On the leapfrogging phenomenon in fluid mechanics

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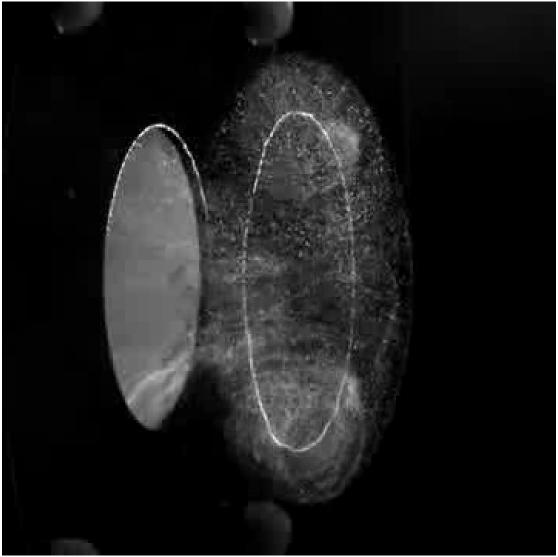
Single vortex ring : a movie example



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Helmholtz master paper on vorticity

Vortex rings :

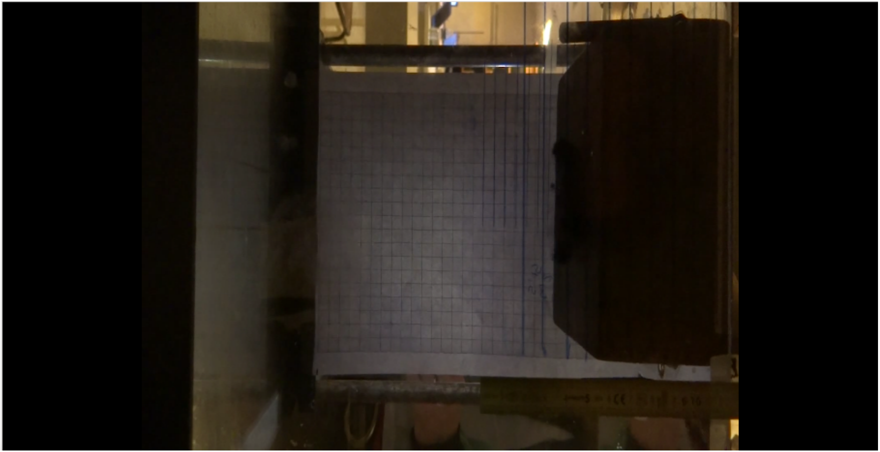
Hence *in a circular vortex-filament of very small section in an indefinitely extended fluid, the centre of gravity of the section has, from the commencement, an approximately constant and very great velocity parallel to the axis of the vortex-ring, and this is directed towards the side to which the fluid flows through the ring.*

Leapfrogging :

We can now see generally how two ring-formed vortex-filaments having the same axis would mutually affect each other, since each, in addition to its proper motion, has that of its elements of fluid as produced by the other. If they have the same direction of rotation, they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster, till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.

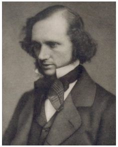
Leapfrogging vortices : a movie example

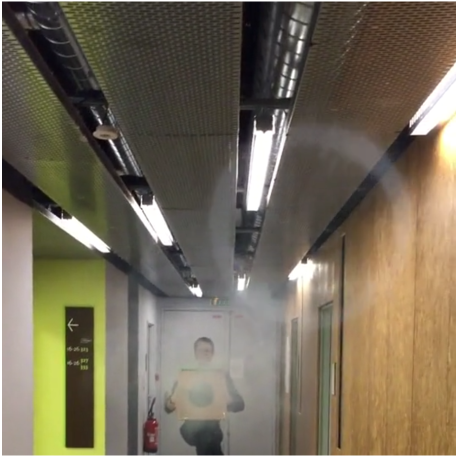
Leapfrogging vortices : a movie example



Existence of vortex rings : a practical point of view

Professor Tait's plan of exhibiting smoke-rings is as follows :— A large rectangular box, open at one side, has a circular hole of 6 or 8 inches diameter cut in the opposite side. A common rough packing-box of 2 feet cube, or thereabout, will answer the purpose very well. The open side of the box is closed by a stout towel or piece of cloth, or by a sheet of india-rubber stretched across it. A blow on this flexible side causes a circular vortex ring to shoot out from the hole on the other side. The vortex rings thus generated are visible if the box is filled with smoke.





Euler equation for incompressible fluids

The 3 dimensional Euler equations read

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = -\nabla p, \\ \mathrm{div}\, v = 0, \end{cases}$$

where

- $v : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is the velocity field,
- $p : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is the pressure field.

For a number of physically meaningful flows, an important quantity is given by the *vorticity*

$$\omega = \mathrm{curl}\, v$$

which satisfies

$$\partial_t \omega + v \nabla \omega = \omega \nabla v.$$

The velocity v can always be recovered from the vorticity ω through the Biot-Savart law

$$v(t,x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{y-x}{\|y-x\|^3} \times \omega(t,y)\, dy.$$

Gross-Pitaevskii equation for quantum fluids

The 3 dimensional Gross-Pitaevskii equation reads

$$i\partial_t u - \Delta u = \frac{1}{\varepsilon^2} u(1 - |u|^2)$$

where

- $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ is the complex field function
- $\varepsilon > 0$ is a given *length-scale*.

In the framework of thin vortex tubes, it bears some resemblance with the Euler equation.
An important quantity here is given by the *vorticity*

$$Ju = \frac{1}{2} \operatorname{curl} j(u) = \operatorname{curl} \left(u \times \nabla u \right).$$

In contrast with the Euler equation, the "current" $j(u)$ is not fully determined by the Jacobian Ju since an additional degree of freedom is present through the *phase*:

$$u = \rho \exp(i\varphi) \Longrightarrow j(u) = \rho^2 \nabla \varphi, \qquad \rho \equiv 1 \Longrightarrow Ju = \frac{1}{2} \operatorname{curl} \nabla \varphi \equiv 0.$$

On the analogy between GP and Euler

For axisymmetric solutions of (GP) we have the equation for the vorticity

$$\frac{d}{dt}\int_{\mathbb{H}} Ju\,\varphi\,drdz = \mathcal{F}(\nabla u, \varphi),$$

where

$$\mathcal{F}(\nabla u, \varphi) := -\int_{\mathbb{H}} \varepsilon_{ij}\frac{\partial_k r}{r}(\partial_j u, \partial_k u)\partial_i\varphi + \int_{\mathbb{H}} \varepsilon_{ij}(\partial_j u, \partial_k u)\partial_{ik}\varphi.$$

For the axisymmetric Euler equation we have

$$\frac{d}{dt}\int_{\mathbb{H}} \omega\,\varphi\,drdz = \mathcal{F}(v, \varphi).$$

In contrast to Euler, for (GP) we also need an equation for the “compressible” part of $j(u)$:

$$\varepsilon\partial_t\frac{|u|^2-1}{\varepsilon} = \frac{2}{r}\mathrm{div}(rj(u)).$$

Existence of vortex rings : a mathematical point of view

One looks for *cylindrically symmetric* traveling wave solutions of the Euler equation.
These can be obtained by the *variational problem*

$$\text{maximize} \quad E := \int_{\mathbb{H}} \int_{\mathbb{H}} \omega(x) G(x, x') \omega(x') \, d\nu d\nu'$$

$$\text{under constrains} \quad \begin{cases} P := \int_{\mathbb{H}} r^2 \omega drdz = \text{given Cst}, \\ \dfrac{\omega}{r} \text{ is a transport measure of } \dfrac{\omega^0}{r}, \text{ where } \omega_0 \text{ is given.} \end{cases}$$

Here, $\mathbb{H} := \{(r, z),\ r \geq 0, z \in \mathbb{R}\}$, $d\nu = r\, dr\, dz$ and G refers to the Green function of the Laplacian in cylindrical coordinates.

Important contributions by Arnold 1964, Fraenkel-Berger 1974, Benjamin 1976, Friedman-Turkington 1981, Burton 2003.

For the Gross-Pitaevskii, a related strategy is given by

$$\text{minimize} \quad E := \int_{\mathbb{H}} [\frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (|u|^2 - 1)^2] \, rdrdz$$

$$\text{under constrain} \quad P := \int_{\mathbb{H}} r^2 Ju \, drdz = \text{given Cst.}$$

Analysis by Bethuel-Orlandi-S. 2003.

Leapfrogging for GP : notion of reference vortex tubes

Let $\mathcal{C}$ be a smooth oriented closed curve in $\mathbb{R}^3$ and let $\vec{\mathcal{J}}$ be the vector distribution corresponding to 2π times the circulation along $\mathcal{C}$, namely

$$\langle \vec{\mathcal{J}}, \vec{X} \rangle = 2\pi \int_{\mathcal{C}} \vec{X} \cdot \vec{\tau} \qquad \forall \vec{X} \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3),$$

where $\vec{\tau}$ is the tangent vector to $\mathcal{C}$. To the "current density" $\vec{\mathcal{J}}$ is associated the "induction" $\vec{B}$, which satisfies the equations

$$\mathrm{div}(\vec{B}) = 0, \qquad \mathrm{curl}(\vec{B}) = \vec{\mathcal{J}} \qquad \text{in } \mathbb{R}^3,$$

and is obtained from $\vec{\mathcal{J}}$ by the Biot-Savart law. To $\vec{B}$ is then associated a vector potential $\vec{A}$, which satisfies

$$\mathrm{div}(\vec{A}) = 0, \qquad \mathrm{curl}(\vec{A}) = \vec{B} \qquad \text{in } \mathbb{R}^3,$$

so that

$$-\Delta \vec{A} = \mathrm{curl}\,\mathrm{curl}(\vec{A}) = \vec{\mathcal{J}} \qquad \text{in } \mathbb{R}^3.$$

For $a \in \mathbb{H}$, let $\mathcal{C}_a$ be the circle of radius $r(a)$ parallel to the xy-plane in $\mathbb{R}^3$, centered at the point $(0, 0, z(a))$, and oriented so that its binormal vector points towards the positive z-axis. By cylindrical symmetry, we may write the corresponding vector potential as

$$\vec{A}_a \equiv A_a(r, z)\vec{e}_\theta.$$

The expression of the vector laplacian in cylindrical coordinates yields the equation for the scalar function A_a :

$$\begin{cases} -\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} + \partial_z^2\right) A_a = 2\pi\delta_a & \text{in } \mathbb{H} \\ A_a = 0 & \text{on } \partial\mathbb{H}, \end{cases}$$

Leapfrogging for GP : notion of reference vortex tubes

or equivalently

$$\begin{cases} -\mathrm{div}\left(\dfrac{1}{r}\nabla\left(rA_a\right)\right) = 2\pi\delta_a & \text{in } \mathbb{H} \\ A_a = 0 & \text{on } \partial\mathbb{H}, \end{cases}$$

which can be integrated explicitly in terms of complete elliptic integrals.
Up to a constant phase factor, there exists a unique unimodular map $u_a^* \in \mathcal{C}^\infty(\mathbb{H}\setminus\{a\}, S^1) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{H}, S^1)$ such that

$$r(iu_a^*, \nabla u_a^*) = rj(u_a^*) = -\nabla^\perp(rA_a).$$

In the sense of distributions in $\mathbb{H}$, we have

$$\begin{cases} \mathrm{div}(rj(u_a^*)) & = 0 \\ \mathrm{curl}(j(u_a^*)) & = 2\pi\delta_a, \end{cases}$$

and the function u_a^* corresponds therefore to a *singular* vortex ring. In order to describe a *reference* vortex ring for the Gross-Pitaevskii equation, we shall make the notion of core more precise. In $\mathbb{R}^2$, the Gross-Pitaevskii equation possesses a distinguished stationary solution called vortex : in polar coordinates, it has the special form

$$u_\varepsilon(r,\theta) = f_\varepsilon(r)\exp(i\theta)$$

where the profile $f_\varepsilon : \mathbb{R}^+ \to [0,1]$ satisfies $f_\varepsilon(0) = 0$, $f_\varepsilon(+\infty) = 1$, and

$$\partial_{rr} f_\varepsilon + \frac{1}{r}\partial_r f_\varepsilon - \frac{1}{r^2} f_\varepsilon + \frac{1}{\varepsilon^2} f_\varepsilon(1 - f_\varepsilon^2) = 0.$$

The reference vortex ring associated to the point $a \in \mathbb{H}$ is defined to be

$$u_{\varepsilon,a}^*(r,z) = f_\varepsilon\big(\|(r,z) - a\|\big)\, u_a^*(r,z).$$

Leapfrogging for GP : notion of reference vortex tubes

More generally, when $a = \{a_1, \cdots, a_n\}$ is a family of n distinct points in $\mathbb{H}$, we set

$$u_a^*(r,z) := \prod_{k=1}^{n} u_{a_k}^*(r,z), \quad \text{and} \quad u_{\varepsilon,a}^*(r,z) := \prod_{k=1}^{n} u_{\varepsilon,a_k}^*(r,z),$$

where the products are meant in $\mathbb{C}$. The field $u_{\varepsilon,a}^*$ hence corresponds to a collection of n reference vortex rings (sharing the same axis and oriented in the same direction), and is the typical kind of object which we shall study the evolution of. It can be shown that

$$\Big\| J u_{\varepsilon,a}^* - \pi \sum_{i=1}^{n} \delta_{a_i} \Big\|_{\dot{W}^{-1,1}(\mathbb{H})} = O(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Finally, classical computations lead to

$$E\big(u_{\varepsilon,a}^*\big) = H_\varepsilon(a_1, \cdots, a_n) + o(1),$$

where

$$H_\varepsilon(a_1, \cdots, a_n) := \sum_{i=1}^{n} r(a_i)\Big[\pi \log\big(\tfrac{r(a_i)}{\varepsilon}\big) + \gamma + \pi\big(3\log(2) - 2\big) + \pi \sum_{j \neq i} A_{a_j}(a_i)\Big],$$

and γ is a numerical constant.

The ODE of leapfrogging

We consider the associated hamiltonian system

$$(LF) \qquad \dot{a}_i(t) = \frac{1}{\pi}\mathbb{J}\nabla_{a_i} H_\varepsilon\big(a_1(t), \cdots, a_n(t)\big), \qquad i = 1, \cdots, n,$$

where, with a slight abuse of notation,

$$\mathbb{J} := \begin{pmatrix} 0 & -\frac{1}{r(a_i)} \\ \frac{1}{r(a_i)} & 0 \end{pmatrix}.$$

In addition to the hamiltonian H_ε, the system (LF) also conserves the momentum

$$P(a_1, \cdots, a_n) := \pi \sum_{k=1}^{n} r^2(a_k),$$

which may be interpreted as the total area of the disks determined by the vortex rings. As a matter of fact, also note that

$$P(u^*_{\varepsilon,a}) := \int_{\mathbb{H}} Ju^*_{\varepsilon,a}\, r^2\, drdz = \pi \sum_{k=1}^{n} r^2(a_k) + o(1),$$

as $\varepsilon \to 0$, and that, at least formally, the momentum P is a conserved quantity for the Gross-Pitaevskii equation.

Mathematical convergence results

We consider axisymmetric initial data u_ε^0 for (GP), we write

$$a_{i,\varepsilon}^0 = a^0 + \frac{b_{i,\varepsilon}^0}{\sqrt{|\log \varepsilon|}}$$

and denote by $a_{i,\varepsilon}^s$ the corresponding solution of $(LF)_\varepsilon$. We let $S > 0$ and fix a constant $K > 0$ such that $|b_{i,\varepsilon}^s| \leq K$ for all $s \in [0, S]$.
We define

- $\left\| Ju_\varepsilon^0 - \pi \sum_{i=1}^n \delta_{a_{i,\varepsilon}^0} \right\|_{\dot{W}^{-1,1}} =: r_a^0$ (concentration scale)
- $\left[E(u_\varepsilon^0) - H_\varepsilon(a_{1,\varepsilon}^0, \cdots, a_{n,\varepsilon}^0) \right]^+ =: \Sigma_a^0$ (energy excess).

Theorem (Jerrard-S.)

Under the above assumptions, there exist constants $\varepsilon_0, \sigma_0, C_0$ (depending only on a^0, n, K) such that if $\varepsilon \leq \varepsilon_0$ and if $\Sigma_a^0 + r_a^0 |\log \varepsilon| \leq \sigma_0$, then

$$r_a^s := \left\| Ju_\varepsilon^s - \pi \sum_{i=1}^n \delta_{a_{i,\varepsilon}^s} \right\|_{\dot{W}^{-1,1}} \leq C_0 \left(r_a^0 + \frac{\Sigma_a^0}{\sqrt{|\log \varepsilon|}} + \frac{C_\delta}{|\log \varepsilon|^{1-\delta}} \right) \exp(C_0 s)$$

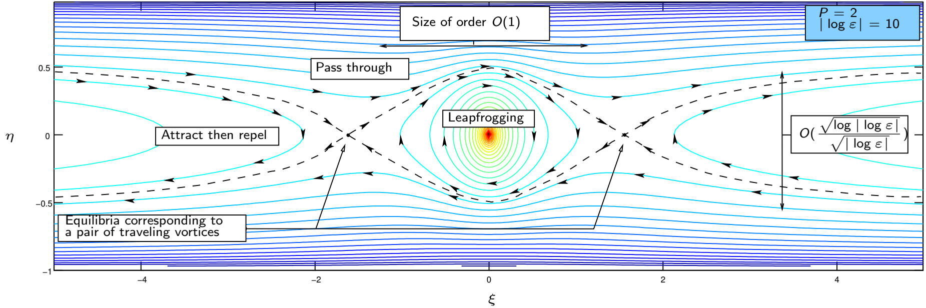
for $s \in [0, S]$, where $\delta > 0$ is arbitrary.

Analysis of the leapfrogging system for two vortex rings

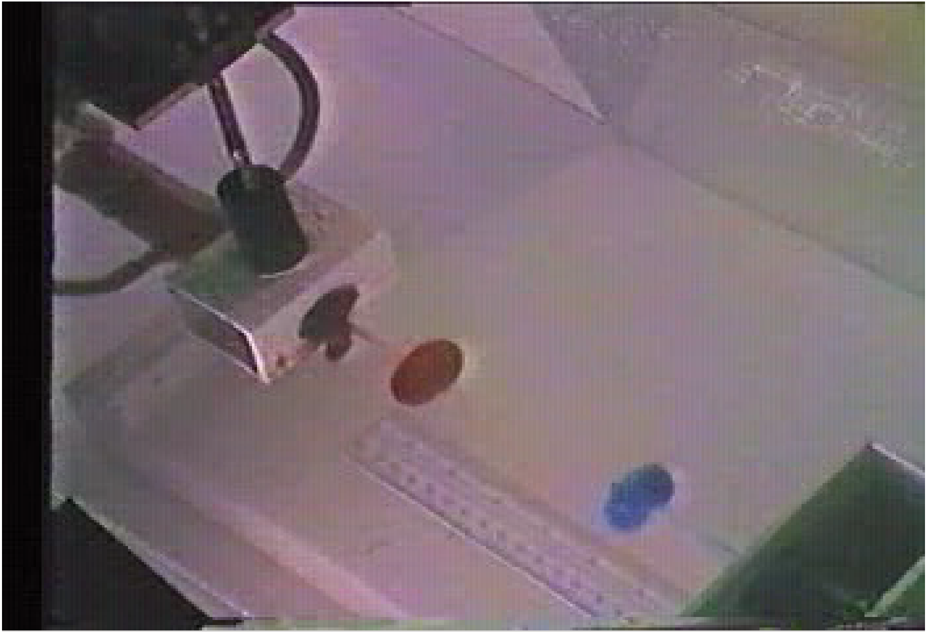
When $n = 2$, the system (LF) may be analyzed in great details. Since P is conserved and since H_ε is invariant by a joint translation of both rings in the z direction, it is classical to introduce the variables (η, ξ) by

$$\begin{cases} r^2(a_1) = \frac{P}{2} - \eta \\ r^2(a_2) = \frac{P}{2} + \eta \end{cases}, \qquad \xi = z(a_1) - z(a_2),$$

and to draw the level curves of the function H_ε in those two real variables, the momentum P being considered as a parameter.



Collision of vortex rings



Strategy of the proof

We rely mostly on the already mentioned evolution equation for the Jacobian

$$\frac{d}{ds}\int_{\mathbb{H}} Ju\,\varphi\,drdz = \frac{\mathcal{F}(\nabla u,\varphi)}{|\log\varepsilon|},$$

where

$$\mathcal{F}(\nabla u,\varphi) := -\int_{\mathbb{H}} \varepsilon_{ij}\frac{\partial_k r}{r}(\partial_j u,\partial_k u)\partial_i\varphi + \int_{\mathbb{H}} \varepsilon_{ij}(\partial_j u,\partial_k u)\partial_{ik}\varphi.$$

We wish to prove that

- Ju remains well concentrated around a sum of Dirac masses
- $\mathcal{F}(\nabla u,\varphi)$ is a good approximation of the ode $(\mathrm{LF})_\varepsilon$.

It is clear that both are not independent and we shall use a common Gronwall argument on r_a^s. We will decompose

$$\begin{aligned}
(\partial_j u,\partial_k u) &= \partial_j|u|\partial_k|u| + \frac{j(u)_j}{|u|}\frac{j(u)_k}{|u|}\\
&= \partial_j|u|\partial_k|u| + \left(\frac{j(u)}{|u|} - j^\natural\right)_j\left(\frac{j(u)}{|u|} - j^\natural\right)_k\\
&\quad + \left(\frac{j(u)}{|u|} - j^\natural\right)_j\left(j^\natural\right)_k + \left(\frac{j(u)}{|u|} - j^\natural\right)_k\left(j^\natural\right)_j\\
&\quad + \left(j^\natural\right)_j\left(j^\natural\right)_k,
\end{aligned}$$

where $j^\natural$ is a "suitable" approximation of $j(u)$.

Some elements in the proof: excess energy and concentration

Proposition

There exist constants $\varepsilon_1, \sigma_1, C_1 > 0$, depending only on n, r_0 and K, with the following properties. If $\varepsilon \leq \varepsilon_1$ and

$$\Sigma_a^r := \Sigma_a + r_a|\log\varepsilon| \leq \sigma_1|\log\varepsilon|,$$

then there exist $\xi_1, \cdots, \xi_n$ in $\mathbb{H}$ such that

$$r_\xi := \|Ju - \pi\sum_{i=1}^n \delta_{\xi_i}\|_{\dot{W}^{-1,1}} \leq C_1\varepsilon|\log\varepsilon|^{C_1} e^{C_1\Sigma_a^r},$$

and

$$\int_{\mathbb{H}\setminus\cup_i B(\xi_i,\varepsilon^{\frac{2}{3}})} r\left[e_\varepsilon(|u|) + \left|\frac{j(u)}{|u|} - j(u_\xi^*)\right|^2\right] \leq C_1\left(\Sigma_\xi + \varepsilon^{\frac{1}{3}}|\log\varepsilon|^{C_1} e^{C_1\Sigma_a^r}\right),$$

where we have written

$$\Sigma_\xi := [E(u) - H_\varepsilon(\xi_1, \cdots, \xi_n)]^+ .$$

Moreover,

$$\Sigma_\xi \leq \Sigma_a + C_1 r_a|\log\varepsilon| + C_1\varepsilon|\log\varepsilon|^{C_1} e^{C_1\Sigma_a^r}.$$

Note the pointwise identity

$$e_\varepsilon(u) = \frac{1}{2}|j(u_\xi^*)|^2 + j(u_\xi^*)\left(\frac{j(u)}{|u|} - j(u_\xi^*)\right) + e_\varepsilon(|u|) + \frac{1}{2}\left|\frac{j(u)}{|u|} - j(u_\xi^*)\right|^2.$$

Some elements in the proof: approximation including the core

Idea : smoothen u_ξ^* not at scale ε but at the larger scale r_ξ (i.e. at the best known localization scale). It actually suffices to regularize $j(u_\xi^*)$, we call it $j^\natural(u_\xi^*)$.

Proposition

In addition to the statement of the previous Proposition,

$$\int_{\mathbb{H}} r\left[e_\varepsilon(|u|) + \left|\frac{j(u)}{|u|} - j^\natural(u_\xi^*)\right|^2\right] \leq C_1\left(\Sigma_a^r + \log|\log\varepsilon|\right).$$

$$e_\varepsilon(u) = \frac{1}{2}|j^\natural(u_\xi^*)|^2 + j^\natural(u_\xi^*)\left(\frac{j(u)}{|u|} - j^\natural(u_\xi^*)\right) + e_\varepsilon(|u|) + \frac{1}{2}\left|\frac{j(u)}{|u|} - j^\natural(u_\xi^*)\right|^2.$$

Some elements in the proof: get rid of the main translation

Recall that

$$H_\varepsilon(a_1, \cdots, a_n) := \sum_{i=1}^{n} r(a_i)\Big[\pi \log\big(\tfrac{r(a_i)}{\varepsilon}\big) + \gamma + \pi\big(3\log(2) - 2\big) + \pi \sum_{j\neq i} A_{a_j}(a_i)\Big],$$

and write

$$a_i := \Big(r_0 + \frac{r(b_i)}{\sqrt{|\log\varepsilon|}}, z_0 + \frac{z(b_i)}{\sqrt{|\log\varepsilon|}}\Big), \qquad i = 1, \cdots, n.$$

Then

$$H_\varepsilon(a_1, \cdots, a_n) = \Gamma_\varepsilon(r_0, n) + W_{\varepsilon, r_0}(b_1, \cdots, b_n) + o(1) \text{ as } \varepsilon \to 0, \tag{1}$$

where

$$W_{\varepsilon, r_0}(b_1, \cdots, b_n) = \pi \sum_{i=1}^{n} r(b_i)\sqrt{|\log\varepsilon|} - \pi r_0 \sum_{i\neq j} \log|b_i - b_j|. \tag{2}$$

Also, expansion of the squares leads directly to

$$P(a_1, \cdots, a_n) = \pi n r_0^2 + 2\pi r_0 \sum_{i=1}^{n} \frac{r(b_i)}{\sqrt{|\log\varepsilon|}} + \pi \sum_{i=1}^{n} \frac{r(b_i)^2}{|\log\varepsilon|},$$

and therefore

$$\Big(H_\varepsilon - \frac{|\log\varepsilon|}{2r_0}P\Big)(a_1, \cdots, a_n) = -\frac{\pi}{2} n r_0 |\log\varepsilon| + \Gamma_\varepsilon(r_0, n) + \pi r_0 W(b_1, \cdots, b_n) + o(1), \quad \text{as } \varepsilon \to 0,$$

where

$$W(b_1, \cdots, b_n) := -\sum_{i\neq j} \log|b_i - b_j| - \frac{1}{2r_0^2} \sum_{i=1}^{n} r(b_i)^2.$$

Some elements in the proof: get rid of the main translation

Proposition

For σ sufficiently small, if

$$\Sigma_a + |H_\varepsilon(a_1, \cdots, a_n) - H_\varepsilon(\xi, \cdots, \xi_n)| \leq \sigma |\log \varepsilon|,$$

then

$$\Sigma_\xi \leq 2\Sigma_a + C\left[r_a \sqrt{|\log \varepsilon|} + \frac{1}{|\log \varepsilon|} + |\log \varepsilon| \, |P(u) - P(a_1, \cdots, a_n)| \right]$$

For a quantity f write $\delta f := |f(a_1, \cdots, a_n) - f(\xi_1, \cdots, \xi_n)|$. By the triangle inequality we have

$$\delta H_\varepsilon \leq \delta \left(H_\varepsilon - \frac{|\log \varepsilon|}{2r_0} P \right) + \frac{|\log \varepsilon|}{2r_0} \delta P,$$

and also

$$\delta P \leq |P(u) - P(\xi_1, \cdots, \xi_n)| + |P(u) - P(a_1, \cdots, a_n)| .$$

In view of the discussion in the previous slide

$$\delta \left(H_\varepsilon - \frac{|\log \varepsilon|}{2r_0} P \right) \leq C |(\xi_1 - a_1, \cdots, \xi_n - a_n)| \sqrt{|\log \varepsilon|}.$$

We also have a good control on P since it involves only the Jacobian :

$$\frac{|\log \varepsilon|}{2r_0} |P(u) - P(\xi_1, \cdots, \xi_n)| \leq \frac{C}{2r_0} \frac{(1 + \Sigma_\xi)^2}{|\log \varepsilon|} \leq \frac{C}{2r_0} \frac{(1 + \Sigma_a + \delta H_\varepsilon)^2}{|\log \varepsilon|}$$

and we may then absorb the last term involving δH_ε in the left-hand side above.