Numerical methods on simulating dynamics of the nonlinear Schrödinger equation with rotation and/or nonlocal interactions

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Outline

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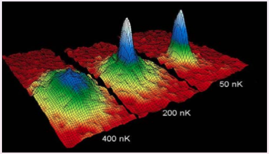
Gross-Pitaevskii/Nonlinear Schrödinger equations

$$i\partial_t \psi(\mathbf{x},t) = -\frac{1}{2}\nabla^2\psi + V(\mathbf{x})\psi + \beta\,|\psi|^2\psi, \qquad \mathbf{x}\in\mathbb{R}^d, \quad t\ge 0. \qquad (1)$$

- $V(\mathbf{x})$: trapping potential.
- β : const. charactering short-range interaction.
- Energy

$$\mathcal{E}(t) = \int_{\mathbb{R}^d}\left[\frac{1}{2}|\nabla\psi|^2 + V(\mathbf{x})|\psi|^2 + \frac{\beta}{2}|\psi|^4\right]d\mathbf{x} \equiv \mathcal{E}(0) \quad (2)$$



BEC @ JILA

- Well studied:
  - Theoretical aspects: T. Cazenave, 03; C. Sulem & P.L. Sulem, 99', E. H. Lieb, R. Seiringer, R. Carl, P. Degond, W. Bao, Y. Cai, H. T. Yau, E. Grenier, F. Poupaud, C. Sparber, B. Guo, C. Miao, P. A. Markowich, P.L. Lious,......
  - Numerical aspects: G. D. Akrivis, C. Bess, X. Antoine, R. Duboscq, I. Danaila, Q. Du, Y. Zhang, H. Wang, Y. Cai T. F. Chan, Q. S. Chang, V. A. Dougalis, L. J. Shen, E. Jia, D. F. Griffiths, M. Delfour, M. Fortin, G. Payre, P. Markowich, S. Jin,......

Existing numerics

- Time splitting spectral method (TSSP): (Bao, Du, Jin, Bess, Markowich, etc)

$$\text{Step 1}: \; i\partial_t \psi = -\frac{1}{2}\nabla^2\psi, \qquad \text{Step 2}: \; i\partial_t \psi = \left(V + \beta|\psi|^2\right)\psi. \tag{3}$$

- **Step 1**: discretised by spectral method and integrated in phase space exactly.
- **Step 2**: nonlinear ODE integrated analytically. ($\rho = |\psi(\mathbf{x}, t)|^2$, $\frac{d\,\rho}{dt} \equiv 0$).

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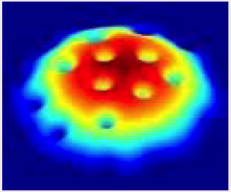
- Good Properties:

  - very easy to implement, explicit, unconditionally stable.
  - mass conserved, time reversible, time transverse invariant.
  - spectral order in spatial, easy to extend to higher order in time.

BEC in rotating frame: generation of vortex lattices

(a) 

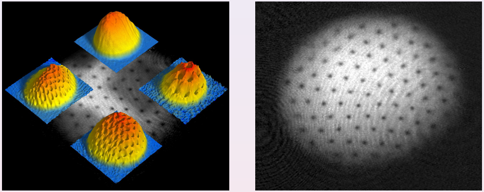
(b) 

Figure: (a) BEC at ENS (b) BEC at MIT

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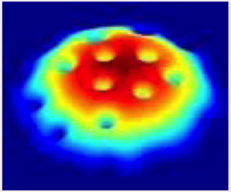
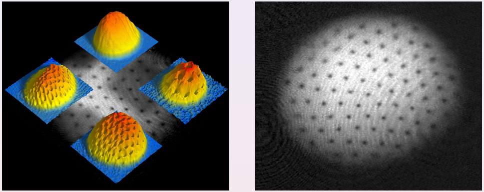
(a)  (b) 

Figure: (a) BEC at ENS (b) BEC at MIT

NLSE/GPE with rotation term

$$i\partial_t\psi(\mathbf{x},t) = \left[-\frac{1}{2}\nabla^2 + V(\mathbf{x}) + \beta\,|\psi|^2 - \omega\,L_z\right]\psi(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^d \tag{4}$$

- $d = 2, 3$, ω : rotating frequency.
- $L_z = -i(x\partial_y - y\partial_x)$: z-component of the angular momentum $\mathbf{L} = \mathbf{x}\times(-i\nabla)$

Numerical methods for rotating GPE/NLSE

Difficulty located in rotating term

$$L_z = -i(x\partial_y - y\partial_x)$$

TSSP Cannot be simply extended to NLSE with rotation term!

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Some existing numerical methods

- Time splitting + ADI[a]: extra splitting error, non-trivial to extent to higher order in time
- Time splitting + polar/cylindrical coordinates[b]: low order accuracy in radial direction.
- Time splitting + Laguerre-Fourier-Hermite[c]: not easy to be implemented

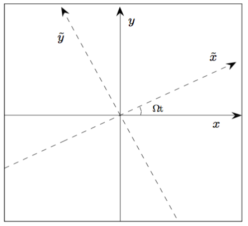
[a] Bao and Wang, JCP, 2006.
[b] Bao, Du and Zhang, SIAP, 2006.
[c] Bao, Li and Shen, SISC, 2009.

A rotating Lagrangian coordinate transformation[1]

- $\widetilde{\mathbf{x}} = \mathbf{A}^T(t)\mathbf{x}$ & $\phi(\widetilde{\mathbf{x}}, t) := \psi(\mathbf{x}, t) = \psi\left(\mathbf{A}(t)\widetilde{\mathbf{x}}, t\right)$,

$$\mathbf{A}(t) = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}, \quad \text{for} \quad \mathrm{d} = 2$$

$$\mathbf{A}(t) = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{for} \quad \mathrm{d} = 3.$$



- GPE in rotating Lagrangian coordinate

$$i\partial_t \phi(\widetilde{\mathbf{x}}, t) = \left[-\frac{1}{2}\nabla^2 + V(\mathbf{A}(t)\widetilde{x}) + \beta|\phi|^2\right]\phi(\widetilde{\mathbf{x}}, t), \qquad \widetilde{\mathbf{x}} \in \mathbb{R}^d. \tag{5}$$

TSSP

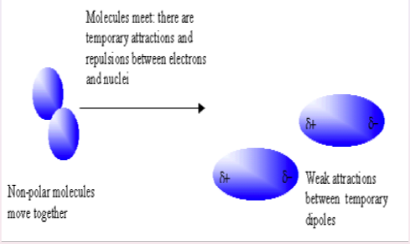
$$\text{Step 1}: \; i\partial_t \phi = -\frac{1}{2}\nabla^2\phi, \qquad \text{Step 2}: \; i\partial_t \phi(\widetilde{\mathbf{x}}, t) = \left[V(\mathbf{A}(t)\widetilde{x}) + \beta|\phi|^2\right]\phi. \tag{6}$$

[1] Bao, Marahrens, Tang & Zhang, SISC, 13; J.Ming, Tang & Zhang, JCP, 13

Dynamics of Vortex lattice

Recent development in dipolar BEC

- Dipolar BEC: degenerate dipolar quantum gas[2]



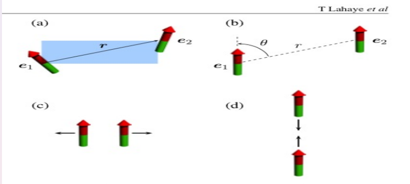
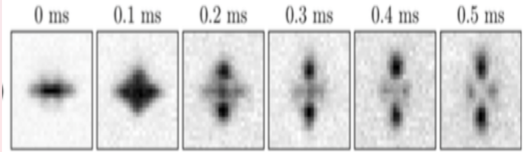


Figure 2. Two particles interacting via the dipole–dipole interaction. (a) Non-polarized case; (b) polarized case; (c) two polarized dipoles side by side repel each other (black arrows); (d) two polarized dipoles in a 'head-to-tail' configuration attract each other (black arrows).

- **Novel phenomena:** e.g. Collapse dynamics[3] —partially attractive & repulsive nature of DDI



[2] A. Griesmaier et al, PRL, 05'; M. Lu et al PRL, 11'; K. Aikawa et al, PRL, 12'.

[3] The physics of dipolar bosonic quantum gases, T Lahaye et al, Rep. Prog. Phys., 09'

NLSE with nonlocal potential

$$i\partial_t\psi = \left[-\frac{1}{2}\nabla^2 + V(\mathbf{x}) + \beta|\psi|^2 + \lambda\Phi\right]\psi, \quad (7)$$

$$\Phi = \mathcal{U} * \rho := \mathcal{U} * |\psi|^2, \qquad \mathbf{x} \in \mathbb{R}^d. \quad (8)$$

where

$$\mathcal{U}(\mathbf{x}) = \begin{cases} -(\mathbf{n}\cdot\mathbf{n})\delta(\mathbf{x}) - 3\,\partial_{\mathbf{nn}}\left(\frac{1}{4\pi|\mathbf{x}|}\right), & 3D \text{ Dipolar}, \\ \frac{1}{2^{d-1}\pi}\,\frac{1}{|\mathbf{x}|}, & 2D/3D \text{ Coulomb}, \\ -\frac{1}{2\pi}\ln|\mathbf{x}|, & 2D \text{ Poisson}. \end{cases} \quad (9)$$

Fields of common interests

- Bose-Einstein Condensates (Dipolar), Many-body system (Coulomb, Poisson)
- Computational physics, chemistry, Density function theory, Surface physics etc.

Dynamics simulation

- Numerics: Time splitting method + Spectral method
- Key algorithm: Effective potential evaluation of Φ with fast-decaying smooth ρ

Difficulty in evaluation of Φ

(1) Nonlocal (2) Singularity: $\mathcal{U}(\mathbf{x})$ at $\mathbf{x} = 0$ and $\widehat{\mathcal{U}}(\mathbf{k})$ at $\mathbf{k} = 0$

Some existing solvers for the nonlocal potential

- Fast Fourier Transform (FFT)[a]
- PDE/pseudo differential equation approach: boundary condition[b]
- NonUniform FFT (NUFFT) based method: Fourier Transform[c]

[a] Ueda et al, Phys. Rev. Lett., 08
[b] Bao et al, JCP 10'; Zhang et al, JCP 11'; Zhang et al, CiCP, 14'
[c] Bao et al SISC 14'; Tang et al, JCP,15', CiCP 16'

NUFFT-based method

Ideas: Fourier transform in spherical/polar coordinates

$$\Phi(\mathbf{x}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\,\mathbf{k}\cdot\mathbf{x}}\,\widehat{\mathcal{U}}(\mathbf{k})\,\widehat{\rho}(\mathbf{k})\,d\mathbf{k} \approx \frac{1}{(2\pi)^d}\int_{|\mathbf{k}|\le P} e^{i\mathbf{k}\cdot\mathbf{x}}\,\widehat{\mathcal{U}}(\mathbf{k})\,\widehat{\rho}(\mathbf{k})\,d\mathbf{k}$$

$$= \frac{1}{(2\pi)^d}\begin{cases}\int_0^P\int_0^\pi\int_0^{2\pi} e^{i\mathbf{k}\cdot\mathbf{x}}\,\widehat{\rho}(\mathbf{k})\,\widehat{\mathcal{U}}(\mathbf{k})\,|\mathbf{k}|^2\,\sin\theta\,d|\mathbf{k}|d\theta d\phi, & d=3,\\ \int_0^P\int_0^{2\pi} e^{i\mathbf{k}\cdot\mathbf{x}}\,\widehat{\rho}(\mathbf{k})\widehat{\mathcal{U}}(\mathbf{k})\,|\mathbf{k}|\,d|\mathbf{k}|d\phi, & d=2,\end{cases}$$

Coulomb/Dipolar: $\widehat{\mathcal{U}}(\mathbf{k}) \sim 1/|\mathbf{k}|^2$, if $d=3$, $\widehat{\mathcal{U}}(\mathbf{k}) \sim 1/|\mathbf{k}|$ if $d=2$.

Question: extension to the 2D Poisson potential??

$$\Phi(\mathbf{x}) \approx \frac{1}{(2\pi)^2}\int_{|\mathbf{k}|\le P}\frac{1}{|\mathbf{k}|^2}\,e^{i\mathbf{k}\cdot\mathbf{x}}\,\widehat{\rho}(\mathbf{k})\,d\mathbf{k}, \qquad \mathbf{x}\in\mathbb{R}^2$$

NUFFT-based method: Bao, Jiang, Tang, Zhang, JCP 15'

Idea

$$\Phi(\mathbf{x}) = \mathcal{U} * \rho = \mathcal{U} * \left[G_1(\mathbf{x}) + \Big(\rho(\mathbf{x}) - G_1(\mathbf{x}) \Big) \right] =: u_1(\mathbf{x}) + u_2(\mathbf{x}), \tag{10}$$

$$u_1(\mathbf{x}) = \mathcal{U} * G_1(\mathbf{x}), \tag{11}$$

$$u_2(\mathbf{x}) = \int_{\mathbb{R}^2} \widehat{\mathcal{U}}(\mathbf{k}) \Big(\widehat{\rho^n} - \widehat{G_1} \Big) d\mathbf{k} = \int_{\mathbb{R}^2} \frac{\widehat{\rho} - \widehat{G}_1}{|\mathbf{k}|} \, \frac{e^{i \ \mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|} d\mathbf{k}. \tag{12}$$

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Proper choice of function $G_1(\mathbf{x})$ s.t

- (1). u_1: integrated analytically. (2). $\widehat{G}_1(\mathbf{k})$: exponentially decay.
- (3). $w(\mathbf{k}) =: \frac{\widehat{\rho}(\mathbf{k}) - \widehat{G}_1(\mathbf{k})}{|\mathbf{k}|}$: smooth at $\mathbf{k} = 0$, $\Longrightarrow$ solve $u_2(\mathbf{x})$ by **NUFFT.**

Example : $G_1(\mathbf{x}) = \widehat{\rho}(\mathbf{0})\, G(\mathbf{x}) - \widehat{(\mathbf{x}\rho)}(\mathbf{0}) \cdot \nabla_{\mathbf{x}} G(\mathbf{x}),$ with $G(\mathbf{x}) = \exp\left\{ -|\mathbf{x}|^2/(2\sigma^2) \right\} / (2\pi\sigma^2).$ (13)

Comparison: NUFFT vs DST (PDE approach with Dirichlet BC)

We take 3D DDI as an example, let $\mathbf{n} = (0,0,1)^T$, $c = 1.4$

$$\rho(\mathbf{x}) = e^{-|\mathbf{x}|^2/c^2} \Longrightarrow \quad \Phi(\mathbf{x}) = -\rho(\mathbf{x}) - 3\,\mathbf{n}^T \mathbf{D}\,\mathbf{n} \tag{14}$$

Table: l^2-errors of Φ on $\mathcal{D} = [-L, L]^3$ computed by **DST** (upper) & **NUFFT** (lower)

	$h = 1$	$h = 1/2$	$h = 1/4$	$h = 1/8$
$L = 8$	6.919E-02	7.720E-02	8.124E-02	8.327E-02
$L = 16$	2.709E-02	2.853E-02	2.925E-02	2.961E-02
$L = 32$	1.008E-02	1.033E-02	1.046E-02	1.052E-02
	$h = 1$	$h = 1/2$	$h = 1/4$	$h = 1/8$
$L = 8$	3.428E-4	9.834E-12	1.601E-14	$<$1E-14
$L = 16$	3.551E-4	1.143E-11	8.089E-15	$<$1E-15

Collapse dynamics for 3D dipolar BEC[4]: Bao, Tang, Zhang, CICP, 16.

[4] T. Lahaye et al, PRL, 08', Rep. Prog. Phys., 09'

Dynamics of 3D dipolar BEC: Bao, Tang, Zhang, CICP, 16.

Tunability of the dipole: dipole orientation

$$\mathbf{n}(t) = (\cos(0.2t), 0, \sin(0.2t))^T.$$

Extenstion to rotating dipolar BEC

NLSE with rotation and nonlocal term

$$i\partial_t \psi(\mathbf{x},t) = \left[-\frac{1}{2}\nabla^2 + V(\mathbf{x}) + \beta\,|\psi|^2 + \lambda\mathcal{U} * |\psi|^2 - \omega\, L_z\right]\psi(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^d \qquad (15)$$

Idea of the Algorithm

- Rotating Lagrangian coordinate transformation $\Longrightarrow$ eliminate L_z

$$i\partial_t \phi(\widetilde{\mathbf{x}},t) = \left[-\frac{1}{2}\nabla^2 + V(\mathbf{A}(t)\widetilde{\mathbf{x}}) + \beta|\phi|^2 + \lambda\widetilde{\mathcal{U}} * |\phi|^2\right]\phi \qquad (16)$$

- Apply TSSP for the new GPE.

$$\begin{aligned} \text{Step 1}: &\quad i\partial_t \phi(\widetilde{\mathbf{x}},t) = -\tfrac{1}{2}\nabla^2\phi(\widetilde{\mathbf{x}},t), \\ \text{Step 2}: &\quad i\partial_t \phi(\widetilde{\mathbf{x}},t) = \left[V(\mathbf{A}(t)\widetilde{x}) + \beta|\phi|^2 + \lambda\widetilde{\mathcal{U}} * |\phi|^2\right]\phi(\widetilde{\mathbf{x}},t). \end{aligned} \qquad (17)$$

- NUFFT-based algorithm for nonlocal term: $\widetilde{\mathcal{U}} * |\phi|^2$.

Dynamics of vortex lattice in 2d rotating dipolar BEC ($\mathbf{n} = (1, 0, 0)^T$)

Extenstion to rotating two component dipolar BECs[5]

- Rotating two-component dipolar BEC: $j = 1, 2$

$$i\partial_t \psi_j(\mathbf{x},t) = \left[-\frac{1}{2}\nabla^2 + V_j(\mathbf{x}) - \Omega \mathrm{L}_z + \sum_{l=1}^{2}\left(\beta_{jl}|\psi_l|^2 + \lambda_{jl}\,\mathcal{U} * |\psi_l|^2\right)\right]\psi_j(\mathbf{x},t). \tag{18}$$

[5] Saito, Kawaguchi & Ueda, PRL, 09', Tang, Zhang & Mauser, 16'

Extenstion to rotating two component dipolar BECs[5]

- Initial data: ground state under parameters

$$\mathbf{n} = (0,0,1), \gamma_x = \gamma_y = \gamma_z = 1, \Omega = 0, \beta_{11} = \beta_{22} = 103.58,$$
$$\beta_{12} = \beta_{21} = \lambda_{12} = \lambda_{21} = 0, \lambda_{11} = \lambda_{22} = 82.864.$$

Dynamics: $\beta_{12} = \beta_{21} = 100$, $\lambda_{22} = 0$.

[5] Saito, Kawaguchi & Ueda, PRL, 09', Tang, Zhang & Mauser, 16'

Outline

Fractional nonlinear Schödinger equation (FNLSE)

$$i\partial_t\psi(\mathbf{x},t) = \left[\frac{1}{2}\left(-\nabla^2+m^2\right)^s + V(\mathbf{x}) + \beta|\psi(\mathbf{x},t)|^2 + \lambda\Phi(\mathbf{x},t) - \omega L_z\right]\psi, \quad (19)$$

$$\Phi(\mathbf{x},t) = \mathcal{U} * |\psi|^2, \qquad \mathbf{x}\in\mathbb{R}^d,\ t>0 \quad (20)$$

- $s>0$: fractional order. $m\geq 0$: scaled particle mass.
- Fractional kinetic operator defined via Fourier integral:

$$\left(-\nabla^2+m^2\right)^s\psi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{\psi}(\mathbf{k})\left(|\mathbf{k}|^2+m^2\right)^s e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}, \quad (21)$$

- $\mathcal{U}$: Kernel of Coulomb potential or dipole potential.

$$\mathcal{U}(\mathbf{x}) = \begin{cases} -(\mathbf{n}\cdot\mathbf{n})\delta(\mathbf{x}) - 3\,\partial_{\mathbf{nn}}\left(\frac{1}{4\pi|\mathbf{x}|}\right), & 3D \text{ Dipolar}, \\ -\frac{3}{2}\left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\nabla_\perp^2\right)\left(\frac{1}{2\pi|\mathbf{x}|}\right), & 2D \text{ Dipolar}, \\ \frac{1}{2^{d-1}\pi}\frac{1}{|\mathbf{x}|}, & 2D/3D \text{ Coulomb}. \end{cases} \quad (22)$$

[6] N. Laskin, Phys. Lett. A., 00', Phys. Rev. E, 02'
[7] E. Lenzmann, Math. Phys. Anal. Geom., 07', etc.
[8] I. Carusotto & C. Ciuti, Rev. Mod. Phys., 13'.

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- N. Laskin[6]: Generalise Feynman path integral to Lévy-like quantum paths $(\lambda=\omega=0)$
- Model for boson stars:[7] $s=\frac{1}{2}$, $\omega=0$, $\mathcal{U}$ kernel of Coulomb potential
- Model for polariton condensates with mass dependent velocity[8]:

$$\frac{1}{2}\left(-\nabla^2+m^2\right)^s\psi \ \rightarrow\ q*\psi, \quad \text{with } \mathcal{F}(q)(\mathbf{k}) = \frac{|\mathbf{k}|^2}{m(\mathbf{k})} \quad (23)$$

[6] N. Laskin, Phys. Lett. A., 00', Phys. Rev. E, 02'

[7] E. Lenzmann, Math. Phys. Anal. Geom., 07', etc.

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Fractional nonlinear Schrödinger equation

- Two conserved quantities: mass and energy

$$\mathcal{N}(t) = \int_{\mathbb{R}^d} |\psi(\mathbf{x},t)|^2 d\mathbf{x}, \tag{24}$$

$$\mathcal{E}(t) = \int_{\mathbb{R}^d} \left[\frac{1}{2}\psi^*(-\nabla^2+m^2)^s\psi + V(\mathbf{x})|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\lambda}{2}\Phi|\psi|^2 - \omega\psi^* L_z\psi\right] d\mathbf{x}. \tag{25}$$

Ground states exists ??

Ground state $\phi_g(\mathbf{x})$: non-convex minimization problem

$$\phi_g(\mathbf{x}) = \arg\min_{\phi\in S} \mathcal{E}(\phi), \quad \text{with} \quad S = \{\phi \mid \|\phi\| = 1, \mathcal{E}(\phi) < \infty\}. \tag{26}$$

Existence of the ground states

$V(\mathbf{x})$: harmonic potential

(i) $s > 1$ & $\beta \geq 0 \Longrightarrow \exists$ ground states for $\forall\ \Omega > 0$ if one of the following holds:

- (A) $\mathcal{U}(\mathbf{x})$ reads as Coulomb-type.
- (B) For 3D DDI: $-\beta/2 \leq \lambda \leq \beta$.
- (C) For 2D DDI: (c1) $\lambda = 0$. (c2) $\lambda > 0$ and $n_3 = 0$. (c3) $\lambda < 0$ and $n_3^2 \geq \frac{1}{2}$.

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Non-existence results when $0 < s < 1$:

(A) $\forall\ \Omega > 0$, $\lambda = 0$.

(B) $\forall\ \Omega > 0$, $\mathcal{U}(\mathbf{x})$ is a Coulomb-type interaction or a 3D DDI.

(C) $\mathcal{U}(\mathbf{x})$ is the 2D DDI, $\forall\ \Omega > \Omega_0 = c|\lambda|^{\frac{2}{5}}$ with $c = \sqrt[5]{(2\pi^2+1)^4\gamma^6/(48e\pi^9)}$.

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- (B) For 3D DDI: $-\beta/2 \leq \lambda \leq \beta$.
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(ii) $\Omega = 0,\ \beta \geq 0 \Longrightarrow \exists$ ground states for $\forall\ s > 0$ if one of following holds

(1) $\mathcal{U}(\mathbf{x})$ reads as Coulomb-type and $\lambda \geq 0$. (2) (B) or (C) in (i) holds.

Non-existence results when $0 < s < 1$:

(A) $\forall\ \Omega > 0,\ \lambda = 0$.

(B) $\forall\ \Omega > 0$, $\mathcal{U}(\mathbf{x})$ is a Coulomb-type interaction or a 3D DDI.

(C) $\mathcal{U}(\mathbf{x})$ is the 2D DDI, $\forall\ \Omega > \Omega_0 = c|\lambda|^{\frac{2}{5}}$ with $c = \sqrt[5]{(2\pi^2+1)^4\gamma^6/(48e\pi^9)}$.

Open Question

How about the smallest possible Ω_0?

Existence of the ground states

$V(\mathbf{x})$: harmonic potential

(i) $s > 1$ & $\beta \geq 0 \Longrightarrow \exists$ ground states for $\forall\ \Omega > 0$ if one of the following holds:

- (A) $\mathcal{U}(\mathbf{x})$ reads as Coulomb-type.
- (B) For 3D DDI: $-\beta/2 \leq \lambda \leq \beta$.
- (C) For 2D DDI: (c1) $\lambda = 0$. (c2) $\lambda > 0$ and $n_3 = 0$. (c3) $\lambda < 0$ and $n_3^2 \geq \frac{1}{2}$.

(ii) $\Omega = 0$, $\beta \geq 0 \Longrightarrow \exists$ ground states for $\forall\ s > 0$ if one of following holds

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Open Question

How about the smallest possible Ω_0? Can the constant Ω_0 be relaxed to 0?

Ground states: rotating FNLSE with $\lambda = 0, \beta = 100$ and $s = 1.2$

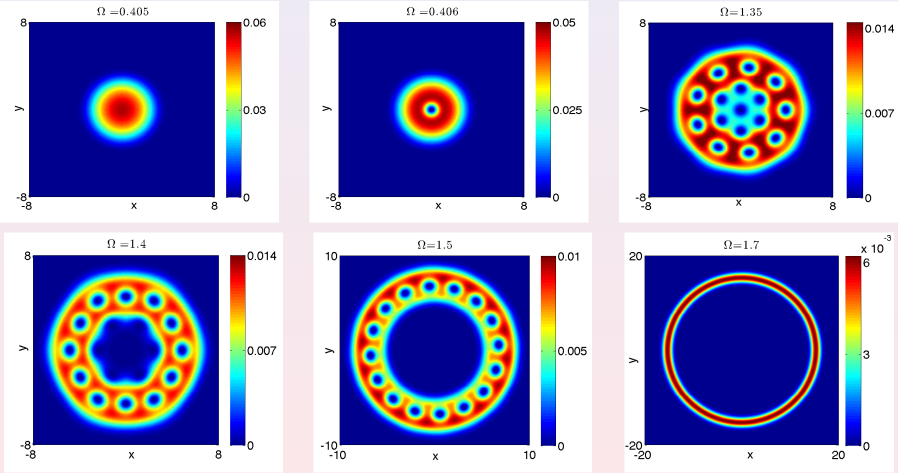

Figure: Contour plots of the density $|\phi_g(\mathbf{x})|^2$ (superdispersion: $s > 1$).

Dynamics simulation

Rotating Lagrangian coordinates transformation: $\widetilde{\mathbf{x}} = \mathbf{A}^T(t)\mathbf{x}$

F-NLSE in rotating Lagrangian coordinate

$$i\partial_t \phi(\widetilde{\mathbf{x}}, t) = \left[\frac{1}{2}(-\nabla^2 + m^2)^s + V(\mathbf{A}(t)\widetilde{\mathbf{x}}, t) + \beta|\phi|^2 + \lambda\widetilde{\Phi}(\widetilde{\mathbf{x}}, t)\right]\phi(\widetilde{\mathbf{x}}, t) \quad (27)$$

TSSP + NUFFT

$$\text{Step 1}: \; i\partial_t \phi(\widetilde{\mathbf{x}}, t) = \frac{1}{2}(-\nabla^2 + m^2)^s, \quad (28)$$

$$\text{Step 2}: \; i\partial_t \phi(\widetilde{\mathbf{x}}, t) = \left[V(\mathbf{A}(t)\widetilde{\mathbf{x}}) + \beta|\phi(\widetilde{\mathbf{x}}, t)|^2 + \lambda\,\widetilde{\Phi}(\widetilde{\mathbf{x}}, t)\right]\phi(\widetilde{\mathbf{x}}, t), \quad (29)$$

Dynamics of vortex cluster: intial: $s = s_0 = 1.2$, $\Omega = 1.35$, $\beta = 100$ and $\lambda = 0$

Perturb fractional order: $s = 0.7$

Dynamics of vortex cluster: intial: $s = s_0 = 1.2$, $\Omega = 1.35$, $\beta = 100$ and $\lambda = 0$

Turn on DDI: $\lambda = 80$, $\mathbf{n} = (1, 0, 0)^T$.

Outline

Summary

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  - Rotating Lagrangian Coordinate transformation technique.
  - NUFFT-based solver for non-local potential
- Extension to fractional NLSE: decoherence & chaotic dynamics

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Thank You For Attention !