday examples of phenomena made from plasma.

Vlasov-Poisson-Fokker-Planck (VPFP) system

The motion of ions in a plasma in the absence of a magnetic field is governed by the following VPFP equations (E. Allen 1994):

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} = ff, \tag{1.1}$$

$$\mathbf{E} = -\nabla \Phi \tag{1.2}$$

$$-\Delta \Phi = \int f(\mathbf{x}, , \mathbf{v}, t) d\mathbf{v} - h(\mathbf{x}), \tag{1.3}$$

where the collisional term

$$ff = \nabla_{\mathbf{v}} \cdot \int \Psi(\mathbf{v} - \mathbf{v}^*)[\nabla_{\mathbf{v}} f(\mathbf{x}, \mathbf{v}, t) f(\mathbf{x}, \mathbf{v}^*, t) - \nabla_{\mathbf{v}^*} f(\mathbf{x}, \mathbf{v}^*, t) f(\mathbf{x}, \mathbf{v}, t)] d\mathbf{v}^*,$$

and $\Psi(\mathbf{v}) = \frac{\gamma}{|\mathbf{v}|} \left(I - \frac{\mathbf{v} \times \mathbf{v}}{|\mathbf{v}|^2} \right)$.

The collision term can be reformulated as

$$ff = \nabla_{\mathbf{v}} \cdot (A(\mathbf{v}, t)\nabla_{\mathbf{v}} f(\mathbf{x}, \mathbf{v}, t)) - b(\mathbf{v}, t) f(\mathbf{x}, \mathbf{v}, t)), \tag{1.4}$$

where the matrix

$$A(\mathbf{v}, t) = \int \Psi(\mathbf{v} - \mathbf{v}^*) f(\mathbf{x}, \mathbf{v}^*, t) d\mathbf{v}^*,$$

and the vector

$$b(\mathbf{v}, t) = \int \Psi(\mathbf{v} - \mathbf{v}^*) \nabla_{\mathbf{v}^*} f(\mathbf{x}, \mathbf{v}^*, t) d\mathbf{v}^*.$$

Vlasov-Poisson-Fokker-Planck (VPFP) system

In the case that $A(\mathbf{v}, t) \approx \alpha$ and $b(\mathbf{v}, t) \approx= -\beta\mathbf{v}$, equations (1.1)-(1.3) are reduced to (E. Allen 1994):

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} = \beta \nabla_{\mathbf{v}} \cdot (\mathbf{v} f) + \alpha \triangle_{\mathbf{v}} f, \qquad (1.5)$$

$$\mathbf{E} = -\nabla \Phi \qquad (1.6)$$

$$-\Delta \Phi = \rho, \qquad (1.7)$$

where β, α are given constants. $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{v} = (v_1, v_2, v_3)$. $\nabla_{\mathbf{v}} = (\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3})$. $\triangle_{\mathbf{v}} = \frac{\partial^2}{\partial (v_1)^2} + \frac{\partial^2}{\partial (v_2)^2} + \frac{\partial^2}{\partial (v_3)^2}$.
$\mathbf{E} = \mathbf{E}(\mathbf{x}, t) = (E_1(\mathbf{x}, t), E_2(\mathbf{x}, t), E_3(\mathbf{x}, t))$ is the electric field, $\Phi = \Phi(\mathbf{x}, t)$ is the self-consistent electrostatic potential and $f(\mathbf{x}, \mathbf{v}, t)$ is the probability distribution function which describes the probability of finding a particle with velocity $\mathbf{v}$ at position $\mathbf{x}$ at time t. The charge density, $\rho = \rho(\mathbf{x}, t) = \int_{\mathbb{R}^d} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v} - h(\mathbf{x})$, and $h(\mathbf{x})$ is a given known function.
Mathematical analysis (F. Bouchut et al. 1995, L. Bonilla et al., 1997).

Vlasov-Poisson-Fokker-Planck (VPFP) system

They are coupled nonlinear partial derivative differential equations, whose analytical solution is available only in a few simplified linear cases. In the nonlinear regime, the VPFP system must be solved numerically to investigate physical phenomena which are interested.
Development of efficient numerical methods for the VPFP system will be also be interested by those hope to solve Boltzmann equation, Vlasov-Maxwell system and many other coupled kinetic equations.
Development of efficient numerical methods for the VPFP system will be helpful for numerical investigation of plasma physics, fluid dynamics, quantum Bose-Einstein condensates at finite temperature (N. Proukakis, P. Vignolo, B. Jackson).

Existing numerical methods for VPFP system

Three kinds of numerical methods:

- Particle based methods, also called the Lagrangian method. They have the flexibility to treat complex problems and has been widely used. Particle based methods are inherently noisy, which becomes problematic when low density or highly turbulent regions are studied.

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- See book by C.K. Birdsall and and A.B. Langdon 1991 and many other latest reference.

- Semi-Lagrangian methods which have first been introduced in meteorology, try to take advantage of both Lagrangian and Eulerian approaches and have been proposed to solve the VPFP system. Indeed, they allow a relatively accurate description of the phase space using a fixed mesh and avoid traditional step size restriction using the invariance of the distribution function along the trajectories.

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- For example J. Qiu 2010, K. Havlak 1999, N. Crouseilles 2009.

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- Spectral method (see A. Klimas 1983, A. Klimas 1994, B. Eliasson 2002, B. Eliasson 2006, S. Bourdiec 2006).

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- Spectral method (see A. Klimas 1983, A. Klimas 1994, B. Eliasson 2002, B. Eliasson 2006, S. Bourdiec 2006).
- We consider splitting Fourier pseudospectral method. The method has wide applications, see the review paper by W. Bao and Y. Cai, 2013.

Basis of spectral method

The basic idea of spectral method is to assume that the unknown $u(x)$ can be approximated by a sum of $N+1$ "basis functions" $\phi_n(x)$:

$$u_N(x) = \sum_{n=0}^{N} a_n \phi_n(x) \tag{3.1}$$

where the basis functions are given and are orthogonal with respect to some weights $\omega(x)$. When this series is substituted into the equation

$$Lu(x) = f(x) \tag{3.2}$$

where L is some differential operator, the result is the so-called "residual function" defined by

$$R(x, a_0, a_1, \cdots, a_N) = Lu_N - f. \tag{3.3}$$

The different spectral and pseudospectral methods differ mainly in choosing the series coefficients $a_n, i = 0, \cdots, N$ so that the residual function is minimized.
Collocation methods impose the condition so that

$$R(x_i, a_1, a_2, \cdots, a_N) = 0, for \quad i = 0, \cdots, N \tag{3.4}$$

for each collocation point x_i. Here the x_i belongs to some suitable set of interpolation or collocation points.
Galerkin spectral methods are obtained by imposing the condition so that

$$(R(x, a_1, a_2, \cdots, a_N), \phi_i(x))_w = 0, for \quad i = 0, \cdots, N, \tag{3.5}$$

where the inner product is defined by

$$(u, v)_w \equiv \int_a^b u(x)v(x)\omega(x)dx \tag{3.6}$$

for a given non-negative weight function $\omega(x)$ and any two functions $u(x)$ and $v(x)$.

Splitting Fourier pseudospectral method: 1D

We propose a splitting Fourier pseudospectral method for VP system in 1D. The VPFP system in 1D is defined as follows:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial v} = \beta\frac{\partial}{\partial v}(vf) + \alpha\frac{\partial^2}{\partial v^2}f, \tag{4.1}$$

$$E = -\frac{\partial \Phi}{\partial x}, \tag{4.2}$$

$$-\frac{\partial^2 \Phi}{\partial x^2} = \rho, \tag{4.3}$$

where $f = f(x, v, t)$, $E = E(x, t)$, $\Phi = \Phi(x, t)$, $\rho = \int_{-\infty}^{\infty} f(x, v, t)dv$, $h = h(x)$ (here we let $x = x_1$ and $v = v_1$ for simplification).

We consider a periodic plasma with period L, and solve the problem on the bounded phase space $(x, v) \in [0, L] \times [a, b]$ and the unknown functions f, E, and Φ are assumed to satisfy the following periodic boundary conditions:

$$
\begin{aligned}
&f(0, v, t) = f(L, v, t), \quad v \in [a, b], \quad t \geq 0, \\
&f(x, a, t) = f(x, b, t), \quad x \in [0, L], \quad t \geq 0, \\
&E(0, t) = E(L, t), \\
&\Phi(0, t) = \Phi(L, t),
\end{aligned}
$$

where L, a, b are some constants.

When $\beta = \alpha = 0$, it is well known that the one-dimensional system (4.1)-(4.3) with periodic boundary conditions has several invariants with respect to time including

$$N = N(t) = \int_0^L \int_a^b f(x, v, t)\, dxdv, \tag{4.4}$$

$$S = S(t) = \int_0^L \int_a^b f^2(x, v, t)\, dxdv, \tag{4.5}$$

$$P = P(t) = \int_0^L \int_a^b vf(x, v, t)\, dxdv, \tag{4.6}$$

$$W = W(t) = \int_0^L \int_a^b \frac{v^2}{2} f(x, v, t)\, dxdv + \frac{1}{2} \int_0^L E^2(x, t)\, dx,$$

which describe the conservation of the total number of particles, the energy norm, total momentum and total energy, respectively.

Splitting in time

For simplification, we solve Eq. (4.1) with first-order splitting technique. From t_n to t_{n+1}, we discretize Eq. (4.1) with the following first-order splitting technique:
Step 1, Given f_{jk}^n, we solve the equation

$$\frac{\partial f}{\partial t} + E\frac{\partial f}{\partial v} = \beta f + \beta v\frac{\partial}{\partial v}(f) + \alpha\frac{\partial^2}{\partial v^2}(f), \quad t_n \le t \le t_{n+1} \quad (4.7)$$

where $E = E(x, t_n)$, and get the intermediate numerical solution f_{jk}^* for all j and k;
Step 2, Taking the intermediate solution f_{jk}^* as the initial data, we solve the equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0, \quad t_n \le t \le t_{n+1}, \quad (4.8)$$

and get the numerical solution f_{jk}^{n+1} for all j and k at time t_{n+1}.

Step 1

Fourier collocation method is applied in velocity direction. Why not Fourier Galerkin method?

To do this, we assume the following Fourier spectral expansion in v-direction

$$f(x,v,t) = \sum_{j=0}^{N-1} f(x,v_j,t) H_j\left(\frac{2\pi(v-a)}{b-a}\right), \tag{4.9}$$

where $H_j(v) = \frac{1}{N}\sin[N\frac{v-v_j}{2}]\cot[\frac{v-v_j}{2}]$ satisfies

$$H_j(v_k) = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}.$$

Then, according to Fourier collocation method, $\frac{\partial f}{\partial v}$ at grid point v_k can be evaluated as (see J. Hesthaven et al., 2007)

$$\frac{\partial f}{\partial v}(x, v, t)|_{v=v_k} = \sum_{j=0}^{N-1} f(x, v_j, t) D_{jk}^{(1)}, \tag{4.10}$$

Similarly, $\frac{\partial^2 f}{\partial v^2}$ at grid point v_k can be evaluated as

$$\frac{\partial^2 f}{\partial v^2}(x, v, t)|_{v=v_k} = \sum_{j=0}^{N-1} f(x, v_j, t) D_{jk}^{(2)}, \tag{4.11}$$

where $D_{jk}^{(2)}$ is element of matrix D^2.

Plugging both (4.10) and (4.11) into (4.7), with collocation method in mind, we get the following matrix formulation

$$\frac{\partial}{\partial t}F + \Lambda_1 F = \left(\beta I + \beta \Lambda_2 D + \alpha D^2\right) F \tag{4.12}$$

where the column vector
$F = (f(x, v_0, t), f(x, v_1, t), \cdots, f(x, v_{N-1}, t))^T$, the diagonal matrix $\Lambda_1 = E(x, t_n)I$ with I being a $N \times N$ identity matrix, and the diagonal matrix $\Lambda_2 = \mathrm{diag}(v_0, v_1, \cdots, v_{N-1})$. Eq. (4.12) can be reduced to

$$\frac{\partial}{\partial t}F = BF, \quad B = -\Lambda_1 + \beta I + \beta \Lambda_2 D + \alpha D^2. \tag{4.13}$$

Using the method of matrix factorization, we can factorize the matrix B into $P\Lambda P^{-1}$ where Λ is a diagonal matrix of $N \times N$, P and its inverse are $N \times N$ matrix, then from the above Eq. (4.13), we can get

$$\begin{aligned}
&\frac{\partial}{\partial t}F = P\Lambda P^{-1}F, \\
&P^{-1}\frac{\partial}{\partial t}F = \Lambda P^{-1}F, \\
&\frac{\partial}{\partial t}(P^{-1}F) = \Lambda(P^{-1}F), \\
&\frac{\partial}{\partial t}\tilde{F} = \Lambda\tilde{F}, \quad \tilde{F} = P^{-1}F,
\end{aligned} \tag{4.14}$$

The above Eq. (4.14) can be decoupled into ordinary differential equations for $\tilde{F}$, which can be solved exactly.

Thus, we get the following spectral algorithm for this step:
For any fixed x_j $(0 \leq j \leq M)$ and t_n $(0 \leq n \leq P)$

- Start from $f(x_j, v_k, t_n)$ $(0 \leq k \leq N-1)$ and define $F = (f(x_j, v_0, t_n), f(x_j, v_1, t_n), \cdots, f(x_j, v_{N-1}, t_n))^T$,
- Factorize matrix B into $P\Lambda P^{-1}$ and compute $\tilde{F} = P^{-1}F$,
- Solve the decoupled Eq. (4.14) exactly over $[t_n, t_{n+1}]$ and get $\tilde{F}$,
- Compute $F = P\tilde{F}$ and obtain F at time $t = t_{n+1}$.

Step 2

We take Fourier spectral approximation in spatial direction. To do that, we assume that $f(x, v, t)$ has the following Fourier spectral approximation in x-direction

$$f(x, v, t) = \sum_{m=-M/2}^{M/2-1} \hat{f}_m(v, t) e^{i\mu_m x}. \tag{4.15}$$

Plugging (4.15) into Eq. $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$ gives us

$$\left(\sum_{m=-M/2}^{M/2-1} \hat{f}_m(v, t) e^{i\mu_m x} \right)_t + v \sum_{m=-M/2}^{M/2-1} i\mu_m \hat{f}_m(v, t) e^{i\mu_m x} = 0. \tag{4.16}$$

Furthermore, we can obtain

$$\frac{\partial}{\partial t} \hat{f}_m(v, t) + i\mu_m v \hat{f}_m(v, t) = 0. \tag{4.17}$$

Thus, we get the following spectral algorithm:
For any fixed $k(0 \leq k \leq N)$ and n $(0 \leq n \leq P)$

- Starting from $f(x_j, v_k, t_n)$ $(0 \leq j \leq M-1)$, do the following one-dimensional forward FFT in x-direction, we get $\hat{f}_m(v_k, t_n)$ for all $-M/2 \leq k \leq M/2-1$, i.e.,

$$\hat{f}_m(v_k, t_n) = \frac{1}{M} \sum_{j=0}^{M-1} f(x_j, v_k, t_n) e^{-i\mu_m x_j}.$$

- Second, from Eq. (4.17), we compute $\hat{f}_m(v_k, t_{n+1})$ for all m $(-M/2 \leq m \leq M/2-1)$.
- Finally, doing the one-dimensional backward FFT in x-direction,i.e.,

$$f(x_j, v_k, t_{n+1}) = \sum_{m=-M/2}^{M/2-1} \hat{f}_m(v_k, t_{n+1}) e^{i\mu_m x_j},$$

we obtain $f(x_j, v_k, t_{n+1})$ $(0 \leq j \leq M-1)$ from $\hat{f}_m(v_k, t_{n+1})$.

Splitting Fourier pseudospectral method: 2D

We propose a splitting Fourier pseudospectral method for VP system in 2D. The VPFP system in 2D is defined as follows:

$$\begin{aligned}
&\frac{\partial f}{\partial t} + v_1\frac{\partial f}{\partial x_1} + v_2\frac{\partial f}{\partial x_2} + E_1\frac{\partial f}{\partial v_1} + E_2\frac{\partial f}{\partial v_2} = \beta\nabla_{\mathbf{v}}\cdot(\mathbf{v}f) + \alpha\triangle_{\mathbf{v}}f,\\
&E_1 = -\frac{\partial\Phi}{\partial x_1}, E_2 = -\frac{\partial\Phi}{\partial x_2},\\
&-\Delta\Phi = \rho,
\end{aligned} \tag{5.1}$$

where $f = f(x_1, x_2, v_1, v_2, t)$, $E_1 = E_1(x_1, x_2, t)$, $E_2 = E_2(x_1, x_2, t)$, $\Phi = \Phi(x_1, x_2, t)$. $\nabla_{\mathbf{v}} = (\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2})$. $\triangle_{\mathbf{v}} = \frac{\partial^2}{\partial(v_1)^2} + \frac{\partial^2}{\partial(v_2)^2}$.

$$\rho = \int_a^b \int_c^d f(x_1, x_2, v_1, v_2, t)dv_1 dv_2 - h(x_1, x_2).$$

Splitting in time

To solve the equation (5.1), we take the following first-order splitting steps:

Step 1, Given f^n_{jklm}, we solve the equation

$$\frac{\partial f}{\partial t} + E_1 \frac{\partial f}{\partial v_1} + E_2 \frac{\partial f}{\partial v_2} = \beta \nabla_{\mathbf{v}} \cdot (\mathbf{v} f) + \alpha \triangle_{\mathbf{v}} f, \tag{5.2}$$

or

$$\frac{\partial f}{\partial t} + E_1 \frac{\partial f}{\partial v_1} + E_2 \frac{\partial f}{\partial v_2} = 2\beta f + v_1 \frac{\partial f}{\partial v_1} + v_2 \frac{\partial f}{\partial v_2} + \alpha \triangle_{\mathbf{v}} f, \tag{5.3}$$

with Fourier collocation method and get the intermediate numerical solution f^*_{jklm}. Here we define $E_1 = E(x_1, x_2, t_n)$ and $E_2 = E(x_1, x_2, t_n)$;

Step 2, Taking the intermediate solution f^*_{jklm} as the initial data, we solve the equation

$$\frac{\partial f}{\partial t} + v_1 \frac{\partial f}{\partial x_1} + v_2 \frac{\partial f}{\partial x_2} = 0 \tag{5.4}$$

with Fourier Galerkin method and get the numerical solution f^{n+1}_{jklm}.

Step 1 in 2D

In Step 1, we solve the equation (5.2) from t_n to t_{n+1} by Fourier collocation method. For shortness of notation, we let $u(v_1, v_2, t) = f(x_1, x_2, v_1, v_2, t)$, then Eq. (5.2) can be reduced to

$$\frac{\partial u}{\partial t} = 2\beta u + (-E_1 + \beta v_1)\frac{\partial u}{\partial v_1} + (-E_2 + \beta v_2)\frac{\partial u}{\partial v_2} + \alpha \triangle u, \quad (5.5)$$

We assume the following Fourier spectral expansion in $v = (v_1, v_2)$-direction

$$u(v_1, v_2, t) = \sum_{j,m} u((v_1)_j, (v_2)_k, t) H_j\left(\frac{2\pi(v_1 - a)}{b - a}\right) \tilde{H}_k\left(\frac{2\pi(v_2 - c)}{d - c}\right), \quad (5.6)$$

where

$$H_j(v_1) = \frac{1}{N_1} \sin[N_1 \frac{v_1 - (v_1)_j}{2}] \cot[\frac{v_1 - (v_1)_j}{2}]$$

and

$$\tilde{H}_k(v_2) = \frac{1}{N_2} \sin[N_2 \frac{v_2 - (v_2)_k}{2}] \cot[\frac{v_2 - (v_2)_k}{2}].$$

Then, according to Fourier collocation method, $\frac{\partial u}{\partial v_1}$ and $\frac{\partial u}{\partial v_2}$ at grid points $((v_1)_I, (v_2)_m)$ can be evaluated as

$$\frac{\partial u}{\partial v_1}|_{((v_1)_I,(v_2)_m)} = \sum_{j=0}^{N_1-1} u((v_1)_j, (v_2)_m, t) D_{jI}^{(1)},$$

$$\frac{\partial u}{\partial v_2}|_{((v_1)_I,(v_2)_m)} = \sum_{k=0}^{N_2-1} u((v_1)_I, (v_2)_k, t) \tilde{D}_{km}^{(1)}, \qquad (5.7)$$

where
Similarly, $\frac{\partial^2 u}{\partial (v_1)^2}$ and $\frac{\partial^2 u}{\partial (v_2)^2}$ at grid points $((v_1)_I, (v_2)_m)$ can be evaluated as

$$\frac{\partial^2 u}{\partial (v_1)^2}|_{((v_1)_I,(v_2)_m)} = \sum_{j=0}^{N_1-1} u((v_1)_j, (v_2)_m, t) D_{jI}^{(2)},$$

$$\frac{\partial^2 u}{\partial (v_2)^2}|_{((v_1)_I,(v_2)_m)} = \sum_{k=0}^{N_2-1} u((v_1)_I, (v_2)_k, t) \tilde{D}_{km}^{(2)}, \qquad (5.8)$$

where $D^{(2)}_{jk}$ is element of matrix D^2. According to collocation method, Eq. (5.5) is collocated at grid points $((v_1)_I, (v_2)_m)$ ($I = 0, 1, \cdots, N_1 - 1$ and $m = 0, 1, \cdots, N_2 - 1$), i.e., Furthermore, plugging both (5.7) and (5.8) into (5.5), we get the following matrix formulation

$$\frac{\partial}{\partial t} U = B_1 U + U B_2, \quad , \tag{5.9}$$

where $B_1 = \beta I_1 - \Lambda_1 \Theta_1 + \beta V_1 \Theta_1 + \alpha \Gamma_1$,
$B_2 = \beta I_2 - \Lambda_2 \Theta_2 + \beta V_2 \Theta_2 + \alpha \Gamma_2$.

Here, we have defined the matrix U as $(u((v_1)_j, (v_2)_k)$ for all $0 \leq j \leq N_1 - 1$, $0 \leq k \leq N_2 - 1$. I_j the $N_j \times N_j$ identity matrix $(j = 1, 2)$. $\Lambda_1 = (E_1(x_1, x_2, t_n)I_1$, $\Lambda_2 = (E_2(x_1, x_2, t_n)I_2$, $N_1 \times N_1$ matrix $\Theta_1 = D$, $N_2 \times N_2$ matrix $\Gamma_2 = D$, $N_2 \times N_2$ matrix $\Theta_2 = \tilde{D}^2$, $N_1 \times N_1$ matrix $\Gamma_2 = \tilde{D}^2$. $N_1 \times N_1$ diagonal matrix $V_1 = diag((v_1)_0, (v_1)_1, \cdots, (v_1)_{N_1-1})$ and $N_2 \times N_2$ diagonal matrix $V_2 = diag((v_2)_0, (v_2)_1, \cdots, (v_2)_{N_2-1})$.

Using the method of matrix factorization again, we can factorize the matrices B_1 and B_2 in Eq. (5.9) into

$$B_1 = P\Lambda_x P^{-1}, \quad B_2 = Q\Lambda_y Q^{-1} \tag{5.10}$$

Then Eq. (5.9) can be reduced to

$$\frac{\partial}{\partial t}U = [P\Lambda_x P^{-1}U + U(Q\Lambda_y Q^{-1})^T]. \tag{5.11}$$

Multiplying the above equation with matrices P^{-1} and $(Q^T)^{-1}$, we get

$$\frac{\partial}{\partial t}[P^{-1}U(Q^T)^{-1}] = \Lambda_x(P^{-1}U(Q^T)^{-1}) + P^{-1}U(Q^T)^{-1}\Lambda_y. \tag{5.12}$$

If we define $\tilde{U} = P^{-1}U(Q^T)^{-1}$, then Eq. (5.12) can be reduced to the following decoupled ordinary differential equations

$$\frac{\partial}{\partial t}\tilde{U} = \Lambda_x\tilde{U} + \tilde{U}\Lambda_y, \tag{5.13}$$

which can also be solved exactly.

Thus, we get the following spectral algorithm:
Let $u(v_1, v_2, t) = f(x_1, x_2, v_1, v_2, t)$. For any fixed j, k $(0 \leq j \leq M_1,\ 0 \leq k \leq M_2)$ and n $(0 \leq n \leq P)$

- Start from $u((v_1)_l, (v_2)_m, t_n)$ $(0 \leq l \leq N_1 - 1,$ $0 \leq m \leq N_2 - 1)$ and define U,
- Factorize matrix B_1 into $P\Lambda P^{-1}$ and matrix B_2 into $Q\Lambda Q^{-1}$ and compute $\tilde{U} = P^{-1}U(Q^T)^{-1}$,
- Solve the decoupled Eq. (5.13) exactly over $[t_n, t_{n+1}]$ and get $\tilde{U}$ at time t_{n+1}.
- Compute $U = P\tilde{U}Q^T$ and obtain U at time $t = t_{n+1}$.

Step 2 in 2D

In Step 2, we assume the following Fourier spectral expansion

$$f(x_1, x_2, v_1, v_2, t) = \sum_{p=-M_1/2}^{M_1/2-1} \sum_{q=-M_2/2}^{M_2/2-1} \hat{f}_{pq}(v_1, v_2, t) e^{i\mu_p x_1 + i\lambda_q x_2}. \tag{5.14}$$

where $\mu_p = \frac{2\pi p}{L_1}$ and $\lambda_q = \frac{2\pi q}{L_2}$ Plugging (5.14) into Eq. (5.4), we can get

$$\left(\sum_{p=-M_1/2}^{M_1/2-1} \sum_{q=-M_2/2}^{M_2/2-1} \hat{f}_{pq}(v_1, v_2, t) e^{i\mu_p x_1 + i\lambda_q x_2} \right)_t$$
$$+ \sum_{p=-M_1/2}^{M_1/2-1} \sum_{q=-M_2/2}^{M_2/2-1} i(v_1\mu_p + v_2\lambda_q) \hat{f}_{pq}(v_1, v_2, t) e^{i\mu_p x_1 + i\lambda_q x_2} = 0.$$

Furthermore, from the above equation, we reach

$$\frac{\partial}{\partial t} \hat{f}_{pq}(v_1, v_2, t) + i(v_1\mu_p + v_2\lambda_q) \hat{f}_{pq}(v_1, v_2, t) = 0. \tag{5.15}$$

we can get the similar spectral algorithm as those presented in 1D.

Extension to solve the equations (1.1)-(1.3)

we use the following first-order splitting steps:
Step 1, Given f^n, we solve the equation

$$\frac{\partial f}{\partial t}+E_1\frac{\partial f}{\partial v_1}+E_2\frac{\partial f}{\partial v_2}=\nabla\cdot(A(\mathbf{v},t_n)\nabla_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t))-b(\mathbf{v},t_n)f(\mathbf{x},\mathbf{v},t)), \tag{5.16}$$

with Fourier collocation method and get the intermediate numerical solution f^*. Here we define $E_1 = E(\mathbf{x}, t_n)$ and $E_2 = E(\mathbf{x}, t_n)$;
Step 2, Taking the intermediate solution f^* as the initial data, we solve the equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{v}}f=0 \tag{5.17}$$

with Fourier Galerkin method and get the numerical solution f^{n+1}.

A theorem

Theorem

The proposed splitting method keep the total number numerically, i.e., for any n, we have

$$||f(\mathbf{x}, \mathbf{v}, t_n)|| = ||f(\mathbf{x}, \mathbf{v}, t_{n+1})||. \tag{5.18}$$

We remark that L. Pareschi et al. proposed Fourier Galerkin method for computing the collision term (1.3) in 2000. However they need solve the resulting highly nonlinear ordinary equations with some modified Runge-Kutta methods. It remains unclear that the proposed method can keep the total number numerically.

Numerical results

In Example 1, we consider $\beta = \alpha = 0$, the VPFP system in 1D is reduced to the VP system in 1D. We take the initial data as $f(x, v, t = 0) = \frac{1}{\sqrt{2\pi}} v^2 e^{-\frac{v^2}{2}} (1 + a\cos(kx))$, $a = 0.01$, $k = 0.5$ and investigate the well-known two-stream-instability problem on the domain $(x, v) \in [0, 4\pi] \times [-6, 6]$. Figure 1 shows us time evolution of the total number $N(t)$, the energy norm $S(t)$, the total momentum $(P(t))$ and the total energy $W(t)$, respectively. From this Figure, we can see these five conservation laws related to the VP system in 1D are kept very well.

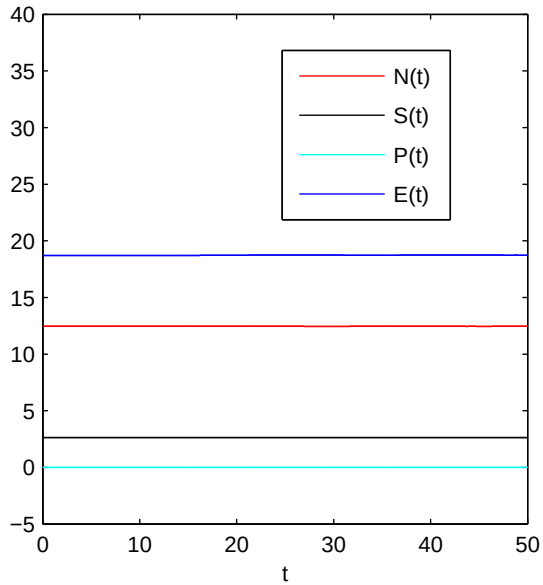
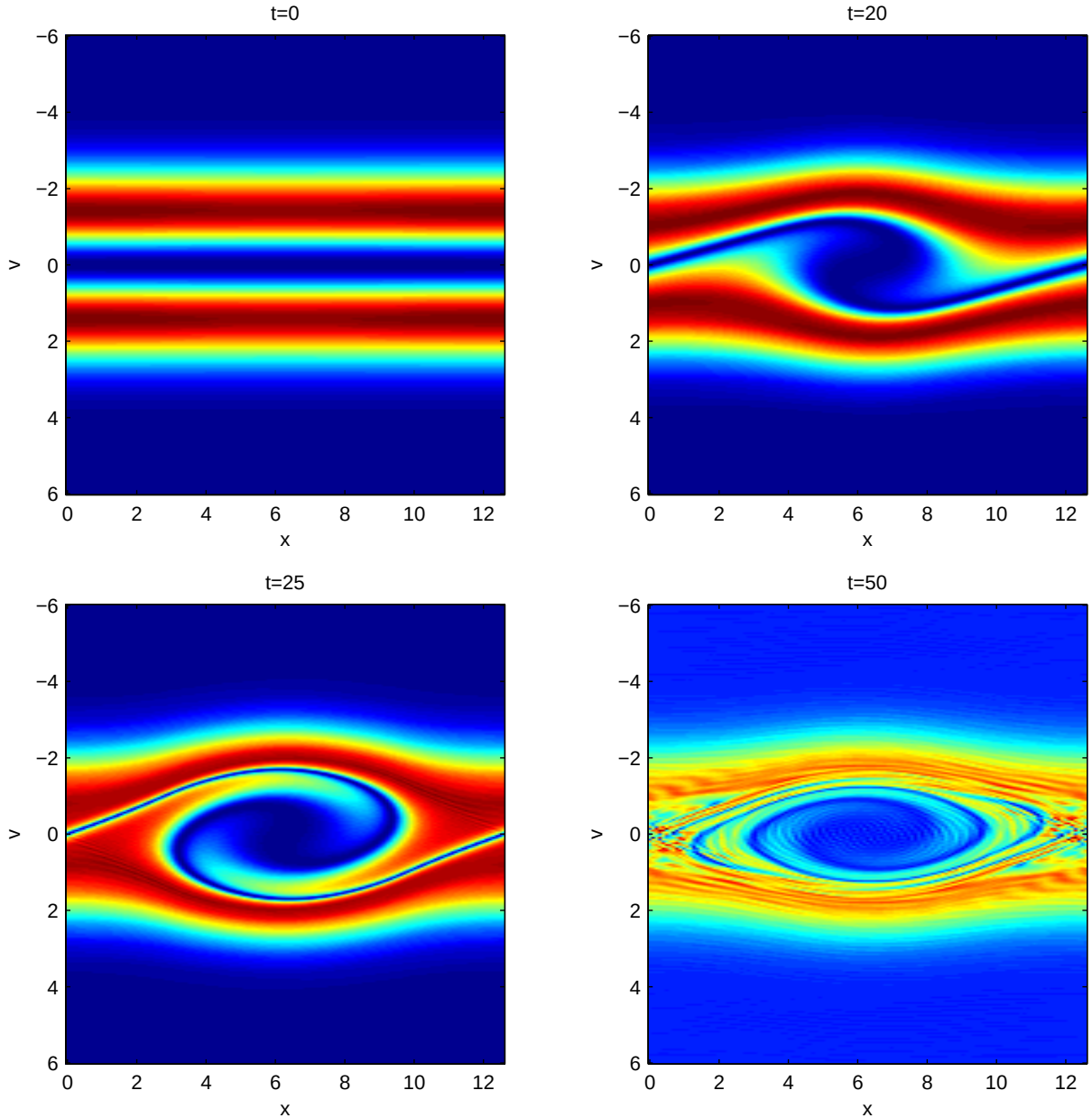

40
35
30
25
20
15
10
5
0
−5
0
10
20
30
40
50
t
N(t)
S(t)
P(t)
E(t)

Figure 2 shows us image plots for the density function $f(x, v, t)$ at different times.



In Example 2, we take the initial data as $f(x, v, t = 0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} (1 + a\cos(kx)), a = 0.5, k = 0.5$ and investigate the well-known strong-Landau damping problem on the domain $(x, v) \in [0, 4\pi] \times [-6, 6]$. Figure 3 shows us time evolution of the total number of electrons $N(t)$, the energy norm $S(t)$, the total momentum $P(t)$ and the total energy $W(t)$, respectively. From this Figure, we can see the conservation laws related to the VP system in 1D are kept well numerically.

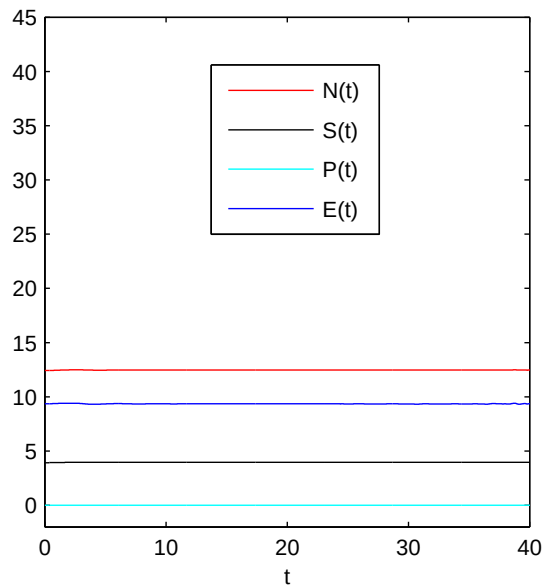
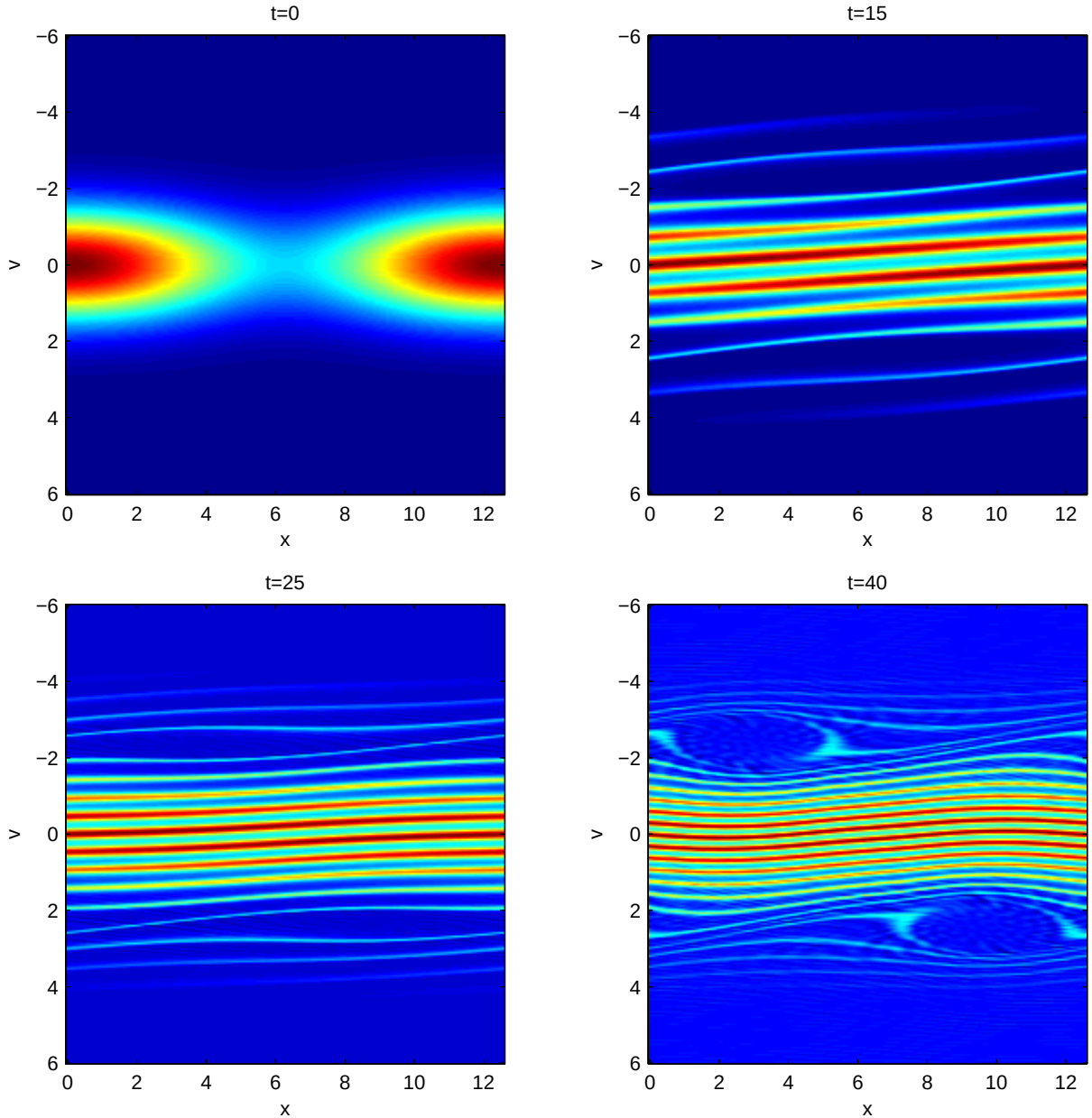


Figure 39 shows us image plots for the density function $f(x, v, t)$ at different times.



In Example 3, we consider the VPFP system in 1D where $\beta = 0.01$ and $\alpha = 0.05\beta$ and we take the initial data as $f(x, v, t = 0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} (1 + a\cos(kx)), a = 0.5, k = 0.5$ on the domain $(x, v) \in [0, 4\pi] \times [-6, 6]$. Figure 4 shows us time evolution of the total number $N(t)$, the energy norm $S(t)$, the total momentum $P(t)$ and the total energy $W(t)$, respectively.

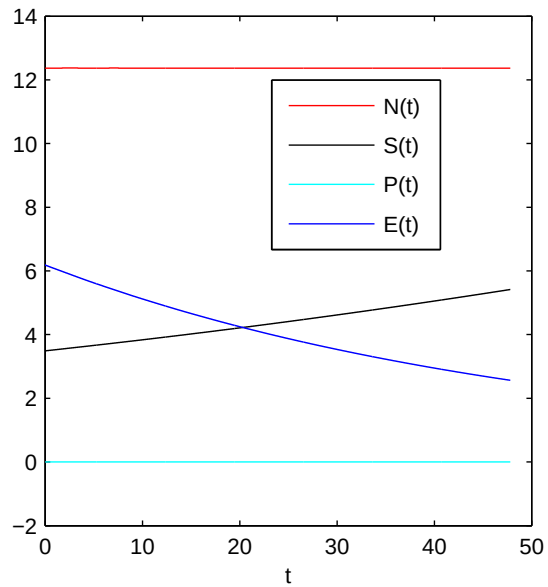
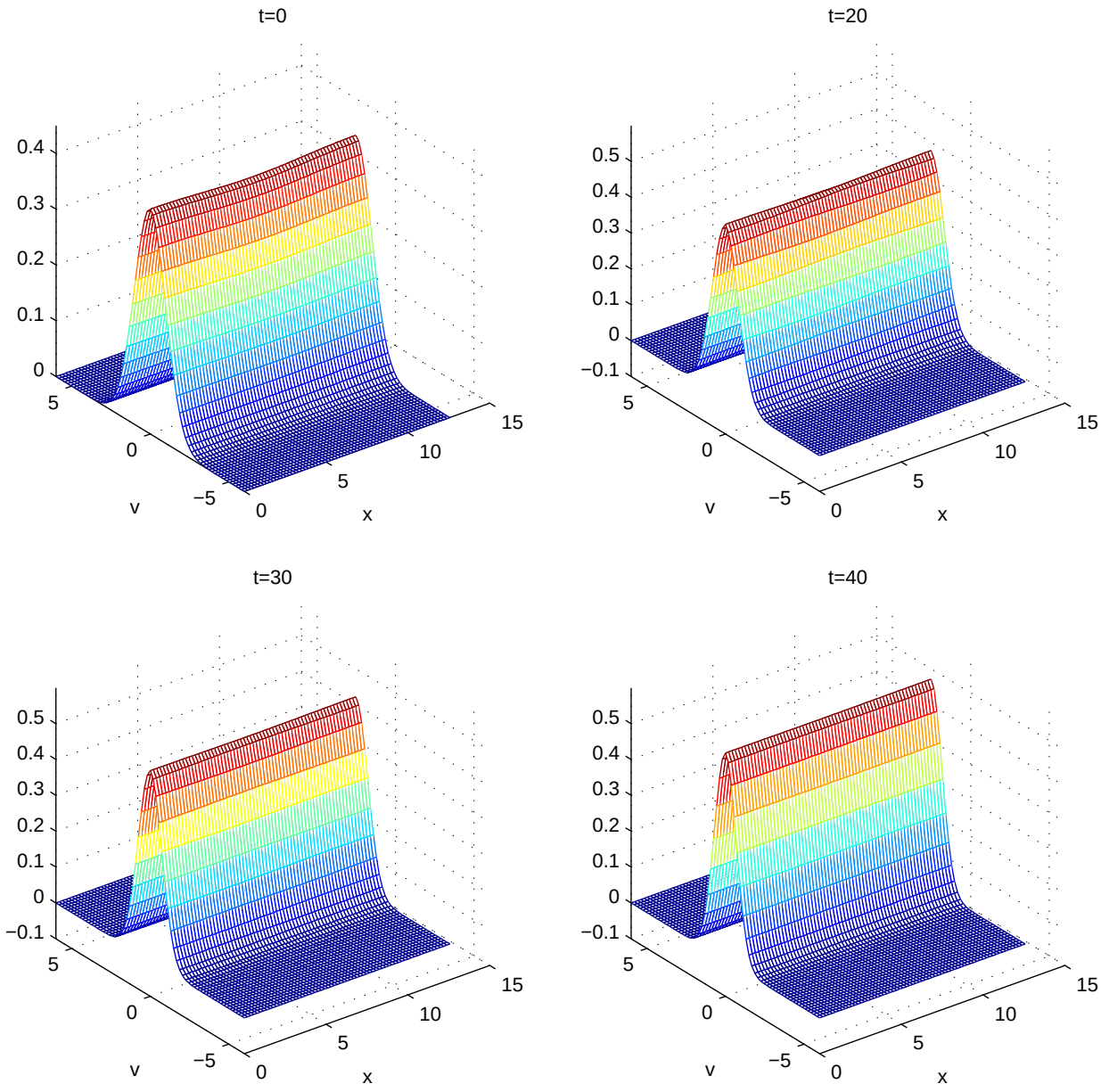
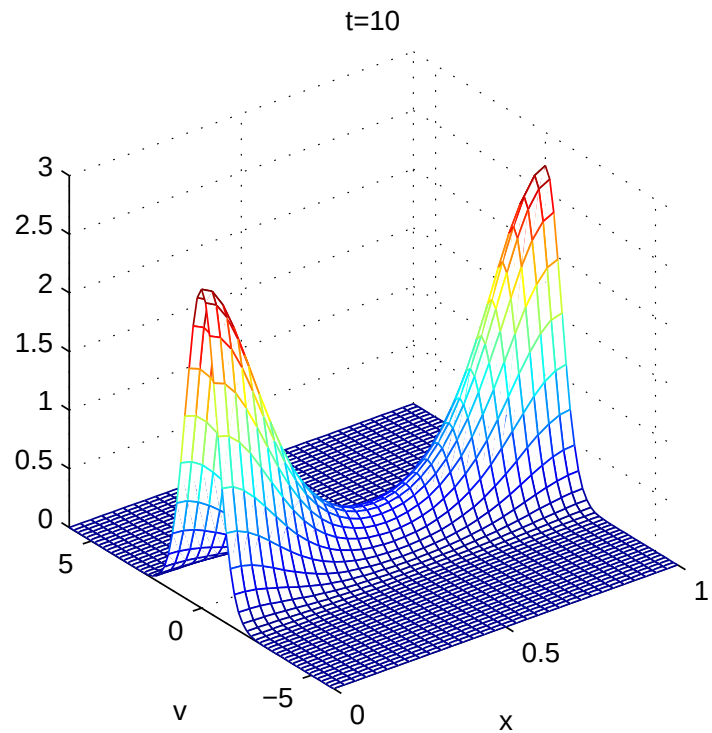


Figure 41 shows us image plots for the density function $f(x, v, t)$ at different times.



In Example 4, we consider the VPFP system in 1D where $\beta = 0.1$ and $\alpha = 0.5\beta$ and we take the initial data as $f(x, v, t = 0) = e^{cos(2*pi*x)-0.5\beta/\alpha v^2}$; and investigate its long time solution on the domain $(x, v) \in [0, 1] \times [-6, 6]$.
Figure 42 shows us image plots for the density function $f(x, v, t)$ at different times.



Conclusions and discussions

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- Numerical results have been shown that the method can keep conservation laws and reproduce the relevant physical phenomena.
- The method might be able to extended to other basis functions such as Chebyshev polynomials, Legendre polynomials as well Hermit polynomials.

Thank you very much for your attention!