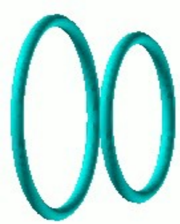
Quantum nature and statistical law in quantum turbulence

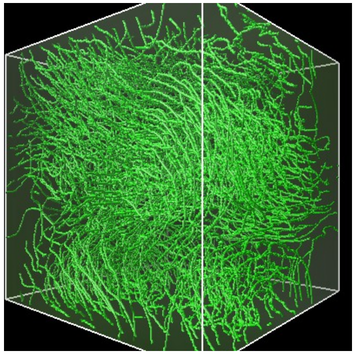
Michikazu Kobayashi (Kyoto University)



July 1, 2016 : CIRM Conference “New Challenges in Mathematical Modelling and Numerical Simulation of Superfluids”

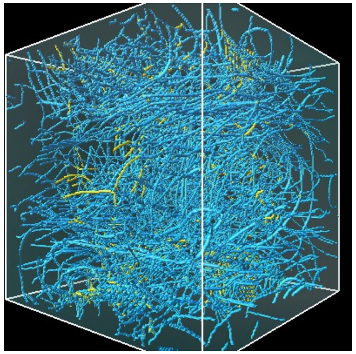
Plan of talk

- Introduction of Gross-Pitaevskii equation and turbulence



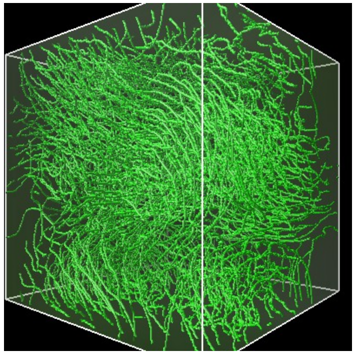
- Directed percolation universality class at turbulent transition

- Topologically protected helicity cascade in non-Abelian quantum turbulence



Plan of talk

- Introduction of Gross-Pitaevskii equation and turbulence



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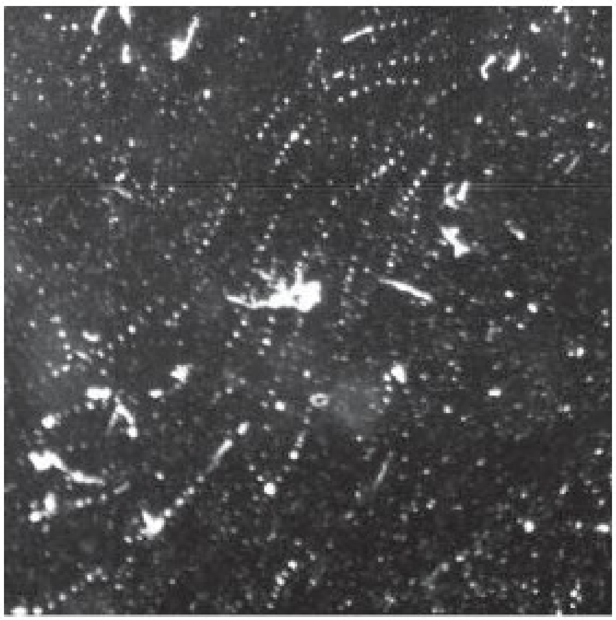
Quantized vortices & quantum turbulence

Quantum turbulence : tangled state of quantized vortices



Lathrop group from Youtube

Motivation of this work : Can we find some universality class of turbulence in quantum turbulence?

Gross-Piteavskii equation : model equation for superfluid system

$$\mathcal{H} = \int d^3x \left\{ \frac{\hbar^2}{2M} |\nabla \psi|^2 + \frac{g}{2} (|\psi|^2 - \bar{\rho})^2 \right\}$$

$$i\hbar\dot{\psi} = \frac{\delta \mathcal{H}}{\delta \psi^*} = \left\{ -\frac{\hbar^2}{2M} \nabla^2 + g(|\psi|^2 - \bar{\rho}) \right\} \psi$$

$\rho = |\psi|^2$: superfluid density

$\boldsymbol{v} = \bar{\kappa} \nabla (\arg[\psi])$: supefluid velocity ($\bar{\kappa} \equiv \hbar/M$)

Quantum hydrodynamic equation

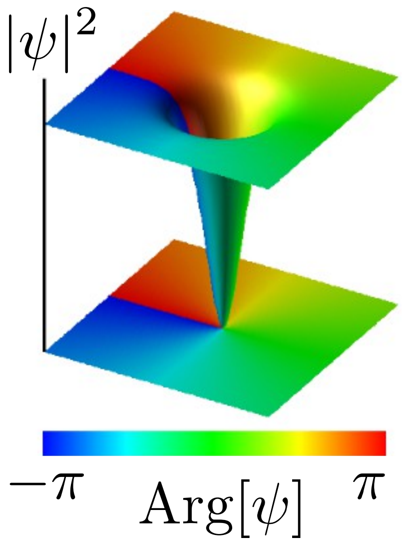
$$\dot{\rho} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad \rho \left(\dot{\boldsymbol{v}} + \frac{\nabla \boldsymbol{v}^2}{2} \right) = -\nabla \left(\frac{g\rho^2}{2M} \right) + \bar{\kappa}^2 \rho \nabla \left(\frac{\nabla^2 \sqrt{\rho}}{2\sqrt{\rho}} \right)$$

Vortex & turbulence

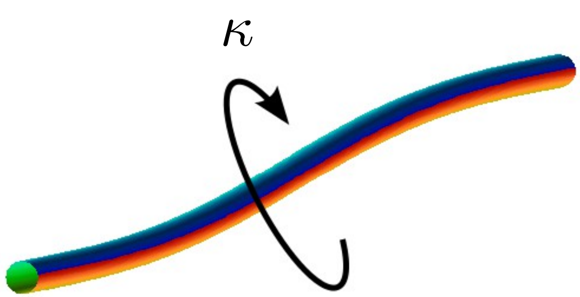
$$\rho = |\psi|^2 \ : \ \text{superfluid density}$$

$$\boldsymbol{v} = \bar{\kappa}\nabla(\arg[\psi]) \ : \ \text{supefluid velocity} \ (\bar{\kappa} \equiv \hbar/M)$$

2D : topological point defect



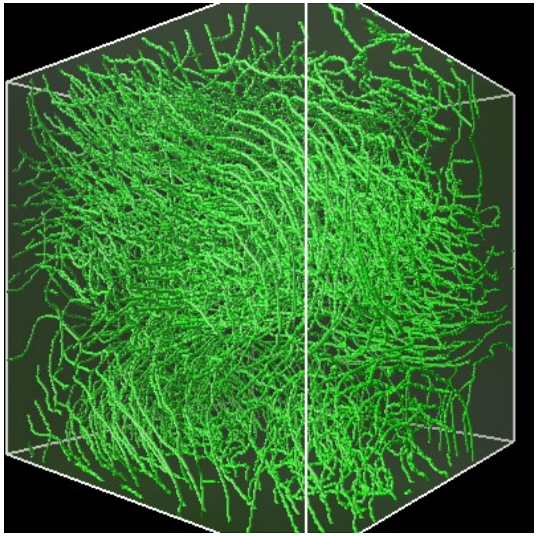
3D : vortex line



quantized circulation

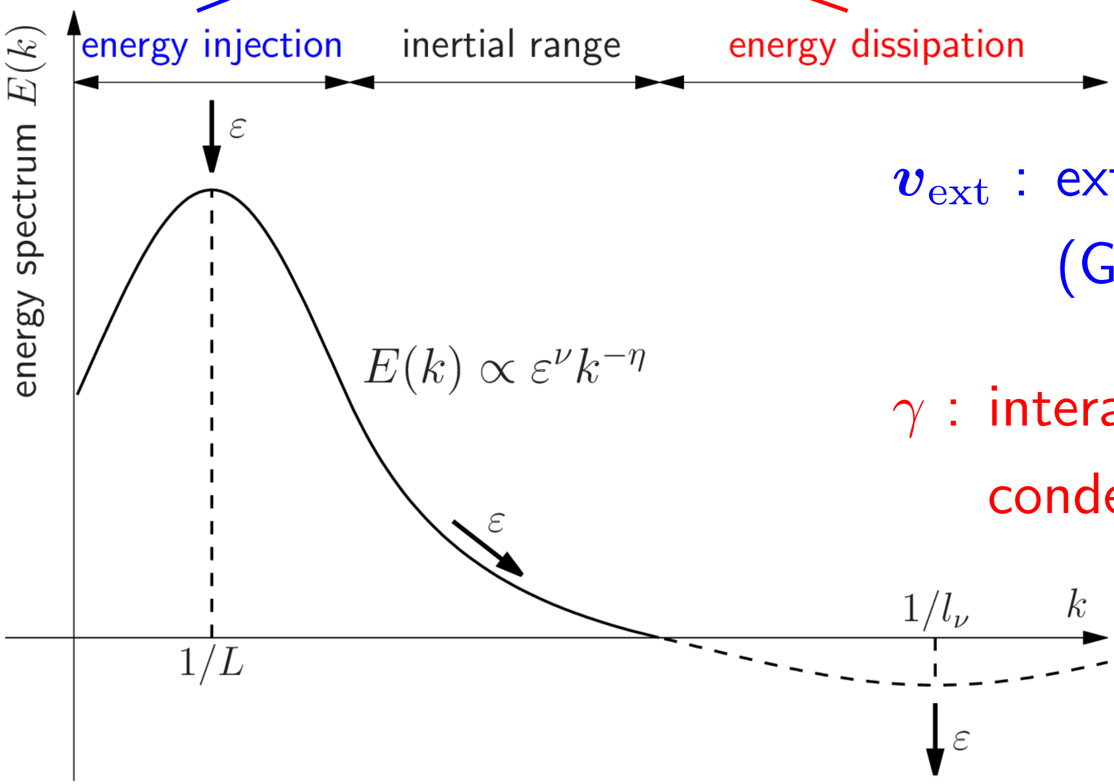
$$\kappa = \oint \boldsymbol{v} \cdot d\boldsymbol{l} = h/M$$

turbulence



How to generate turbulence in GP equation

$$(i\hbar - \gamma)(\partial_t - \boldsymbol{v}_{\rm ext}\cdot\nabla)\psi = \left\{-\frac{\hbar^2}{2M}\nabla^2 + g(|\psi|^2 - \bar{\rho})\right\}\psi$$



$\boldsymbol{v}_{\rm ext}$: external current

(Galillean transformation)

γ : interaction between moving condensate and static thermal gas

Origin of energy dissipation

E. Zaremba, T. Nikuni, and A. Griffin, JLTP **116**, 277 (1999)

thermal gas : Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p} \cdot \nabla f}{m} - 2g\nabla n \cdot \nabla_{\boldsymbol{p}} f = C_{12}(f) + C_{22}(f)$$

condensate : GP equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left\{ -\frac{\hbar^2}{2M}\nabla^2 + g(|\psi|^2 + 2\tilde{n}) + i\Gamma_{12}(f) \right\} \psi$$

exchange between thermal gas and condensate

Assumption : random and Markov process

$$(i\hbar - \gamma)\frac{\partial \psi}{\partial t} = \left\{ -\frac{\hbar^2}{2M}\nabla^2 + g(|\psi|^2 - \bar{\rho}) \right\} \psi + \sqrt{\gamma k_{\mathrm{B}} T}\xi$$

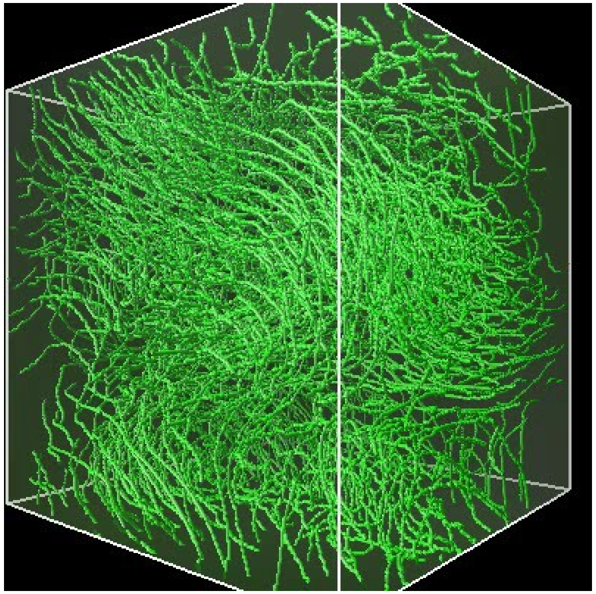
Stochastic GP Eq.

Shifts from Gaussian noise ξ can be renormalized to γ (CLT)

Fully developed quantum turbulence

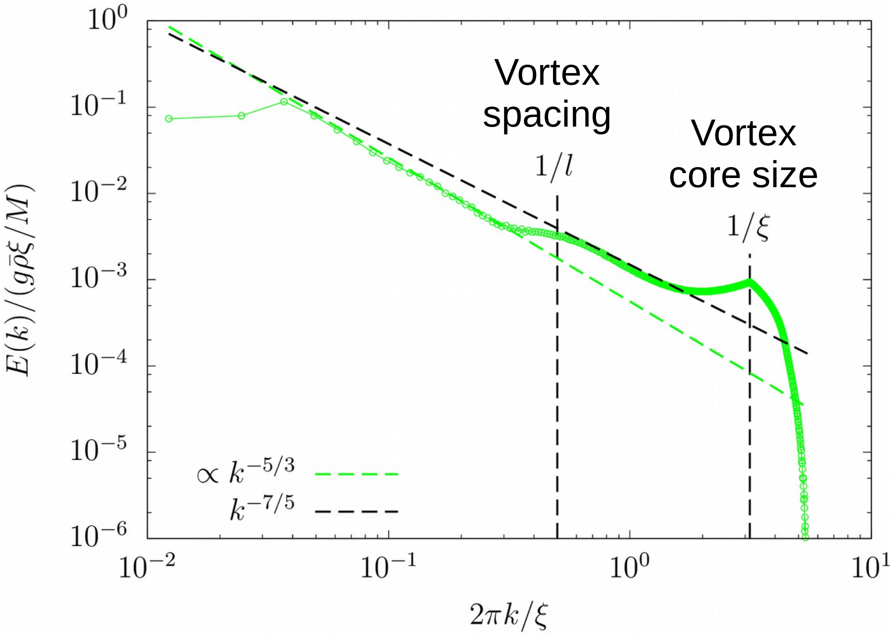
Large scale random current : $(v_{\mathrm{ext}})_{x,y,z} = \bar{v} \sum_{0 \le |\boldsymbol{n}| \le 2} \cos\left(\frac{2\pi \boldsymbol{n} \cdot \boldsymbol{x}}{L} + \theta(\boldsymbol{n})_{x,y,z} \right)$

$$(i\hbar - \gamma)(\partial_t - \boldsymbol{v}_{\mathrm{ext}} \cdot \nabla)\psi$$
$$= \left\{ -\frac{\hbar^2}{2M}\nabla^2 + g(|\psi|^2 - \bar{\rho}) \right\} \psi$$

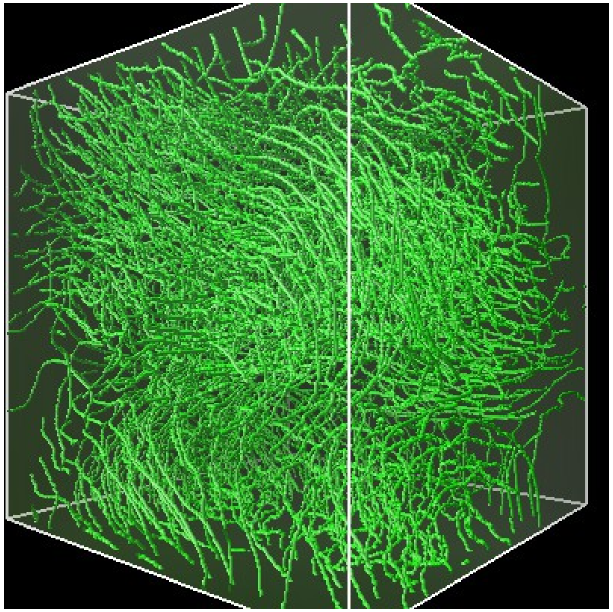


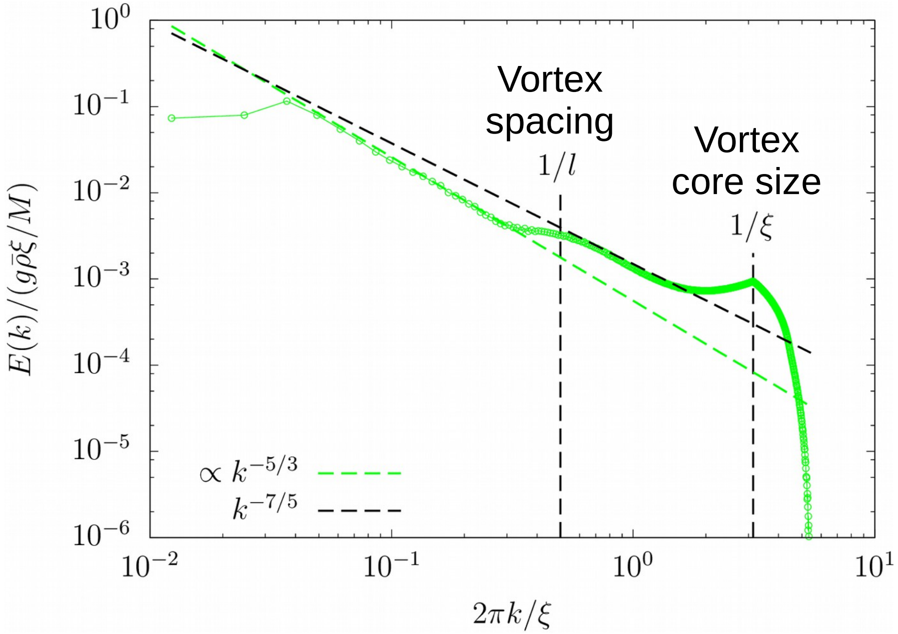
Kinetic energy and its spectrum

$$E = \int d^3x \, |\psi|^2 |\nabla \mathrm{Arg}[\psi]|^2 = \int dk \, E(k)$$



Fully developed quantum turbulence





- Kolmogorov's power law is widely universal over classical and quantum turbulence.
- Quantum nature of vortices is hidden in the inertial range ($k < 1/l$).

We here consider quantum turbulence in which quantum nature of quantized vortices appears.

Plan of talk

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Turbulence with weak energy injection

strength of turbulence (Reynolds number if classical fluid)

static laminar flow without vortices

turbulence with vortices

fully developed turbulence (K41)

Laminar-turbulent transition (nonequilibrium critical state)

(X) Whether vortices can be nucleated under the flow?

(O) Whether vortices can survive under the flow?

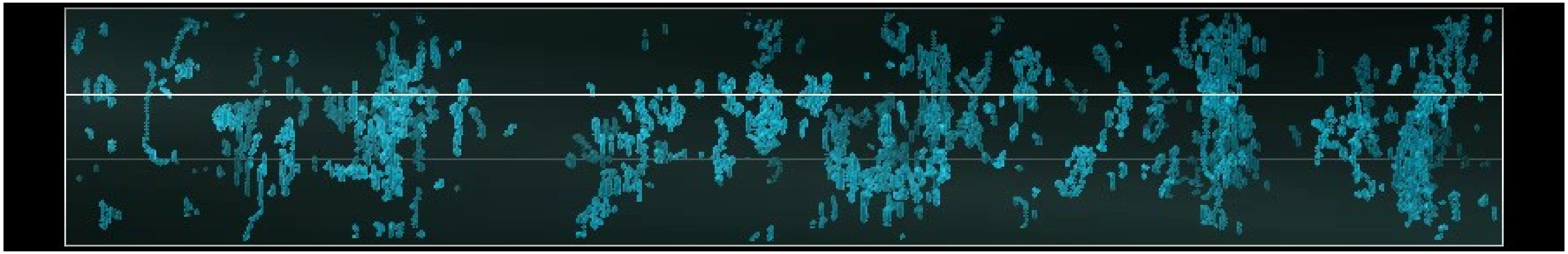
pipe-flow simulation : $V = 512\xi \times 64\xi \times 64\xi$ with periodic BC

$$\boldsymbol{v}_{\mathrm{ext}} = (v_x, 0, 0) \quad \gamma = 0.1\hbar$$

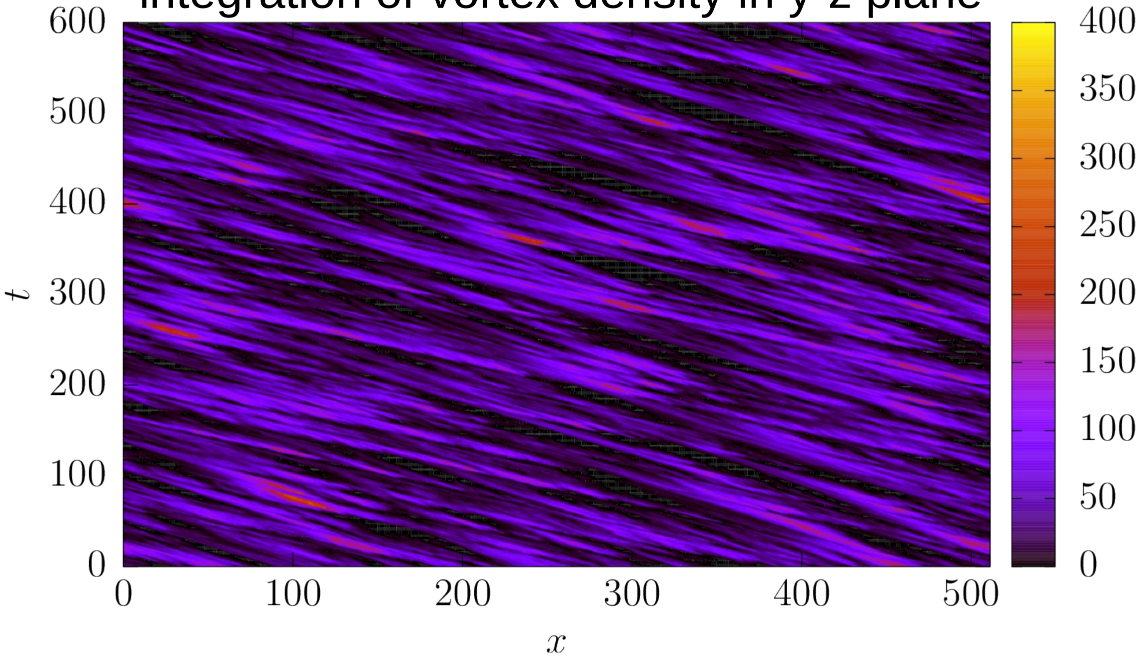
$$(i\hbar - \gamma)(\partial_t - \boldsymbol{v}_{\mathrm{ext}} \cdot \nabla)\psi = \left\{ -\frac{\hbar^2}{2M}\nabla^2 + g(|\psi|^2 - \bar{\rho}) \right\} \psi$$

Pipe flow turbulence simulation

$$\boldsymbol{v}_{\mathrm{ext}} = (0,0,v_0) \quad v_0 = 0.85c_{\mathrm{s}} \quad c_{\mathrm{s}} = \sqrt{g\bar{\rho}/M} \ : \text{ sound velocity}$$

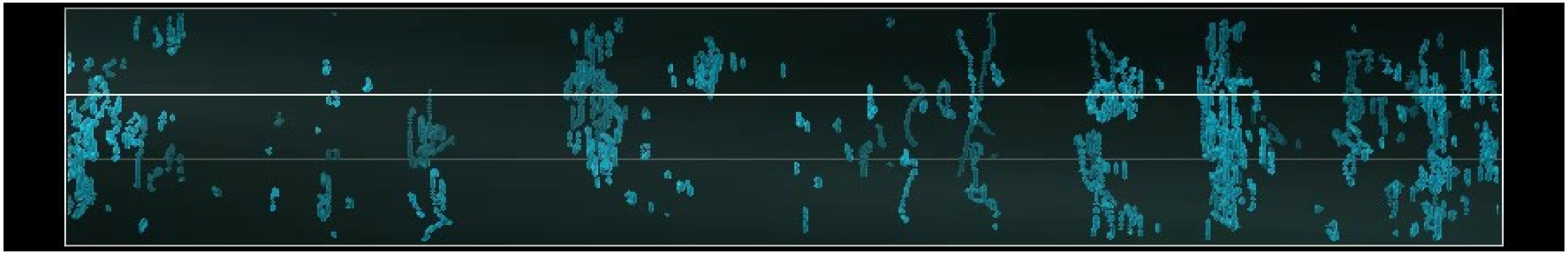


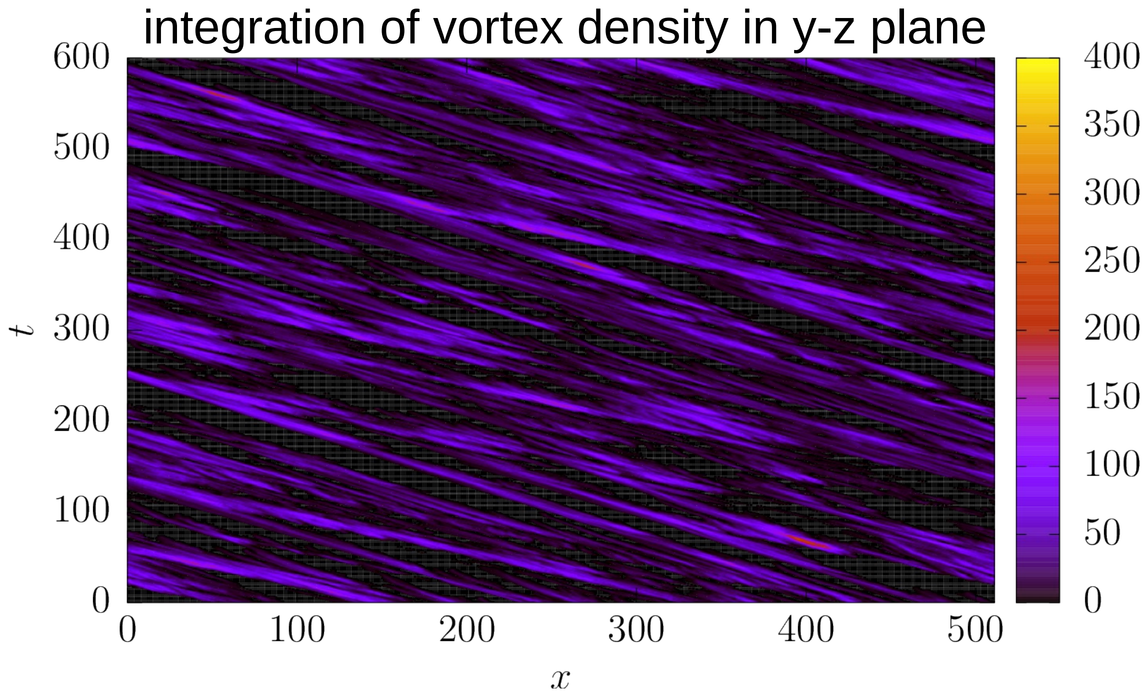
integration of vortex density in y-z plane



Pipe flow turbulence simulation

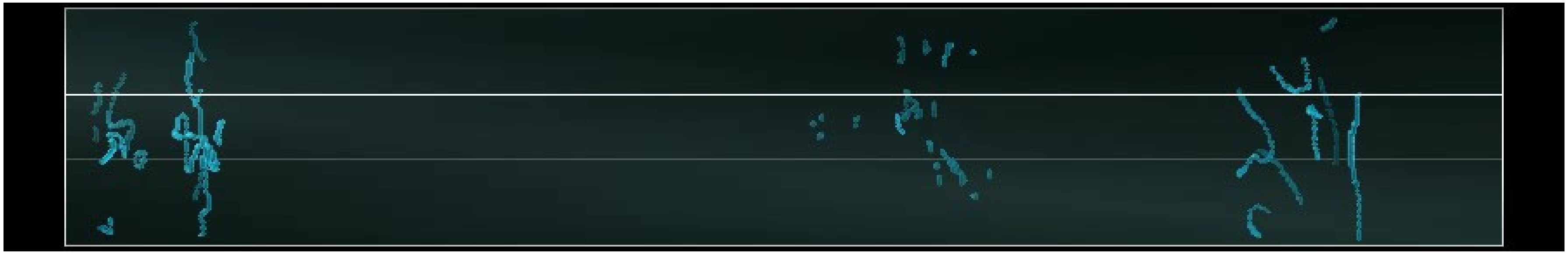
$$\boldsymbol{v}_{\text{ext}} = (0, 0, v_0) \quad v_0 = 0.8c_{\text{s}} \quad c_{\text{s}} = \sqrt{g\bar{\rho}/M}$$: sound velocity

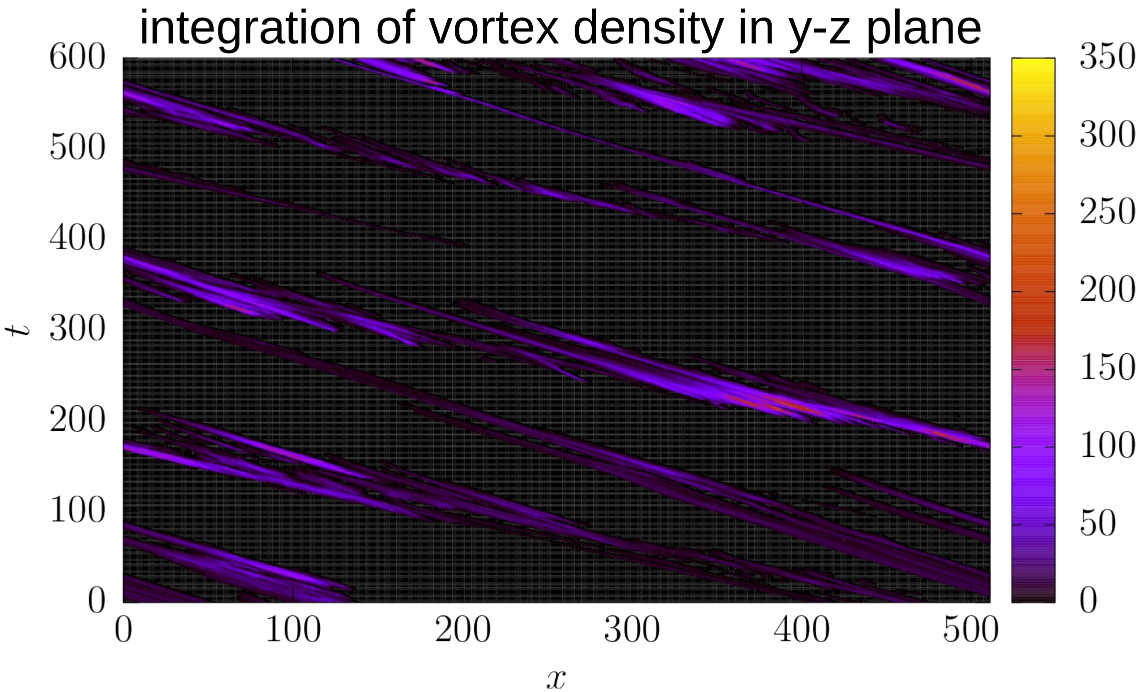




Pipe flow turbulence simulation

$$\boldsymbol{v}_{\rm ext} = (0, 0, v_0) \quad v_0 = 0.75 c_{\rm s} \quad c_{\rm s} = \sqrt{g\bar{\rho}/M} \text{ : sound velocity}$$

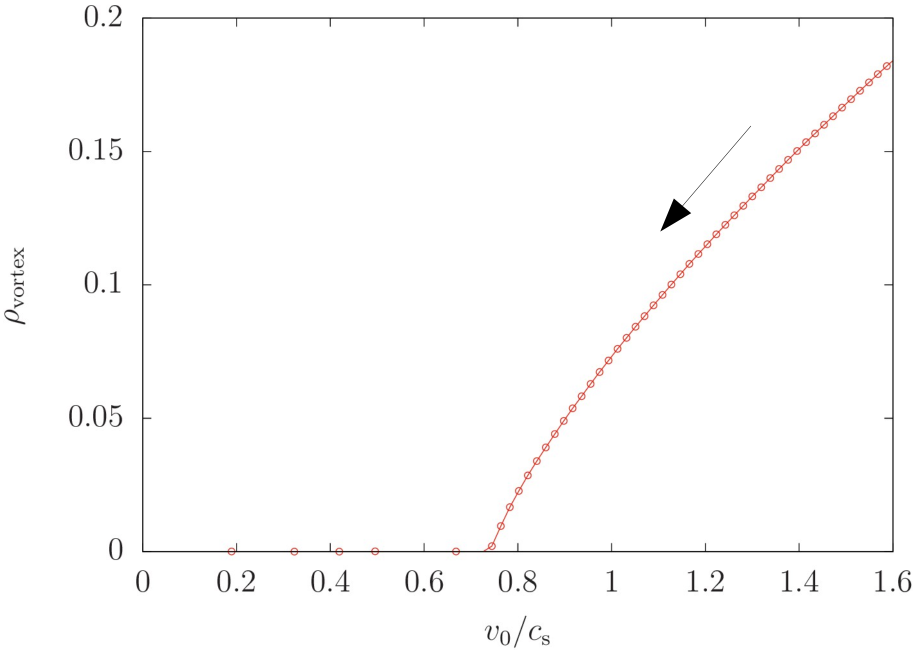


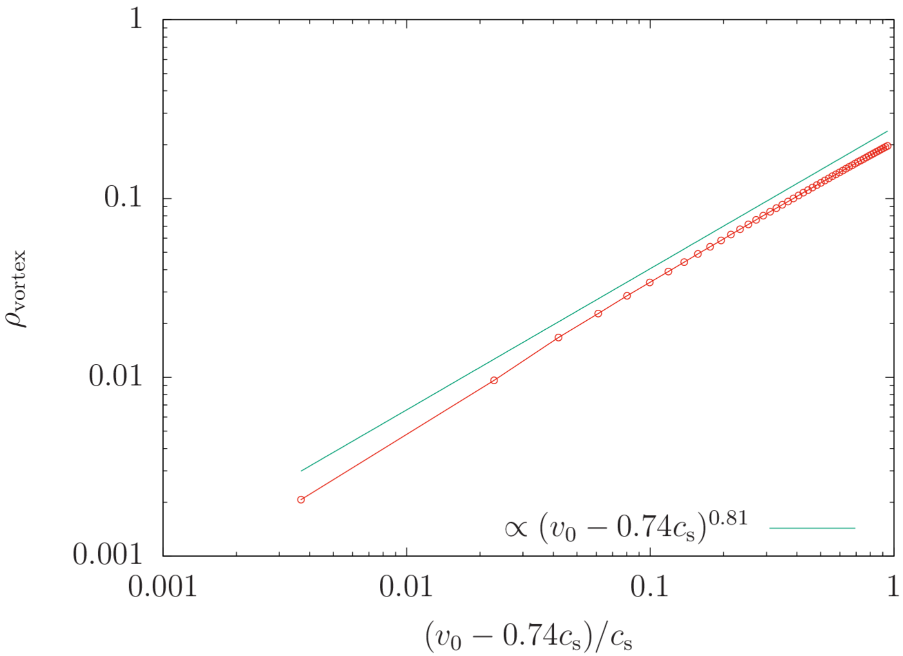


- Turbulent domains are localized.
- A domain sometimes splits into two domains, or is annihilated in a stochastic manner.
- An annihilated domain never returns to turbulent domain (finite energy gap).

Order parameter of turbulence

Order parameter of turbulence : vortex density ρ_{vortex}



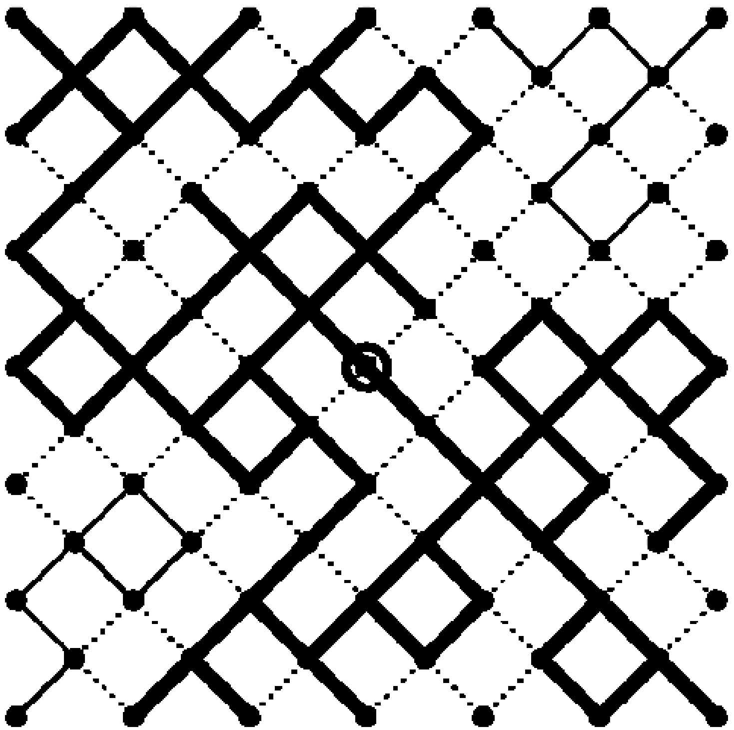


Transition looks like second ordered thermodynamic transition in equilibrium (critical exponent : $\beta = 0.81$)

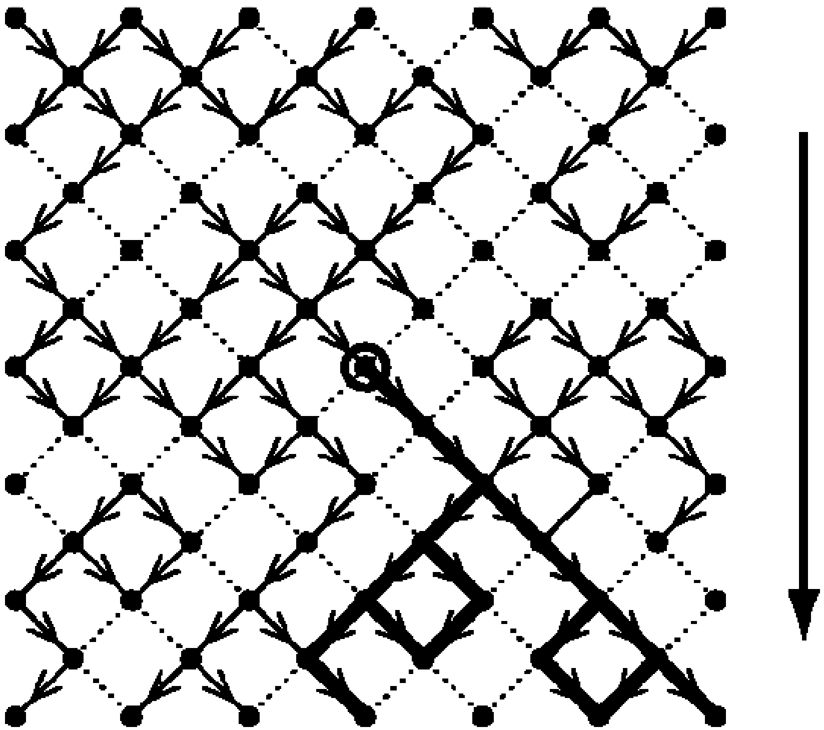
Directed percolation

$\beta = 0.81$: 1+3D directed percolation universality class

Directed percolation : percolation is directed in 1 dimension



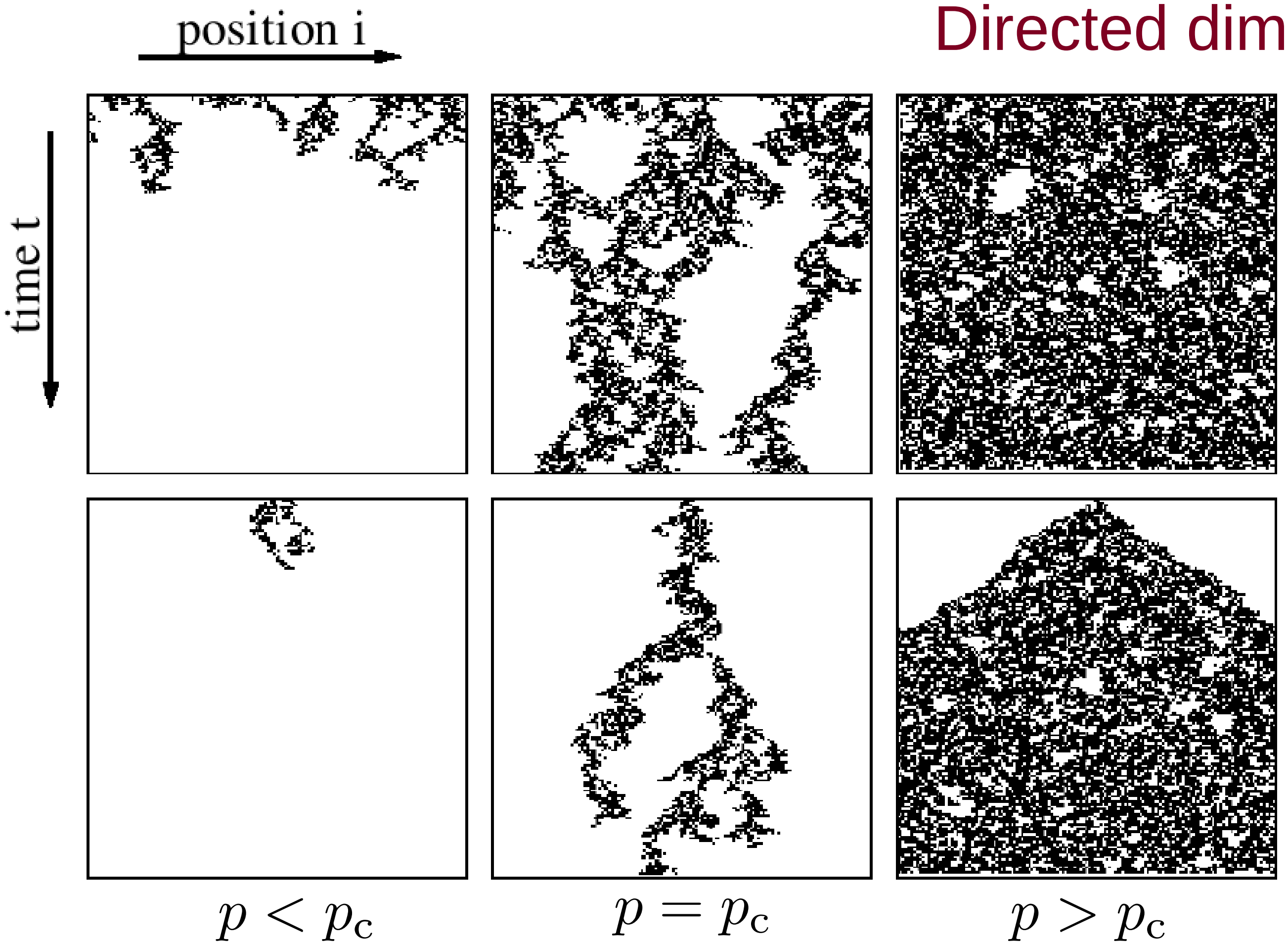
2D isotropic bond percolation



1+1D directed bond percolation

Directed percolation & quantum turbulence

Directed dimension is set to time



$p < p_c$:
Percolation in time is stopped

$v_0 < v_c$:
When turbulent region is annihilated, it never go back to turbulence

p : percolation probability

Directed percolation & quantum turbulence

Vortex is a topological defect with a finite energy gap and is never nucleated after annihilation (at least for $v_0 < c_\mathrm{s}$)
→ Nature of directed percolation universality for the laminar-turbulent transition

We are now checking other critical exponents (especially η_time & η_space)

M. Takahashi, M. Kobayashi, and K. Takeuchi, will be appeared in arXiv

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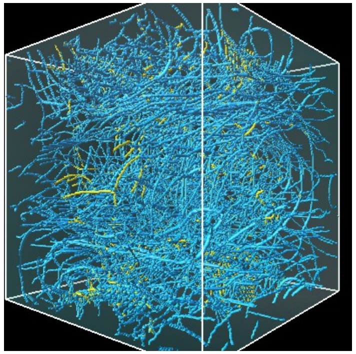
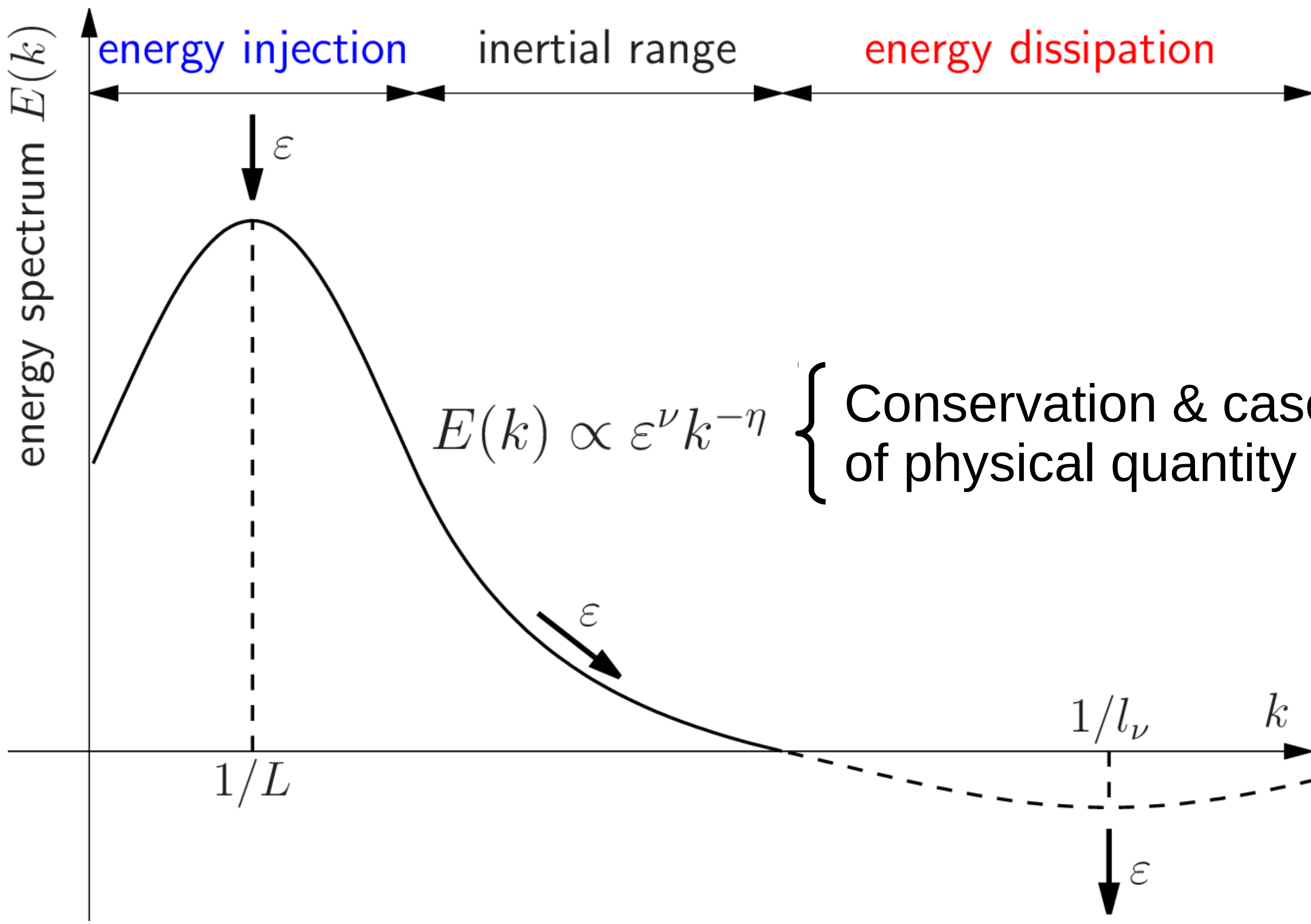


Universality of fully developed turbulence



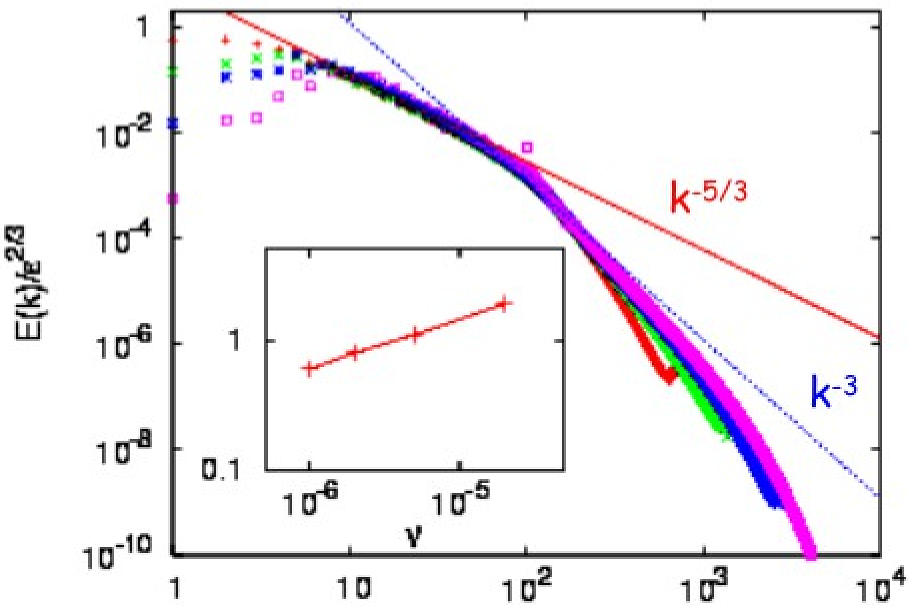
Conservation & cascade
of physical quantity

Conserving quantity in hydrodynamic equation

$$E = \frac{1}{2}\int d\boldsymbol{x}\, \boldsymbol{v}^2 = [L^2 T^{-2}] \Rightarrow E(k) \propto \varepsilon_E^{2/3} k^{-5/3} \text{ : energy (2D, 3D)}$$

$$\Omega = \frac{1}{2}\int d\boldsymbol{x}\, (\nabla \times \boldsymbol{v})^2 = [T^{-2}] \Rightarrow E(k) \propto \varepsilon_\Omega^{2/3} k^{-3} \text{ : enstrophy (2D)}$$

$$H = \int d\boldsymbol{x}\, \boldsymbol{v} \cdot (\nabla \times \boldsymbol{v}) = [L T^{-2}] \Rightarrow E(k) \propto \varepsilon_H^{2/3} k^{-7/3} \text{ : helicity (3D)}$$



G. Boffetta and A. Celani, 2005

Energy spectrum with the helicity cascade : $E(k) \propto k^{-7/3}$ has never been observed in 3D classical turbulence

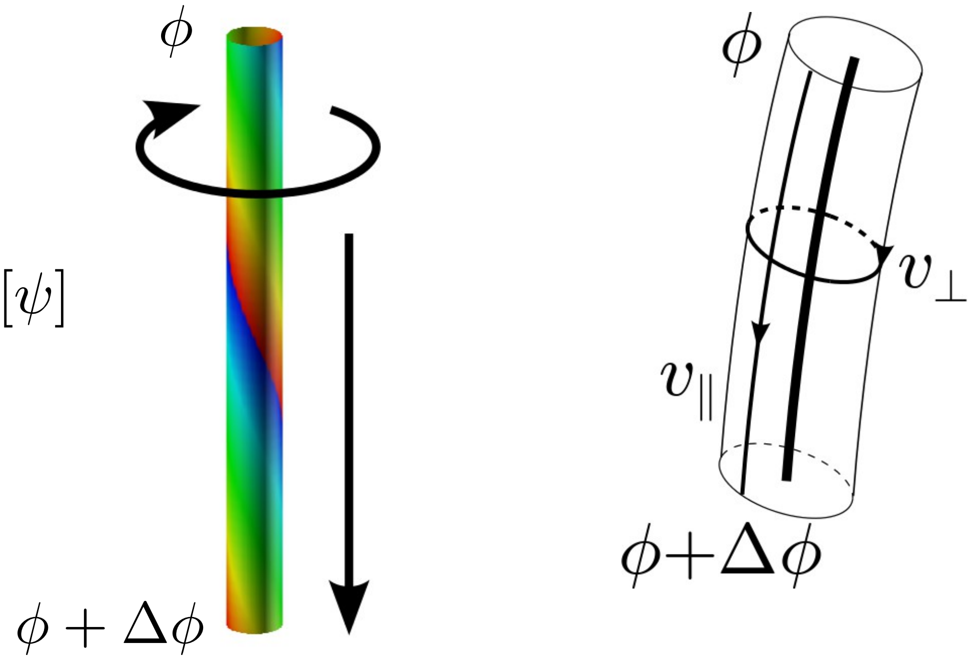
→Helicity and its cascade seem not to be so important in turbulence. Why?

Helicity in quantum fluid

In quantum fluid, helicity consists of two parts.

$$H = \int d\boldsymbol{x}\, \boldsymbol{v} \cdot (\nabla \times \boldsymbol{v}) = H_{\mathrm{twist}} + H_{\mathrm{writhe}}$$

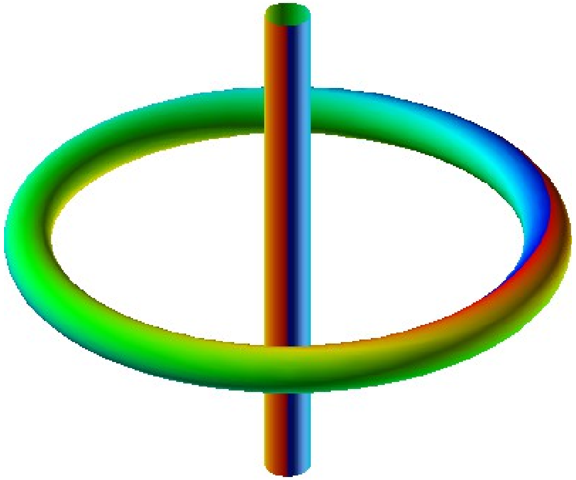
$$\phi \equiv \mathrm{Arg}[\psi]$$

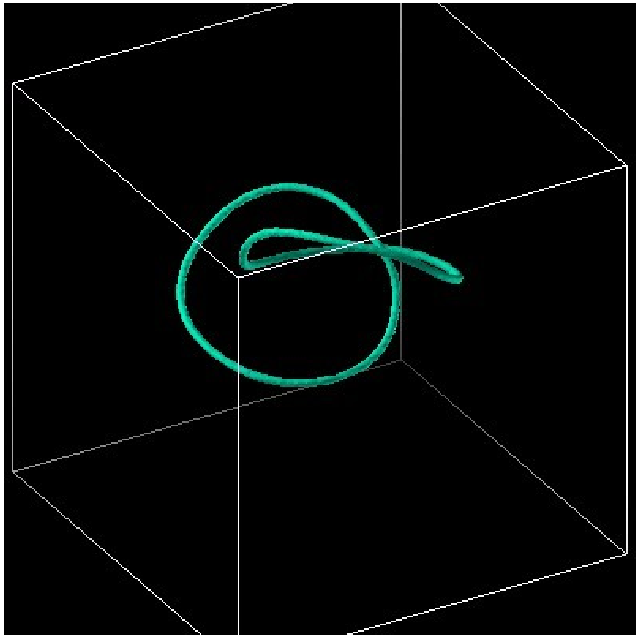


$$H_{\mathrm{twist}} \equiv \frac{\kappa^2}{2\pi} \int ds\, \partial_\parallel \phi(\boldsymbol{x} = \boldsymbol{x}_{\mathrm{vortex}}(s) + \boldsymbol{\epsilon}) = \kappa^2 \frac{\Delta\phi}{2\pi}$$

Helicity in quantum fluid

- H_{twist} is quantized by κ^2 for closed loops
- A loop should be linked or knotted to have a finite H_{twist} (H_{twist} corresponds to linking number of vortices)
- Vortex reconnection can cause the change of the helicity





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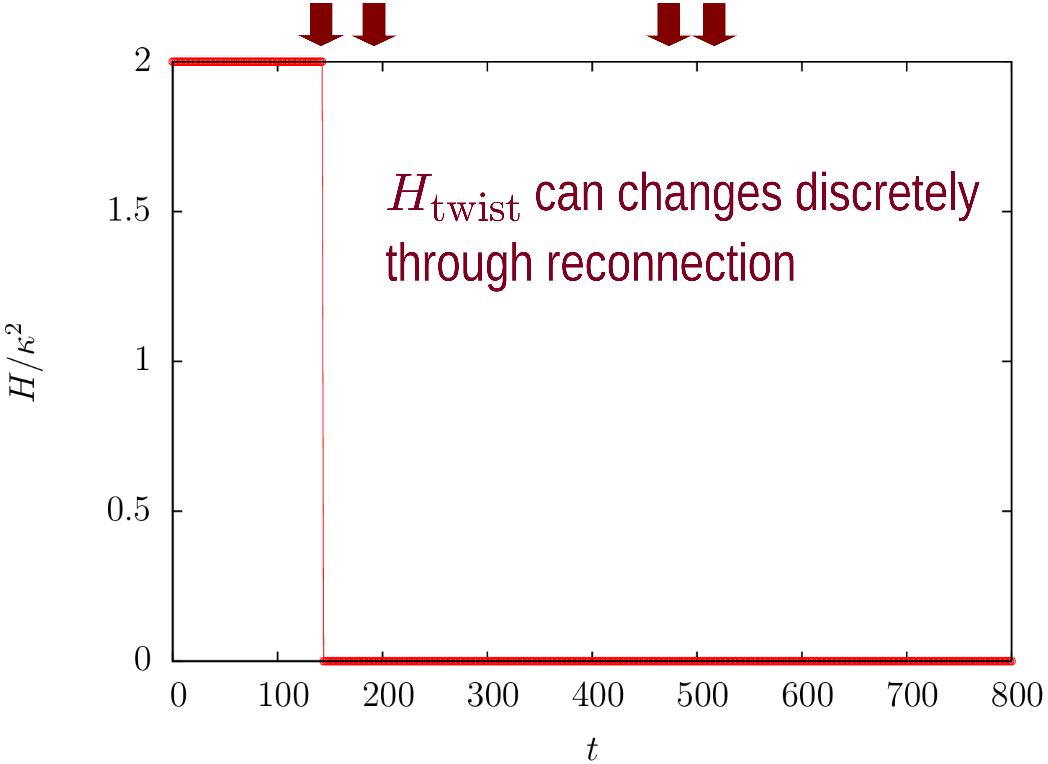
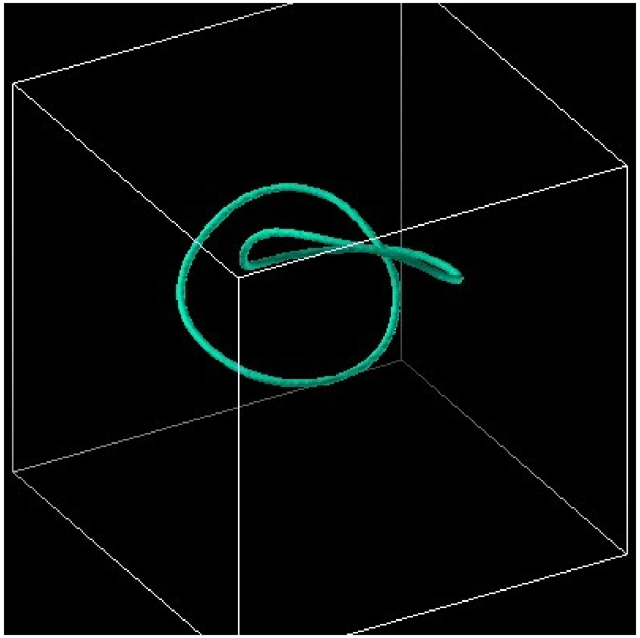
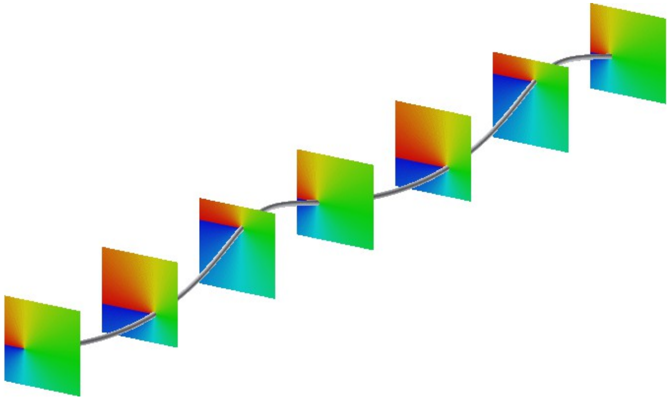




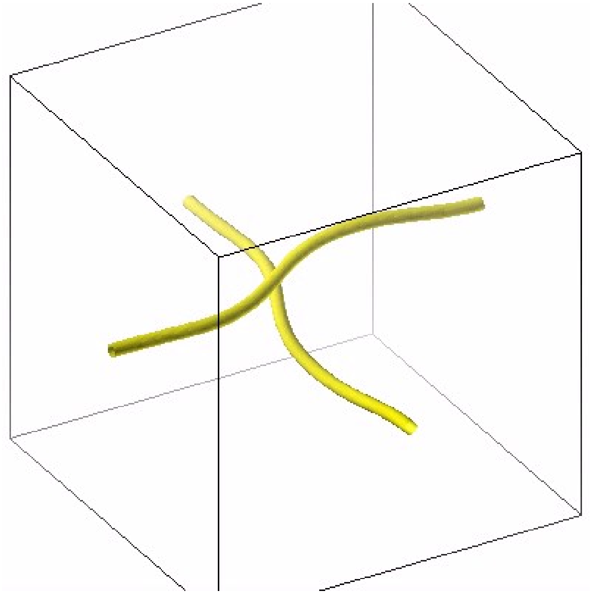
Helicity in quantum fluid

- $H_{\rm writhe}$ is the other contribution of the helicity (spiral Kelvin waves)
- $H_{\rm writhe}$ is not the topological quantity (Thank you very much for Prof. M. E. Brachet).
- Vortex reconnection transfer from $H_{\rm twist}$ to $H_{\rm writhe}$



Helicity in quantum fluid

- H_{writhe} is the other contribution of the helicity (spiral Kelvin waves)
- H_{writhe} is not the topological quantity and is easily dissipated (Thank you very much for Prof. M. E. Brachet).
- Vortex reconnection transfer from H_{twist} to H_{writhe}



Can we suppress vortex reconnection?

Key idea : non-Abelian vortex in spinor condensate

spin-S spinor wave function : $\psi = \begin{pmatrix} \psi_{-S} & \cdots & \psi_0 & \cdots & \psi_S \end{pmatrix}^T$

$$\mathcal{H} = \int d^3x \left[\frac{\hbar^2}{2M} \sum_{m=-S}^{S} |\nabla \psi_m|^2 \right.$$
$$\left. + \frac{1}{2} \sum_{L=0}^{4} g_L \sum_{\mu=-L}^{L} \sum_{m_1,\cdots,m_4=-S}^{S} C^{L\mu}_{m_1 m_2} \left(C^{L\mu}_{m_3 m_4} \right)^* \psi^*_{m_1} \psi^*_{m_2} \psi_{m_3} \psi_{m_4} \right]$$

- Symmetry of the Hamiltonian : $U(1) \times SO(3)$
- Classification of topological charges of vortex : discrete subgroup of $U(1) \times SO(3)$

Possible discrete subgroup of $SO(3)$

L. Michel, Rev. Mod. Phys. **52**, 617 (1980)

S	C_2	C_3	C_4	C_5	D_1	D_2	D_3	D_4	D_5	T	O	Y
1	×	×	×	×	○	○	×	×	×	×	×	×
2	○	○	×	×	○	○	○	×	×	○	×	×
3	○	×	×	×	○	○	○	○	×	×	○	×
4	○	○	○	○	○	○	○	○	○	×	×	×
5	○	○	×	×	○	○	○	○	○	○	○	○

Abelian (or solvable)

Non-Abelian

Atomic species for ultracold BEC in an nonmagnetic optical trap

^{87}Rb, ^{23}Na, ^{7}Li, ^{41}K	$S = 1, 2$
^{85}Rb	$S = 2, 3$
^{133}Cs	$S = 3, 4$
^{52}Cr	$S = 3$

Spin-2 condensate

$$\psi = \begin{pmatrix}\psi_{-2} & \psi_{-1} & \psi_0 & \psi_1 & \psi_2\end{pmatrix}^T$$

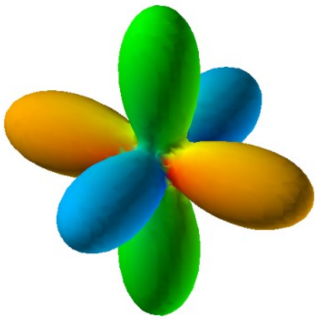
$$H = \int d^3x \left[\frac{\hbar^2}{2M} \sum_{m=-2}^{2} |\nabla \psi_m|^2 + \frac{c_0}{2}(\rho - \bar{\rho})^2 + \frac{c_1}{2}|A|^2 + \frac{c_2}{2}\boldsymbol{S}^2 \right]$$

$$\rho = \sum_{m=-2}^{2} \psi_m^* \psi_m \quad A = \sum_{m=-2}^{2} (-1)^m \psi_m \psi_{-m} \quad \boldsymbol{S} = \sum_{m,n=-2}^{2} \psi_m^* \hat{\boldsymbol{S}}_{mn} \psi_n$$

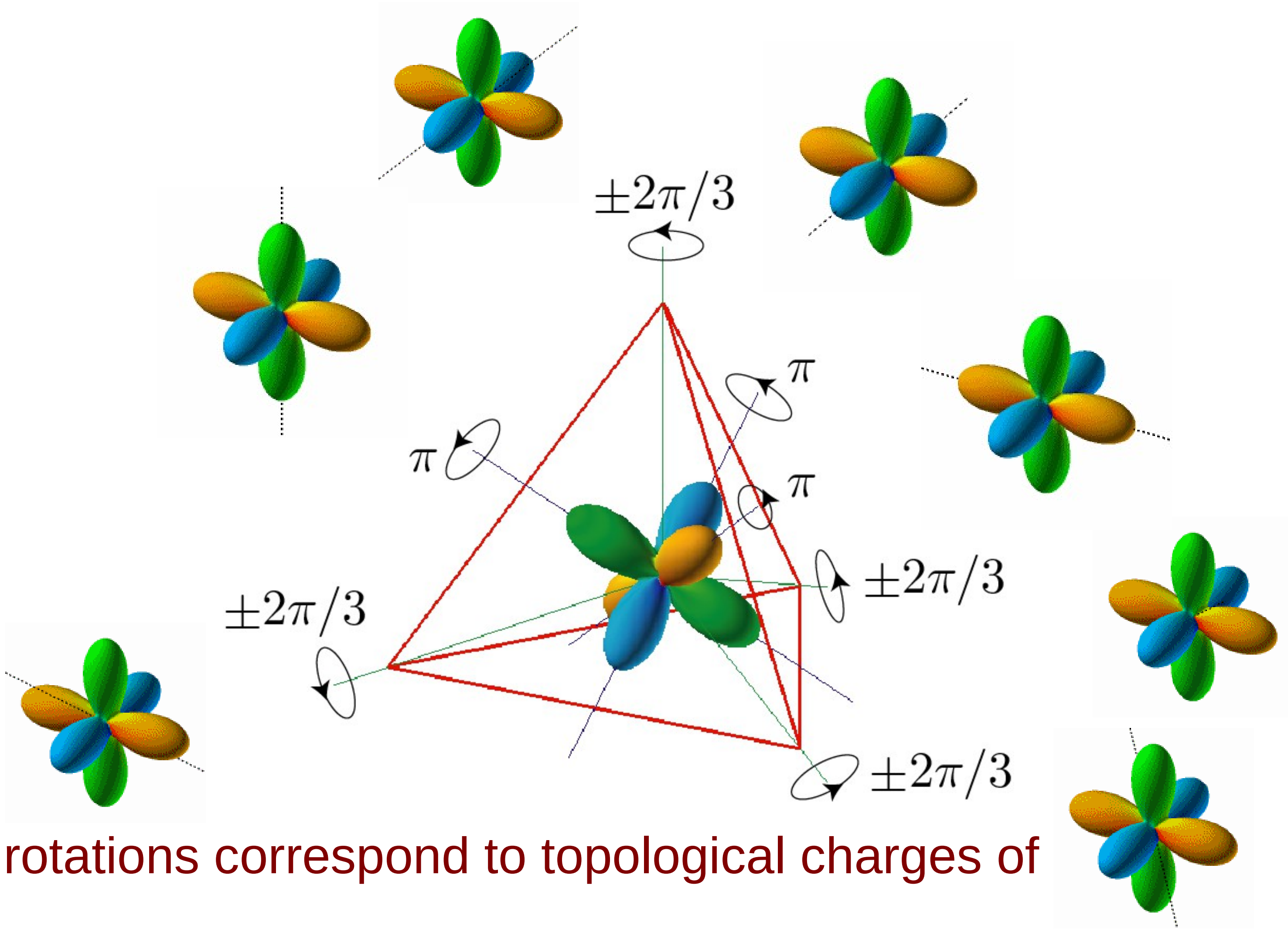
$$c_0 = \frac{4g_2 + 3g_4}{7} \quad c_1 = \frac{7g_0 - 10g_2 + 3g_4}{35} \quad c_2 = \frac{g_4 - g_2}{7}$$

With $c_{0,1,2} > 0$, non-Abelian tetrahedral symmetry for topological charges of vortices becomes possible

ground state : $\psi = \frac{\sqrt{\bar{\rho}}}{2}\begin{pmatrix} i & 0 & \sqrt{2} & 0 & i \end{pmatrix}^T$



Tetrahedral symmetry



12 spin rotations correspond to topological charges of vortices

Gross-Pitaevskii Equation

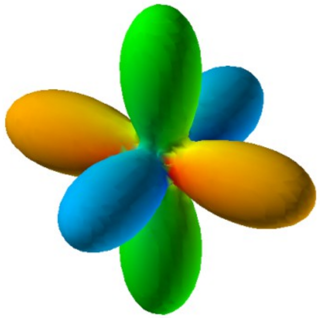
$$\text{GP equation}: \ i\hbar\dot{\psi}_m = \frac{\delta\mathcal{H}}{\delta\psi_m^*}$$

$$i\hbar\dot{\psi}_{\pm2} = -\frac{\hbar^2}{2M}\nabla^2\psi_{\pm2} + c_0(\rho-\bar{\rho})\psi_{\pm2} + c_1(S_\mp\psi_{\pm1} \pm 2S_z\psi_{\pm2}) + c_2A\psi_{\mp2}^*$$

$$i\hbar\dot{\psi}_{\pm1} = -\frac{\hbar^2}{2M}\nabla^2\psi_{\pm1} + c_0(\rho-\bar{\rho})\psi_{\pm1} + c_1\left(\frac{\sqrt{6}}{2}S_\mp\psi_0 + S_\pm\psi_{\pm2} \pm S_z\psi_{\pm1}\right) - c_2A\psi_{\mp1}^*$$

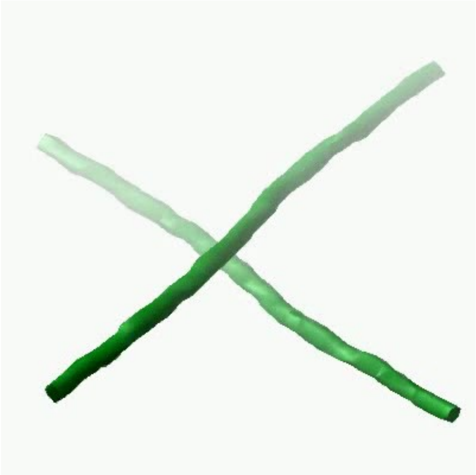
$$i\hbar\dot{\psi}_0 = -\frac{\hbar^2}{2M}\nabla^2\psi_0 + c_0(\rho-\bar{\rho})\psi_0 + \frac{\sqrt{6}}{2}c_1(S_-\psi_{-1} + S_+\psi_1) + c_2A\psi_0^*$$

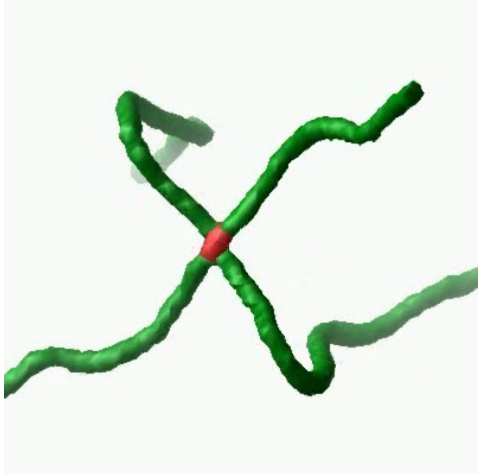
$$\text{ground state}: \ \psi = \frac{\sqrt{\bar{\rho}}}{2}\begin{pmatrix} i & 0 & \sqrt{2} & 0 & i \end{pmatrix}^T$$



Collision dynamics of non-Abelian vortices

$$\text{GP equation : } i\hbar\dot{\psi}_m = \frac{\delta \mathcal{H}}{\delta \psi_m^*}$$



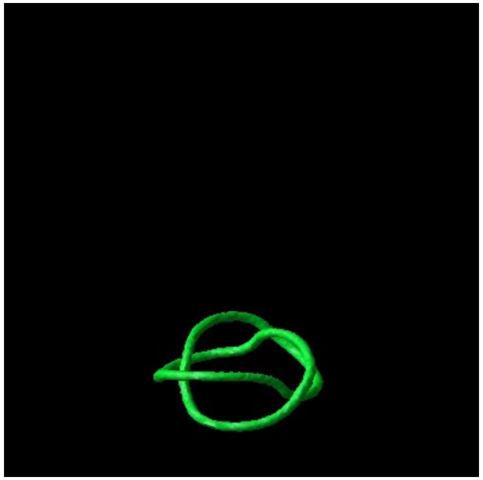


Topological configuration of vortices are kept (as a spin helicity) through the collision due to the formation of a new “rung” vortex

Helicity conservation is topologically protected through the collision

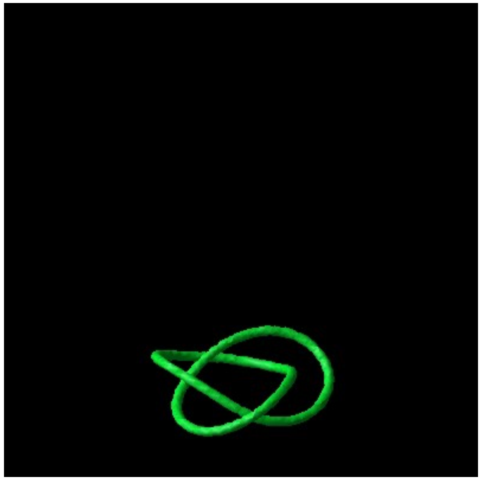
Topological stability of non-Abelian vortex knot

Abelian trefoil



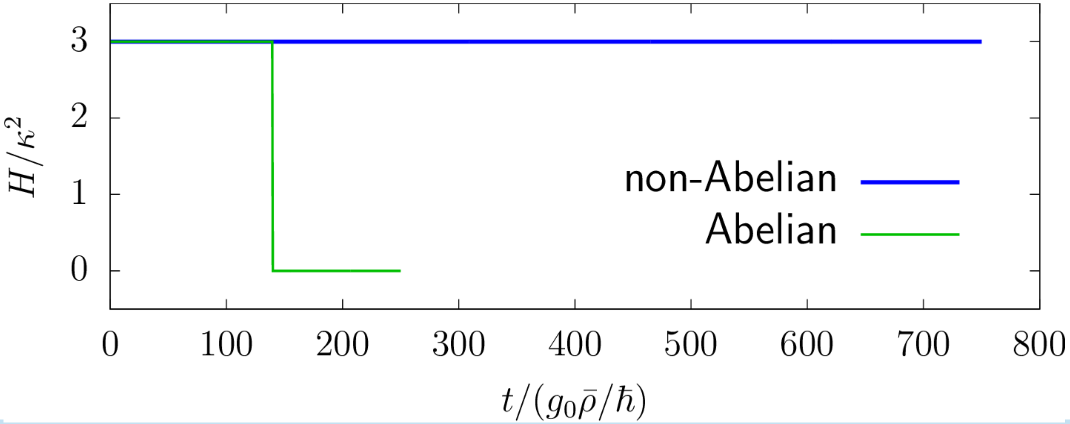
Knot is unstable

Non-Abelian trefoil



Knot is stable

C. F. Barenghi, Milan J. Math 75, 177 (2007)

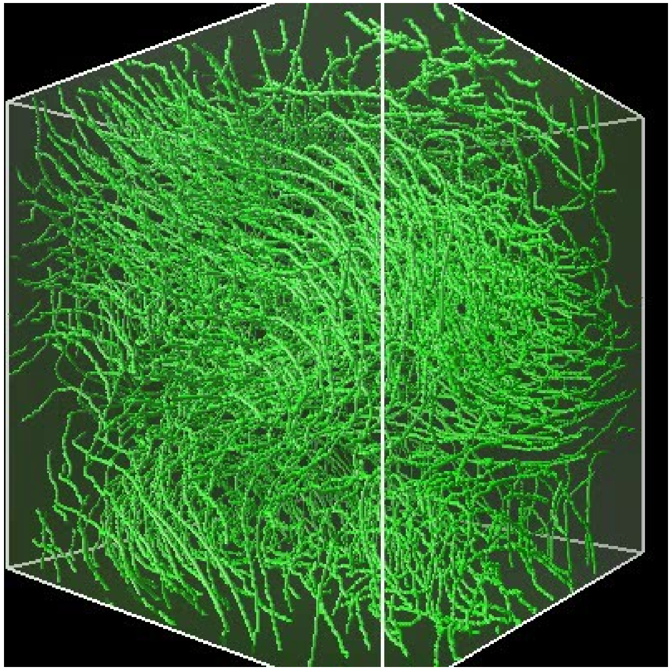


- Abelian trefoil breaks two rings and loses H_{twist}
- Non-Abelian trefoil keeps its topological structure and H_{twist}

Non-Abelian quantum turbulence

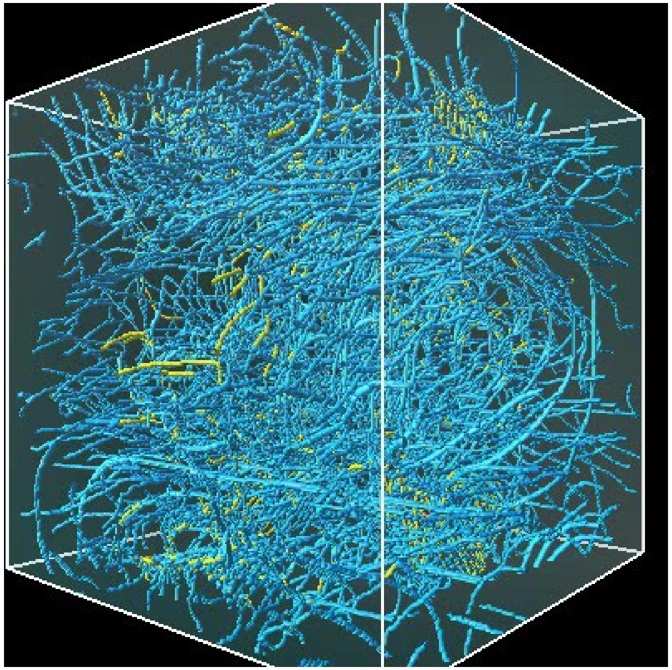
$$(i\hbar - \gamma)(\partial_t - \boldsymbol{v}_{\mathrm{ext}} \cdot \nabla)\psi_m = \frac{\delta \mathcal{H}}{\delta \psi_m^*} \qquad v_{x,y,z} = \bar{v} \sum_{0 \leq |\boldsymbol{n}| \leq 2} \cos\left(\frac{2\pi \boldsymbol{n} \cdot \boldsymbol{x}}{L} + \theta(\boldsymbol{n})_{x,y,z}\right)$$

Abelian turbulence



Vortices are globally dynamic

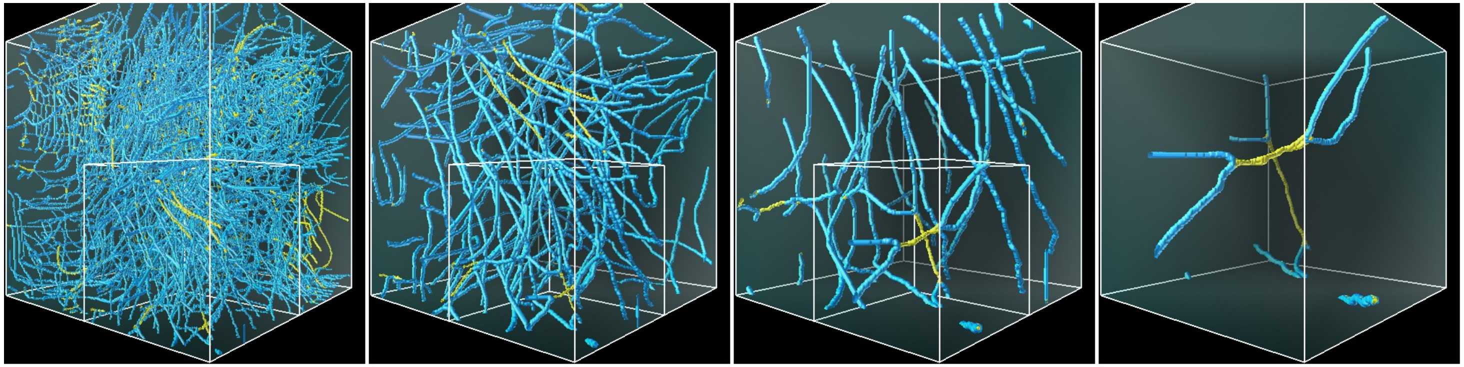
Non-Abelian turbulence



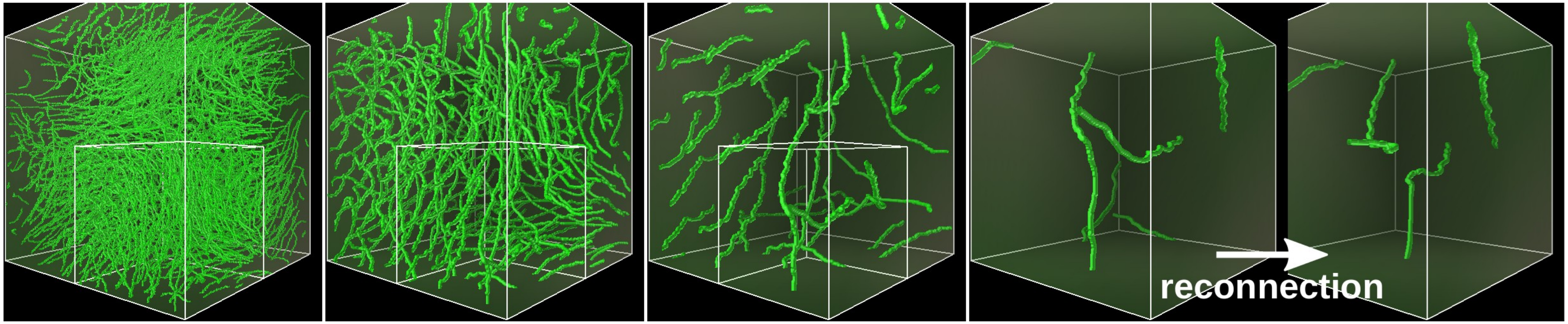
Vortices are locally dynamic

Non-Abelian quantum turbulence

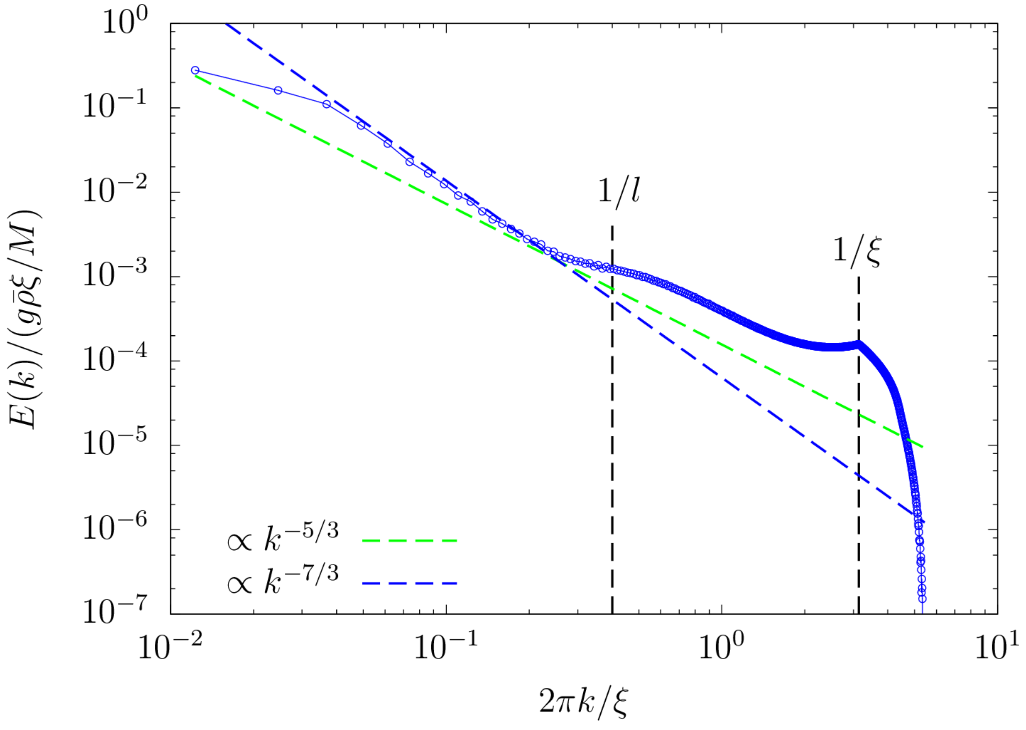
$$(i\hbar - \gamma)(\partial_t - \boldsymbol{v}_{\mathrm{ext}} \cdot \nabla)\psi_m = \frac{\delta \mathcal{H}}{\delta \psi_m^*} \qquad v_{x,y,z} = \bar{v} \sum_{0 \le |\boldsymbol{n}| \le 2} \cos\left(\frac{2\pi \boldsymbol{n} \cdot \boldsymbol{x}}{L} + \theta(\boldsymbol{n})_{x,y,z}\right)$$



Almost all vortices are connected through rung vortices



Quantum turbulence comprised of non-Abelian vortices



It apparently deviates from -5/3 power law (energy resides in large-scale network structures of vortices)

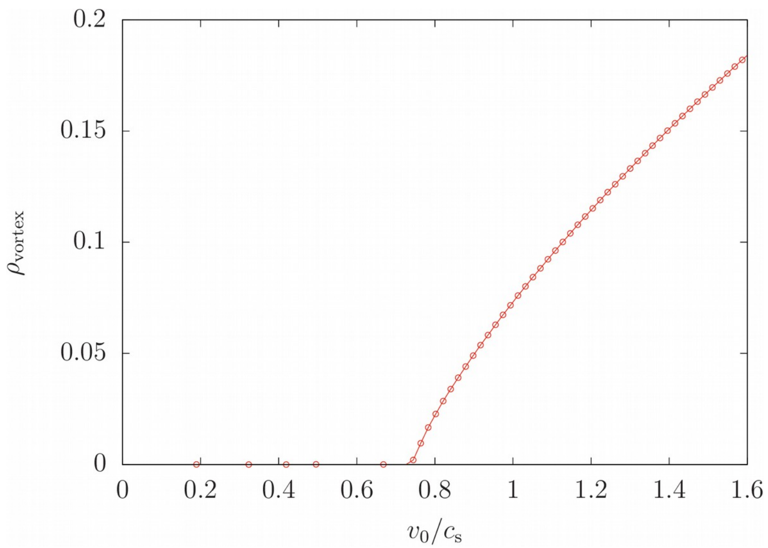
Slope is -7/3 : consistent with the universality with the helicity cascade in turbulence

M. Kobayashi and M. Ueda, arXiv:1606.07190

Summary

We have observed two new universality class in quantum turbulence.

(1) : Directed percolation universality class at laminar-turbulent transition



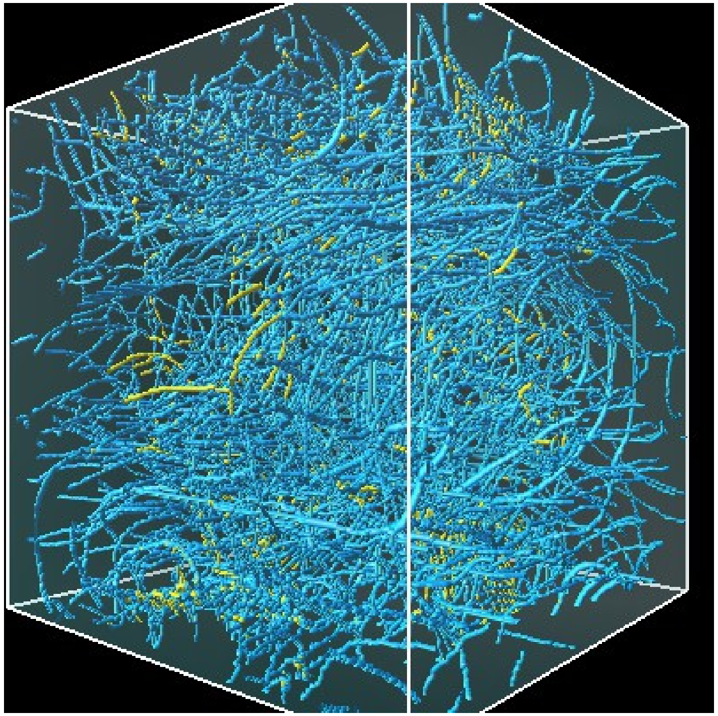
Turbulent domain is never relived after its annihilation due to the finite energy gap of quantized vortex

(Sad news) : DP universality has been observed in classical fluid
→ not specific to quantum fluid

Summary

We have observed two new universality class in quantum turbulence.

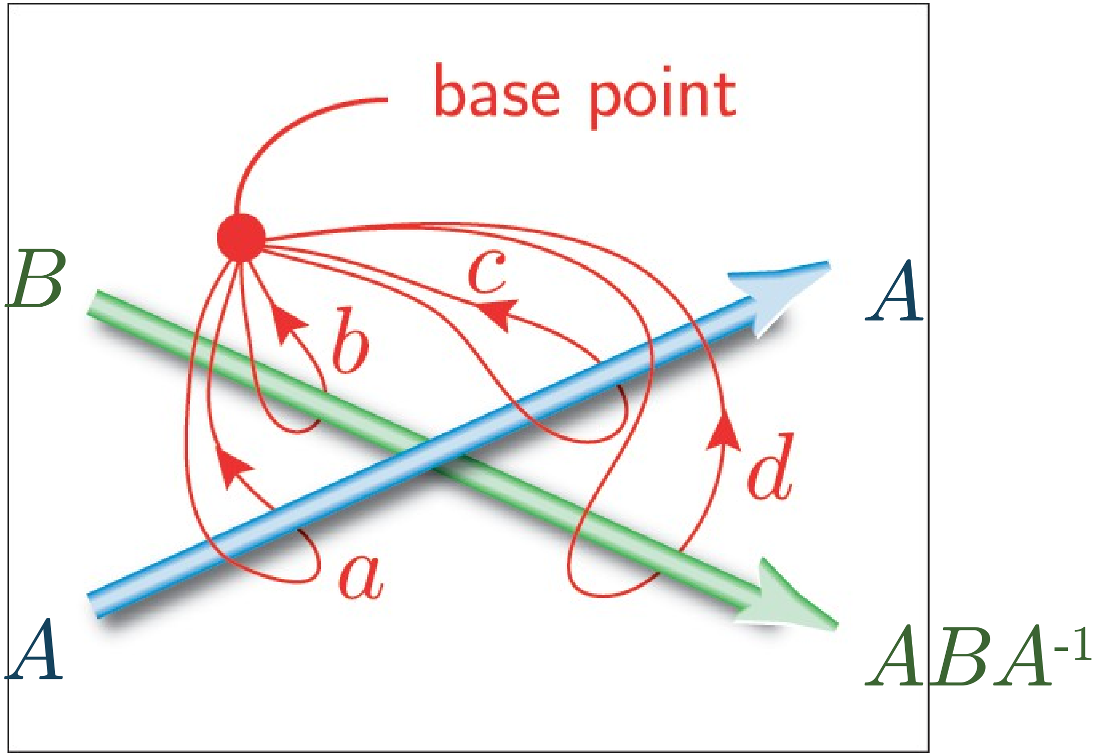
(2) : Topologically protected helicity cascade
in non-Abelian quantum turbulence



By topologically suppressing vortex reconnections, we obtain new universality class $E(k) \propto k^{-7/3}$ characterized by the helicity cascade in fully developed turbulence.

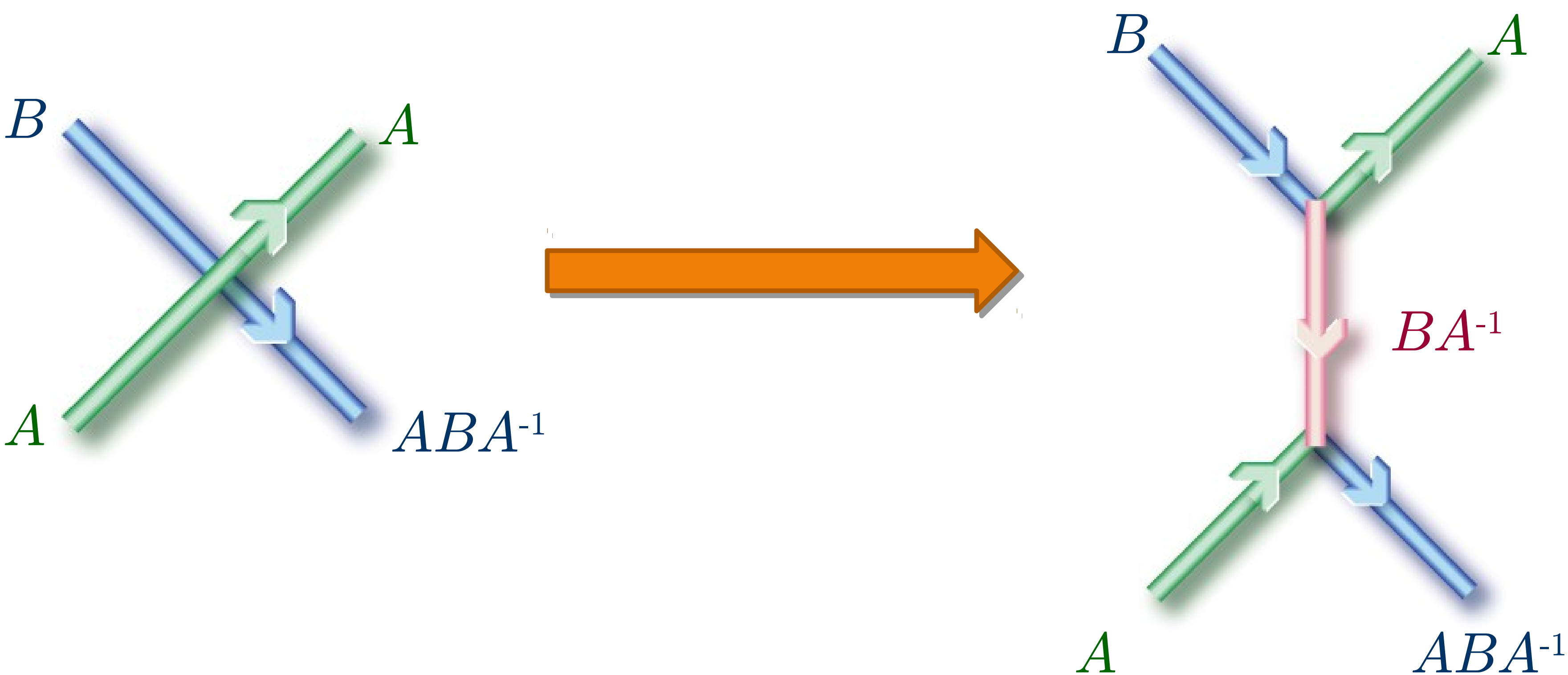
Thank you very much for your attention

Algebra



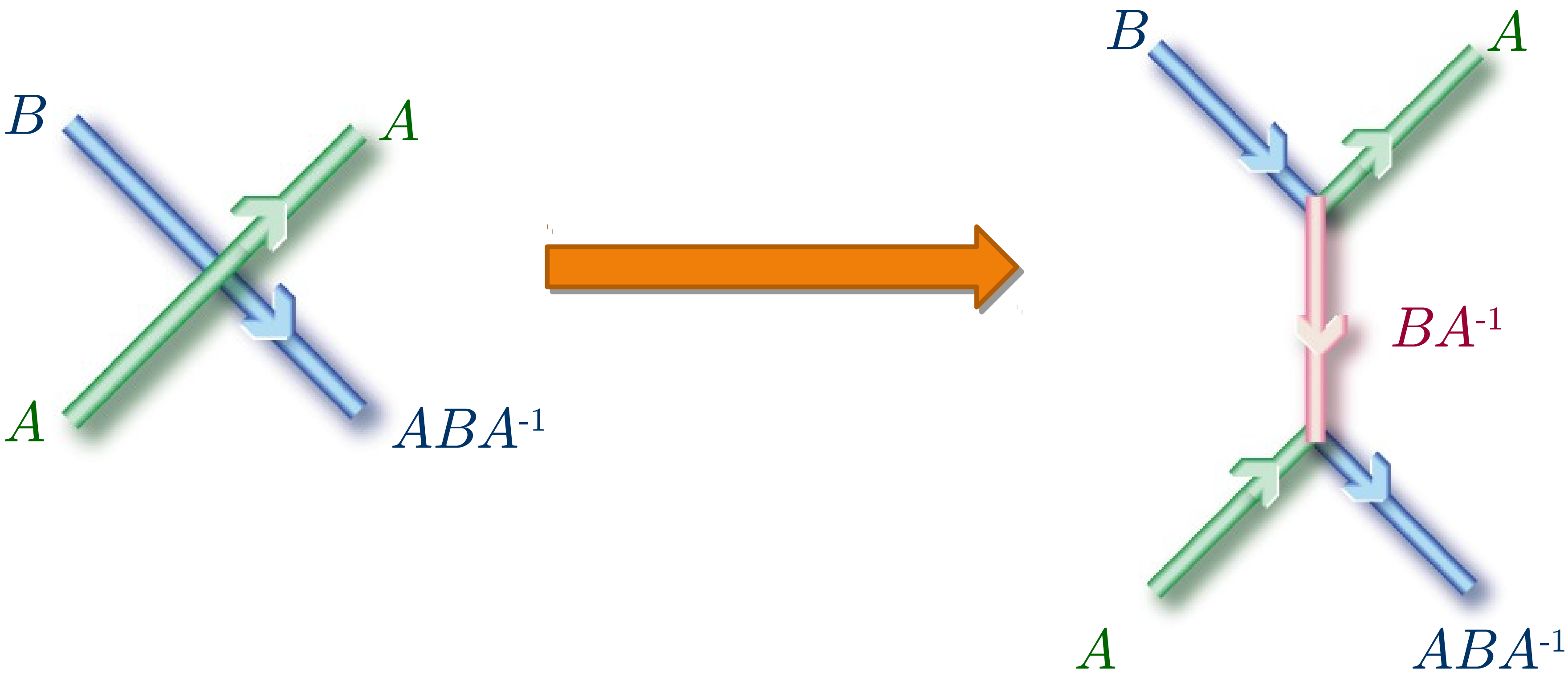
Topological charge of vortex can be fixed by a closed path encircling the vortex

Collision of Vortex



Rung BA^{-1} is formed through the collision.

Collision of Vortex



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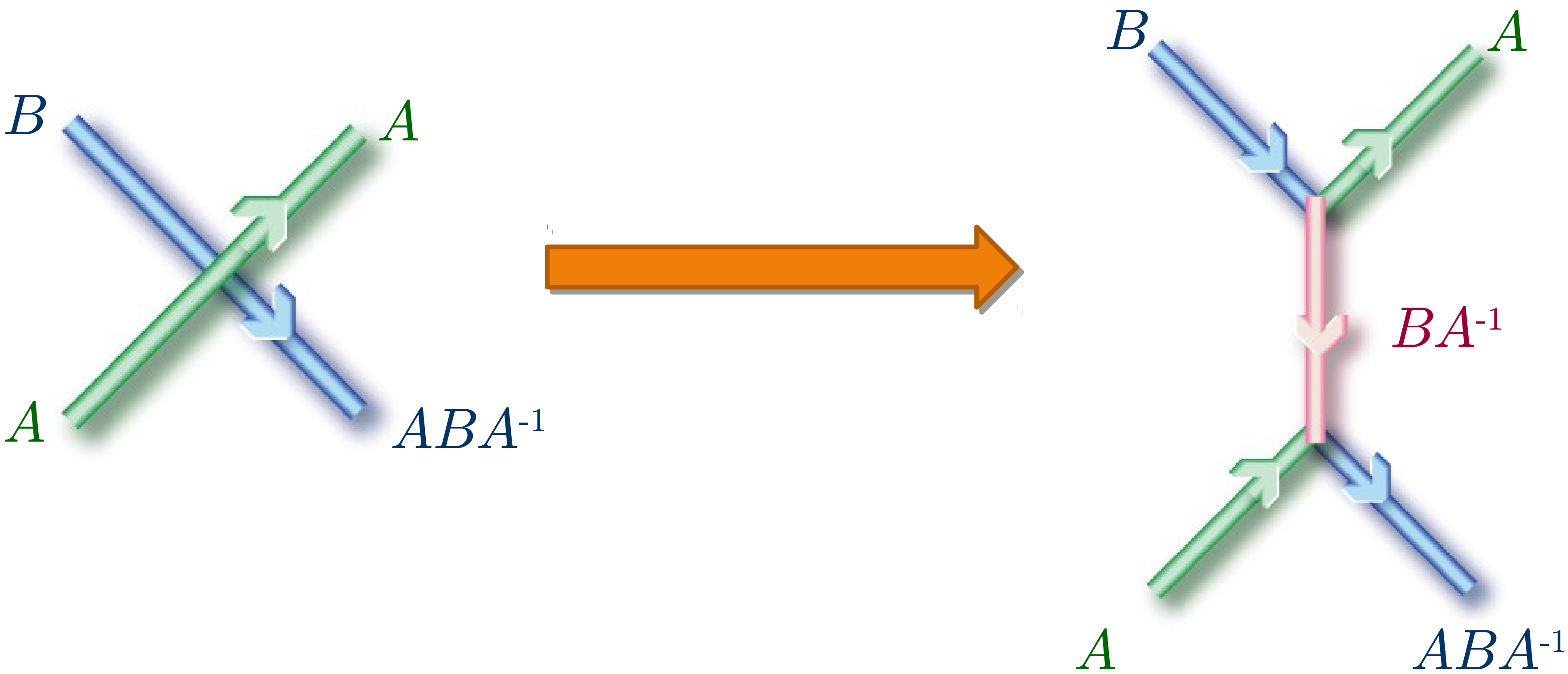
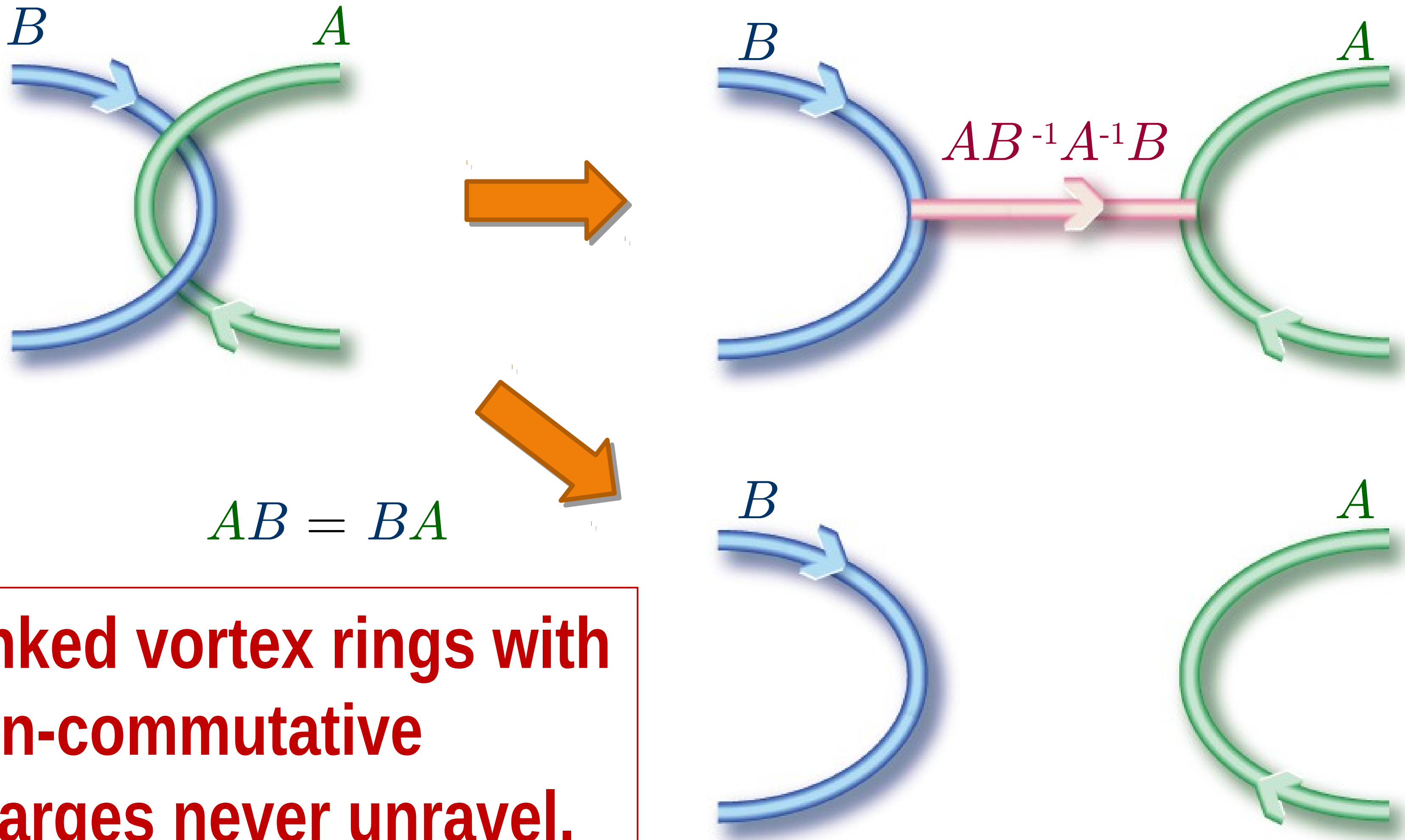
Collision of Vortex



Rung BA^{-1} is formed through the collision.

Linked Vortex Rings



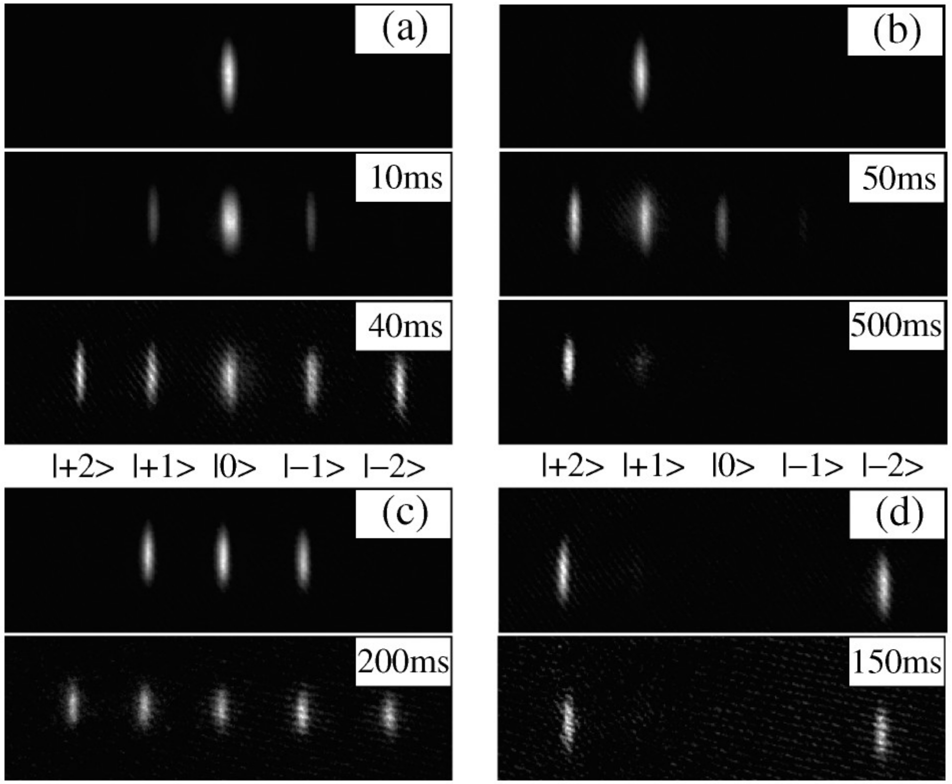
$AB = BA$

Linked vortex rings with non-commutative charges never unravel.

Spin-2 Spinor BEC

5 - component BEC : $\psi = (\psi_2, \psi_1, \psi_0, \psi_{-1}, \psi_{-2})^T$

$S = 2$ ^{87}Rb BEC and its spin dynamics is observed

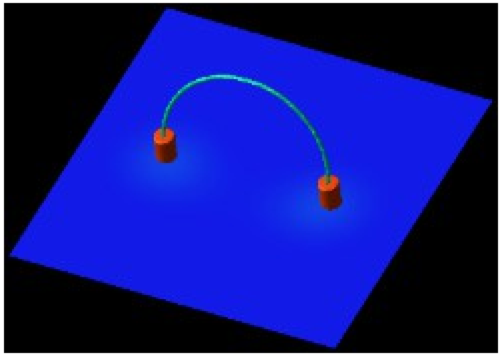


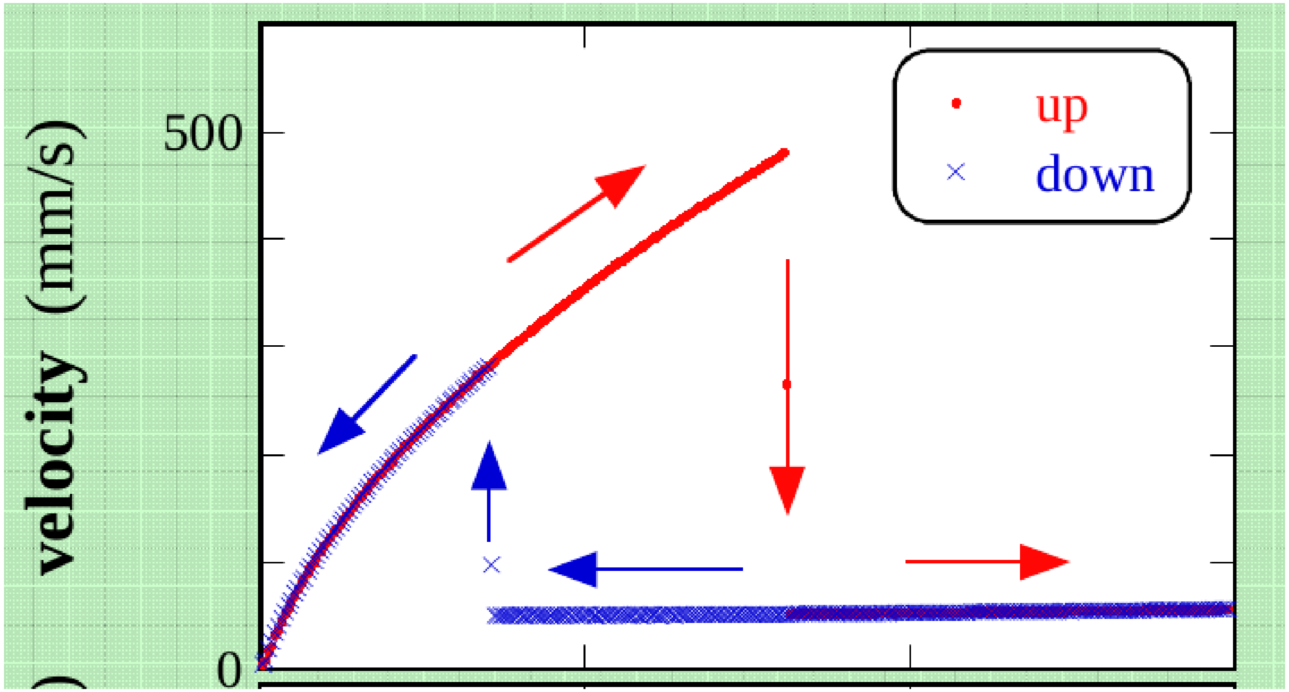
H. Schmaljohann et al. PRL **92**, 040402 (2004)

Vortex free – turbulent transition

Observation of vortices by vibrating wire in ^{4}He

Yano, 2007





Energy from vortex free to turbulence > Energy keeping turbulence

Gross-Piteavskii equation with dissipation

$$(i\hbar-\gamma)\dot{\psi}=\left\{-\frac{\hbar^2}{2M}\nabla^2+g(|\psi|^2-\bar{\rho})\right\}\psi$$

$\rho=|\psi|^2$: superfluid density

$\boldsymbol{v}=\bar{\kappa}\nabla(\arg[\psi])$: supefluid velocity ($\bar{\kappa}\equiv\hbar/M$)

$$\rho\left(\dot{\boldsymbol{v}}+\frac{\nabla\boldsymbol{v}^2}{2}\right)=-\nabla\left(\frac{g\rho^2}{2M}\right)+\bar{\kappa}^2\rho\nabla\left(\frac{\nabla^2\sqrt{\rho}}{2\sqrt{\rho}}\right)+\frac{\gamma\rho}{2M}\nabla\left\{\frac{1}{\rho}\nabla\cdot(\rho\boldsymbol{v})\right\}$$