Localization of nonlocal continuum models

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IN THE CITY OF NEW YORK

Motivation: choices in modeling

- Differential equation $\quad -\dfrac{d^2u}{dx^2}(x) = f(x),\ x \in \mathbb{R}.$

- Difference equation $\quad -D_h^2 u(x) = -\dfrac{u(x+h) - 2u(x) + u(x-h)}{h^2} = f(x).$

... all models are wrong, but some are useful. However, the approximate nature of the model must always be borne in mind ...

An alternative (and more general) modeling choice

- Integral/nonlocal equation $-\mathcal{L}_\delta u_\delta(x) = f(x),\ x \in \mathbb{R}$, which uses, for a given kernel $\underline{\omega}_\delta$ and a nonlocal horizon δ, a nonlocal (integral) operator defined by

$$\mathcal{L}_\delta u\,(x) = \int_0^\delta \frac{u(x+s) - 2u(x) + u(x-s)}{s^2} \underline{\omega}_\delta(s) ds.$$

- Long history (Rayleigh, van de Walls, Korteweg, and Leibniz, L′Hopital,...);
- Generic feature of model reduction (Mori-Zwanzig/Dyson/Durhamel, ...).

- Choices for δ, $\underline{\omega}_\delta \Rightarrow$ local continuum (δ=0), discrete ($\underline{\omega}_\delta$=Dirac measure at h), global (δ=∞) and fractional ($\underline{\omega}_\infty(s)$=$s^{1-2\alpha}$, 0<$\alpha$<1) interactions.

- Allowing singular solutions (to better represent reality, e.g. cracks/fractures) !

Local/nonlocal modeling

- Motivated by applications such as studies of anomalous diffusion processes and mechanics of fractures, our main interests in the mathematical development of nonlocal models are mostly on:

  - systems of nonlocal models with vector/tensor quantities of interest[1];
  - dependence of model properties on the range of nonlocal interactions[2];
  - localization of nonlocal models, coupling of nonlocal/local models[3];
  - effective and asymptotically compatible numerical discretization[4], ...

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[1] Du-Gunzburger-Lehoucq-Zhou, Nonlocal vector calculus, M3AS 2013
[2] D-G-L-Z SIAM Rev 2012; Mengesha-Du 2013, 2014, 2015, Tian-Du 2014, 2015
[3] Tian-Du 2016, Du-Tao-Tian 2016
[4] Du-Zhou 2011, Du-Ju-Tian-Zhou 2012, Tian-Du 2014, 2015, Du-Yang, 2016

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Connecting local and nonlocal models

- Nonlocal operator $-\mathcal{L}_\delta$ is connected to various mathematical concepts, in particular, the δ-dependence allows us to study the local limit $\delta \to 0$.

  Formally, $$\mathcal{L}_\delta u\,(x) = \int_0^\delta \frac{u(x+s) - 2u(x) + u(x-s)}{s^2} \underline{\omega}_\delta(s) ds$$
  $$= \frac{d^2 u}{dx^2}(x) \int_0^\delta \omega_\delta(s) ds + c_2 \delta^2 \frac{d^4 u}{dx^4}(x) + \cdots \quad \text{for } u \text{ smooth.}$$

- The operator $\mathcal{L}_\delta$ is also associated with $\mathcal{S}_\delta(\Omega)$, the closure of $C_0^\infty(\Omega)$ wrt
  $$|u|^2_{\mathcal{S}_\delta(\Omega)} = \int_\Omega \int_{|\mathbf{s}|<\delta} \omega_\delta(|\mathbf{s}|) \frac{|u(\mathbf{x}+\mathbf{s}) - u(\mathbf{x})|^2}{|\mathbf{s}|^2} d\mathbf{s} d\mathbf{x} < \infty\,.$$

  Bougain-Brezis-Mironescu 2001, Ponce 2004: as $\delta \to 0$, $\mathcal{S}_\delta(\Omega) \to H_0^1(\Omega)$ for L^1 density $\omega_\delta(|\mathbf{s}|)$ that approximates the Diract measure at the origin.

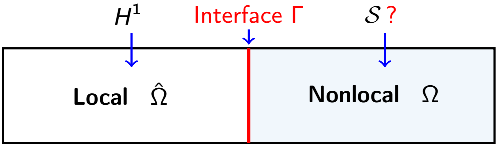
  Nonlocal characterization of local spaces $\Rightarrow$ localization of nonlocal spaces.

- More on localization: Mengesha-Du 2013, 2014, 2015, Tian-Du 2015, 2016.

Coupling of local and nonlocal models

It may be effective to couple local/nonlocal models together in practice.

E.g. on $\hat{\Omega}$ a local 2nd order elliptic equation solutions in $H^1(\hat{\Omega})$, and on Ω a nonlocal model with less regular solutions, say, only in L^2 inside Ω.



Question: can such a coupled local and nonlocal model be well-defined? Particularly, is there an $\mathcal{S}$ with functions that allow possible discontinuities anywhere in Ω, and have traces on Γ to match with their local counterparts?

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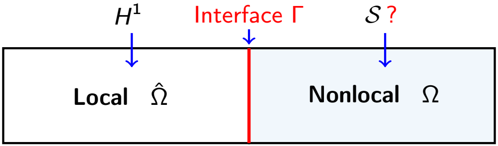

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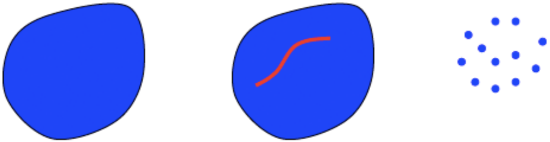
Motivation: peridynamics (PD) by Silling 2001

A nonlocal alternative to classical mechanics by Silling (2015 Belytschko prize), replacing spatial derivatives in Newton's law by nonlocal/integral operators:

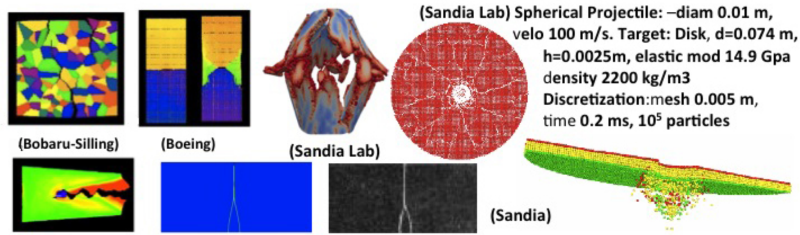
$$\mathcal{L}\mathbf{u}(\mathbf{x}) = \int \left\{ \underline{\mathrm{T}}\langle \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{y}), \mathbf{x}, \mathbf{y} \rangle - \underline{\mathrm{T}}\langle \mathbf{u}(\mathbf{y}), \mathbf{u}(\mathbf{x}), \mathbf{y}, \mathbf{x} \rangle \right\} d\mathbf{y}.$$



(Silling)

Motivation: classical/local continuum models are in question near materials defects such as cracks; multiscale coupling of MD/CM remains challenging.

For recent reviews, see Handbook of Peridynamic Modeling, 2016, CRC Press. (edited by Bobaru, Foster, Geubelle and Silling)



There have been significant code development efforts (PDLAMMPS, PERIDIGM...)

Peridynamics, Fracture, and Nonlocal Continuum Models
By Qiang Du, Robert Lipton

Peridynamics (PD) vs PDEs

- Peridynamics (PD) is formulated as a set of partial-integral equations

Relation	*Peridynamic theory*	*Standard theory*
Kinematics	$\underline{\mathbf{Y}}\langle \mathbf{q}-\mathbf{x}\rangle = \mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{x})$	$\mathbf{F}(\mathbf{x}) = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}(\mathbf{x})$
Linear momentum balance	$\rho\ddot{\mathbf{y}}(\mathbf{x}) = \int_{\mathcal{H}} \big(\mathbf{t}(\mathbf{q},\mathbf{x}) - \mathbf{t}(\mathbf{x},\mathbf{q})\big)\, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x})$	$\rho\ddot{\mathbf{y}}(\mathbf{x}) = \nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) + \mathbf{b}(\mathbf{x})$
Constitutive model	$\mathbf{t}(\mathbf{q},\mathbf{x}) = \underline{\mathbf{T}}\langle \mathbf{q}-\mathbf{x}\rangle, \qquad \underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})$	$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{F})$
Angular momentum balance	$\int_{\mathcal{H}} \underline{\mathbf{Y}}\langle \mathbf{q}-\mathbf{x}\rangle \times \underline{\mathbf{T}}\langle \mathbf{q}-\mathbf{x}\rangle\, dV_{\mathbf{q}} = \mathbf{0}$	$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$
Elasticity	$\underline{\mathbf{T}} = W_{\underline{\mathbf{Y}}}$ (Fréchet derivative)	$\boldsymbol{\sigma} = W_{\mathbf{F}}$ (tensor gradient)
First law	$\dot{\varepsilon} = \underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}} + q + r$	$\dot{\varepsilon} = \boldsymbol{\sigma} \cdot \dot{\mathbf{F}} + q + r$

(Silling)

WIthout spatial derivatives, cracks (singularities) are part of the solution.

A simple example: linear bond-based PD

Eg., force balance for a continuum of (linear/isotropic) Hookean springs:

$$\mathcal{L}_\delta \mathbf{u}_\delta(\mathbf{x}) = \int_{\Omega\cup\Omega_\delta} \omega_\delta(|\mathbf{y}-\mathbf{x}|)\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^2}\left(\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^2}\cdot\big(\mathbf{u}_\delta(\mathbf{y})-\mathbf{u}_\delta(\mathbf{x})\big)\right)d\mathbf{y}\,.$$

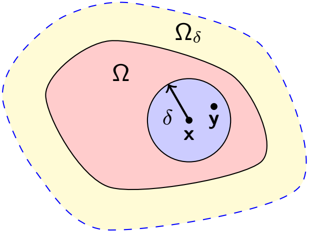
$\mathbf{u}_\delta(\mathbf{x})$: displacement at $\mathbf{x}$;

linear spring $\forall \mathbf{y}\in B_\delta(\mathbf{x})$;

$\mathbf{y}$-$\mathbf{x}$: bond direction;

δ: nonlocal horizon;

$\omega_\delta(|\mathbf{r}|)$: nonlocal kernel, with support in $B_\delta(0)$.



In Ω:

$$\boxed{-\mathcal{L}_\delta \mathbf{u}_\delta = \mathbf{b}}$$

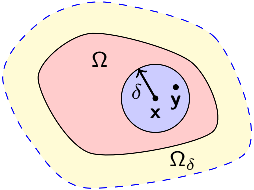
and on Ω_δ:

$$\boxed{\mathbf{u}_\delta = 0}$$

Nonlocal/volumetric constraint in $\Omega_\delta=\{\mathbf{x}\in\Omega^c,\ d(\mathbf{x},\partial\Omega)<\delta\}$, analog of local BC.

Reformulation

Prob.: find $\mathbf{u}_\delta$, $-\mathcal{L}_\delta \mathbf{u}_\delta = \mathbf{b}$ in Ω, $\mathbf{u}_\delta=0$ in Ω_δ.



Reformulation $-\mathscr{D}\big(\omega_\delta\, \mathscr{D}^*\big)\mathbf{u}_\delta = \mathbf{b}$ where

$$\mathscr{D}^*(\mathbf{u})(\mathbf{x},\mathbf{y}) = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^2} \cdot \big(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})\big)$$ linear nonlocal volumetric strain.

$\mathscr{D}$: dual/adjoint operator of $\mathscr{D}^*$, $< \mathscr{D}(\varphi), \mathbf{u} >=< \varphi, \mathscr{D}^*(\mathbf{u}) >$, $\forall\ \varphi, \mathbf{u}$.

$$\Rightarrow \quad \mathscr{D}\big(\omega_\delta\, (\mathscr{D}^*(\mathbf{u}))\big)(\mathbf{x}) = \int \omega_\delta(|\mathbf{y}-\mathbf{x}|)\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^2}\mathscr{D}^*(\mathbf{u})(\mathbf{x},\mathbf{y})\, dy\,.$$

Operators $\mathscr{D}^*$, $\mathscr{D}$, and integral identities: part of nonlocal vector calculus,

$\Rightarrow$ nonlocal calculus of variations, asymptotically compatible schemes:

Systematic/axiomatic framework, mimicing classical/local calculus/PDEs[5].

[5] D-G-L-Z 2012, 2013; Mengesha-Du 2013, 2014, 2015, Tian-Du 2015, ...

Nonlocal vector calculus

Inspired by earlier works on continuum mechanics (Silling), image/data analysis (Gilboa-Osher, Smale et al), nonlocal space (Bougain-Brezis-Mironescu, Ponce)

Newton's vector calculus	$\Leftrightarrow$	Nonlocal vector calculus
Local balance (PDE)	$\Leftrightarrow$	Nonlocal balance (PD)
Differential operators	$\Leftrightarrow$	Nonlocal operators
$-\nabla \cdot (K \nabla u) = f$	$\Leftrightarrow$	$-\mathscr{D} \cdot (\omega_\delta \mathscr{D}^* u) = f$
Boundary conditions	$\Leftrightarrow$	Volumetric constraints
$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\partial_n v - v\partial_n u$	$\Leftrightarrow$	$\iint u\mathscr{D}(\mathscr{D}^* v) - v\mathscr{D}(\mathscr{D}^* u) = 0$

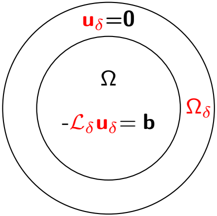
Main distinction: systems (vectors/tensors), δ-dependence, minimal regularity[6]

[6] D-G-L-Z 2012, 2013; Mengesha-Du 2013, 2014, 2015, 2016, Tian-Du 2015, 2016, ...

Nonlocal models and local limits

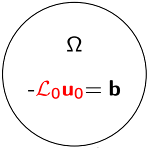
Nonlocal problem $\mathbf{u}_\delta \in \mathcal{S}_\delta$

Local PDE limit $\mathbf{u}_0 \in \mathcal{S}_0$



$\leftarrow$ Volumetric constraint

$\leftarrow$ Well-posed with a unique solution[7] $\rightarrow$



$\partial\Omega$

Boundary condition $\rightarrow$ $\mathbf{u}_0 = \mathbf{0}$

$$|\mathbf{u}|^2_{\mathcal{S}_\delta} = \iint \omega_\delta(|\mathbf{x}-\mathbf{y}|)|\mathcal{D}^*\mathbf{u}(\mathbf{x},\mathbf{y})|^2$$

$$|\mathbf{u}|^2_{\mathcal{S}_0} = 2|\mathrm{Sym}\nabla\mathbf{u}|^2_{L^2} + |\operatorname{div}\mathbf{u}|^2_{L^2}$$

$$\mathcal{D}^*\mathbf{u}(\mathbf{x},\mathbf{y}) = \frac{\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\cdot\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$$

$$\mathrm{Sym}\nabla\mathbf{u} = \frac{1}{2}\left(\nabla u + (\nabla u)^{\mathrm{T}}\right)$$

Key: nonlocal Kohn's inequality and nonlocal Poincare inequalities, ...

[7] D-G-L-Z 2013 J. Elasticity; Mengesha-Du 2013 J. Elasticity, 2015 Nonlinearity.

Nonlocal models and local limits

Localization of bond-based linear peridynamics:

- $\star$ As $\delta \to 0$, $\mathcal{S}_\delta \to H_0^1$, e.g., for $\omega_\delta(r) = \hat{\omega}(r/\delta)\delta^{-d}$ or more generally a sequence of densities approaching the Dirac-measure[8].
- $\star$ $\mathbf{u}_\delta \in \mathcal{S}_\delta$ (a much larger space) $\xrightarrow{L^2}$ a more regular $\mathbf{u}_0 \in \mathcal{S}_0$ ($=H_0^1$).
- $\star$ $\mathcal{L}_0$: Navier operator of linear elasticity with a Poisson's ratio $1/4$.

$\Rightarrow$ consistency/compatibility of nonlocal/local models on the continuum level.

These results have been further extended[9] to stated-based linear peridynamics (for a general Poisson's ratio), more general nonlocal volumetric constraints (boundary conditions), and for certain nonlinear hyperelastic materials.

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[8] Extending works of Bougain-Brezis-Mironescu, Ponce, ... to vector-fields/nonlocal-systems.

[9] Mengesha-Du, 2014 J. Elasticity, Proc. Roy Soc., 2015 Nonlinearity, 2016 Nonlinear Analysis....

Nonlocal coupling

From MD to PD:

Parks-Lehoucq-Plimpton-Silling (2008),

Seleson-Parks-Gunzburger-Lehoucq (2009),

Rahman-Foster-Haque (2014), ...

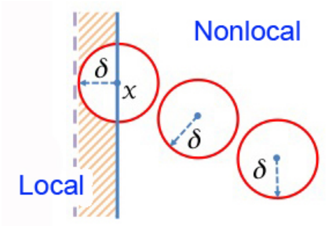
From CE to PD:

Seleson-Beneddine-Prudhomme (2013),

Seleson-Gunzburger-Parks (2013),

DElia-Perego-Bochev-Littlewoood (2015),

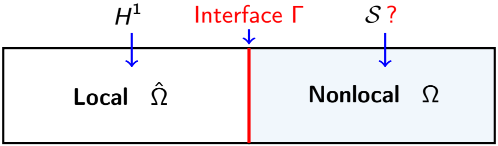
Costa-Bond-Littlewoood (2016),...



Popular local/nonlocal coupling: sharp transition, blending, overlapping...

Tian-Du, Du-Tao-Tian: heterogeneously localized nonlocal interactions.

Coupling a 2nd order elliptic equation on $\hat{\Omega}$ with a nonlocal model on Ω.



Goal: develop a nonlocal model allowing solutions with possible discontinuities anywhere inside Ω but having traces on Γ matching with the local counterparts.

Conventional nonlocal spaces

Recall the typical scalar nonlocal space on $\Omega \subset \mathbb{R}^d$,

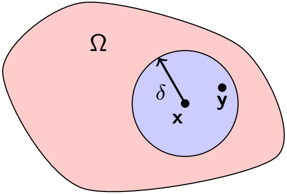
$$\mathcal{S}(\Omega) = \{u : \|u\|^2_{\mathcal{S}(\Omega)} = \|u\|^2_{L^2(\Omega)} + |u|^2_{\mathcal{S}(\Omega)} < \infty\} \qquad \text{where}$$

$$|u|^2_{\mathcal{S}(\Omega)} = \int_\Omega \int_{|\mathbf{x}-\mathbf{y}|<\delta} \omega_\delta(|\mathbf{x}-\mathbf{y}|)\,|u(\mathbf{y}) - u(\mathbf{x})|^2\, d\mathbf{y}d\mathbf{x},$$

$$\omega_\delta(|\mathbf{z}|) = \frac{1}{\delta^{2+d}}\hat{\omega}(\frac{|\mathbf{z}|}{\delta}), \quad \int_0^1 \hat{\omega}(r) r^{d+1} dr < \infty,$$

$$\operatorname{supp}\hat{\omega} \subset [0,1), \quad \hat{\omega} \geq 0.$$

As $\delta \to 0$, $\mathcal{S}(\Omega) \to H^1(\Omega)$ (BBM 2001) !



For finite δ, $\mathcal{S}(\Omega)$ is a space between $L^2(\Omega)$ and $H^1(\Omega)$ depending on $\hat{\omega}$. It does not meet our goal with the nonlocal interaction being spatially homogeneous.

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$$\omega_\delta(|\mathbf{z}|) = \frac{1}{{\delta_x}^{2+d}} \hat{\omega}\Big(\frac{|\mathbf{z}|}{\delta_x}\Big), \quad \int_0^1 \hat{\omega}(r) r^{d+1} dr < \infty,$$

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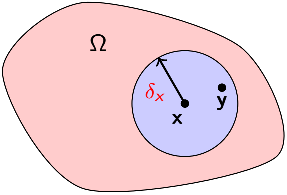
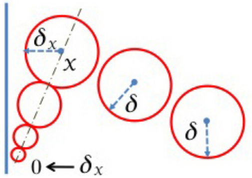


For finite δ, $\mathcal{S}(\Omega)$ is a space between $L^2(\Omega)$ and $H^1(\Omega)$ depending on $\hat{\omega}$. It does not meet our goal with the nonlocal interaction being spatially homogeneous. But it natural leads to the idea of a spatially heterogenous nonlocal interaction.

New nonlocal function space

$$|u|^2_{\mathcal{S}(\Omega)} = \int_\Omega \int_{\Omega\cap\{|\mathbf{y}-\mathbf{x}|<\delta_\mathbf{x}\}} \gamma(\mathbf{x},\mathbf{y})(u(\mathbf{y})-u(\mathbf{x}))^2 d\mathbf{y}d\mathbf{x}, \quad \gamma(\mathbf{x},\mathbf{y}) = \frac{1}{|\delta_\mathbf{x}|^{2+d}}\hat{\omega}\left(\frac{|\mathbf{y}-\mathbf{x}|}{\delta_\mathbf{x}}\right)$$

where $\delta_\mathbf{x} = \min\{\delta, \sigma\,\mathrm{dist}(\mathbf{x},\Gamma)\}$, $\sigma \in (0,1]$, i.e., a variable horizon (a concept that was studied in Silling-Littlewood-Seleson previously). Key to our work: by making $\delta_\mathbf{x} \to 0$, we are able to achieve heterogeneous localization.



E.g.: $\hat{\omega}(r) = |r|^{-\lambda}\chi_{\{|r|\le 1\}}$, χ: characteristic function, $\lambda \in [0, d+2)$.

New trace theorem

Theorem (Tian-Du 2016, Trace Theorem for a Nonlocal Space)

For $d \geq 2$, $\Omega \subset \mathbb{R}^d$ *bounded, simply connected, Lipschitz,* $\exists C(\Omega) > 0$,

$\Rightarrow$ $$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C(\Omega)\|u\|_{\mathcal{S}(\Omega)}\,, \quad \forall u \in \mathcal{S}(\Omega)\,.$$

This extends a classical trace theorem for the Sobolev space $H^1(\Omega)$. Indeed,

Proposition (T-D, continuous imbedding of heterogeneous nonlocal space)

$$H^1(\Omega) \hookrightarrow \mathcal{S}(\Omega): \quad \exists C(\Omega) > 0, \Rightarrow \|u\|_{\mathcal{S}(\Omega)} \leq C(\Omega)\|u\|_{H^1(\Omega)}, \forall u \in H^1(\Omega).$$

The imbedding also extends a well-known result by BBM[10] for constant horizon to the case that allows variable horizon and heterogeneous localization.

[10] Bourgain-Brezis-Mironescu, Another look at Sobolev spaces, 2011

New trace theorem

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$\Rightarrow$
$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C(\Omega)\|u\|_{\mathcal{S}(\Omega)}, \quad \forall u \in \mathcal{S}(\Omega).$$

Among extensions of Sobolev space: Morrey, Besov, Campanato, Triebel, variable-order Sobolev, ..., none contains theorems of the type presented here.

The result was expected but its proof, as it turned out, is highly non-trivial, relying on substantially more involved estimates than the local counterpart.

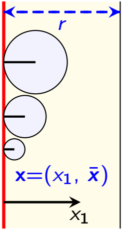
Some of the technical results used in the proof are of independent interests. For examples, a new nonlocal Hardy type inequality has been established.

Proving the trace theorem

A couple of key strategies/ingredients to make the proof more accessible:

1. first consider a strip, then more general domains via partition of unity;
2. use a simple (constant) kernel for $\hat{\omega}$ first, then generalize.

$\Omega = (0, r) \times \mathbb{R}^{d-1}$



$\Gamma = \{0\} \times \mathbb{R}^{d-1}$

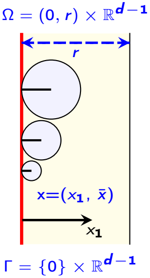
$$\delta_{\mathbf{x}} = |x_1|, \quad \gamma(\mathbf{x}, \mathbf{y}) = |x_1|^{-2-d} \chi_{\{|\mathbf{y}-\mathbf{x}| \le x_1\}},$$

$$|u|^2_{\mathcal{S}(\Omega)} = \int_\Omega \int_{\Omega \cap \{|\mathbf{y}-\mathbf{x}| < x_1\}} \frac{(u(\mathbf{y}) - u(\mathbf{x}))^2}{|x_1|^{2+d}} d\mathbf{y} d\mathbf{x}.$$

$\Rightarrow \mathcal{S}(\Omega)$ contains all functions in $L^2(\tilde{\Omega})$, $\forall \tilde{\Omega} \Subset \Omega$.

In contrast $$|u|^2_{H^\alpha(\Omega)} = \int_\Omega \int_\Omega \frac{(u(\mathbf{y}) - u(\mathbf{x}))^2}{|\mathbf{y} - \mathbf{x}|^{2\alpha+d}} d\mathbf{y} d\mathbf{x}.$$

Special trace theorem



A more special but precise form of the trace theorem:

Theorem (Tian-Du 2016 Special Case)

$$\|u\|_{L^2(\Gamma)} \leq c(d)\,(r^{-1/2}\|u\|_{L^2(\Omega)} + r^{1/2}|u|_{\mathcal{S}(\Omega)}) \quad (\star)$$

$$|u|_{H^{1/2}(\Gamma)} \leq c(d)\,(r^{-1}\|u\|_{L^2(\Omega)} + |u|_{\mathcal{S}(\Omega)}) \quad (\star\star)$$

While one may use interior extensions to derive the desired inequalities, there are a few unexpected complications.

Sketch of the proof:

Step 1 for $(\star)$, standard extension gives $\|u\|^2_{L^2(\Gamma)} \leq c(d)\,(r^{-1}\|u\|_{L^2(\Omega)}{}^2 + r|u|^2_w)$

where $$|u|^2_w = \int_\Gamma \int_0^r \frac{|u(x_1, \bar{\mathbf{x}}) - u(0, \bar{\mathbf{x}})|^2}{|x_1|^2} dx_1 d\bar{\mathbf{x}}\,.$$

$\Rightarrow$ a classical version of $(\star)$ follows via a Hardy's inequality $|u|^2_w \leq C\|\partial_{x_1} u\|^2_{L^2(\Omega)}$,

but, we need, for $(\star)$ and $(\star\star)$, a more refined version $|u|^2_w \leq C|u|^2_{\mathcal{S}(\Omega)}$ $(\#)$

Step 2 To bound $|u|_{\mathrm{w}}$, we establish an extension of classical Hardy inequality:

Lemma (Tian-Du 2016 A New Nonlocal Hardy Type Inequality)

$$\int_{\mathcal{D}} \frac{|u(\boldsymbol{x})|^2}{(dist(\boldsymbol{x}, \partial\mathcal{D}))^2} d\boldsymbol{x} \leq C(\mathcal{D})|u|^2_{\mathcal{S}(\mathcal{D})} \ \ (\leq C(\mathcal{D})\|\nabla u\|^2_{L^2(\Omega)}), \ \ \forall u \in C^1_0(\bar{\mathcal{D}}).$$

The 1d nonlocal Hardy is needed to derive a bound for $|u|^2_{\mathrm{w}}$:

$$\int_{\Omega} \frac{|u(x_1, \bar{\boldsymbol{x}}) - u(0, \bar{\boldsymbol{x}})|^2}{|x_1|^2} d\boldsymbol{x} \leq C \int_{\Omega} \fint_{ax_1}^{bx_1} \frac{|u(y_1, \bar{\boldsymbol{x}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^3} dy_1 d\boldsymbol{x}\,,$$

which leads to an object resembling a norm of normal derivative/difference.

Nonlocal norms of directional derivatives/differences

Note: it is trivial that $\|\partial_{x_1} u(x_1, \bar{\mathbf{x}})\|_{L^2}^2 + \|\partial_{\bar{x}} u(x_1, \bar{\mathbf{x}})\|_{L^2}^2 = \|\partial_{\mathbf{x}} u(\mathbf{x})\|_{L^2}^2$.

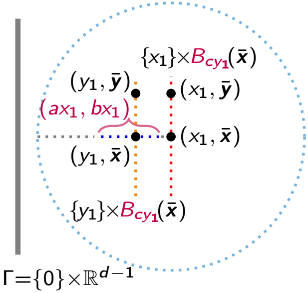
Norms of directional differences may be defined as, for $c \in (0,1)$, $0 \le a < b \le 1$,

$$|u|_n^2 = \int_\Omega \fint_{ax_1}^{bx_1} \frac{|u(y_1, \bar{\mathbf{x}}) - u(x_1, \bar{\mathbf{x}})|^2}{|x_1|^{2+1}} dy_1 d\mathbf{x}$$

$$\left(\le C \|\partial_{x_1} u(x_1, \bar{\mathbf{x}})\|_{L^2(\Omega)}^2\right),$$

$$|u|_t^2 = \int_\Omega \fint_{B_{cx_1}(\bar{x})} \frac{|u(x_1, \bar{\mathbf{y}}) - u(x_1, \bar{\mathbf{x}})|^2}{|x_1|^{2+d-1}} d\bar{\mathbf{y}} d\mathbf{x}$$

$$\left(\le C \|\partial_{\bar{x}} u(x_1, \bar{\mathbf{x}})\|_{L^2(\Omega)}^2\right).$$



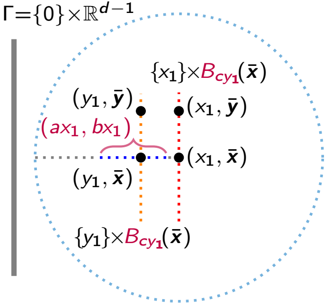
Easy to see $|u|_n + |u|_t \le C \|\partial_{\mathbf{x}} u\|_{\mathcal{L}^2(\Omega)}$, but we need its nonlocal version.

Step 3 a much more involved proof leads to:

Lemma (Tian-Du 2016)

For suitable a, b, c, $\exists\ C=C(a,b,c)>0 \Rightarrow$

$$|u|_n + |u|_t \leq C|u|_{\mathcal{S}(\Omega)}\,,\ \forall u \in \mathcal{S}(\Omega).$$



This new nonlocal estimate is derived from

$$|u|_n \leq \alpha|u|_t + C|u|_{\mathcal{S}(\Omega)}\,,$$
$$|u|_t \leq \beta|u|_n + C|u|_{\mathcal{S}(\Omega)}\,.$$

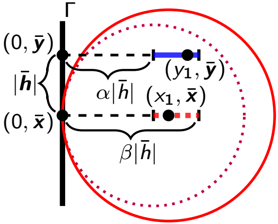
Careful estimate leads to $\alpha\beta < 1$ (a small miracle, with suitable a, b, c).

Putting together, we get $(\#)$, and then $(\star)$ of the special trace theorem.

Proving the trace theorem

Step 4 showing $(\star\star)$ is more delicate; separate far-away interactions (easy) from nearby interactions in $\Gamma_r^2 = \Gamma^2 \cap \{\bar{\boldsymbol{y}}\text{-}\bar{\boldsymbol{x}}{=}\bar{\boldsymbol{h}}, |\bar{\boldsymbol{h}}|{\le}r/2\}$ (more challenging).

Constructing suitable extension of boundary points $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{y}}$ on Γ_r^2 to Ω along the normal direction, i.e., x_1, $y_1{\in}I_{\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}}^{\alpha,\beta}{=}\{\alpha|\bar{\boldsymbol{h}}|{\le}z{\le}\beta|\bar{\boldsymbol{h}}|\}$ for $1{<}\alpha{<}\beta{\le}2$, averaging over $I_{\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}}^{\alpha,\beta}$, we get



$$
\begin{aligned}
|u(0,\bar{\boldsymbol{y}}) - u(0,\bar{\boldsymbol{x}})|^2 \le & \frac{3}{(\beta-\alpha)|\bar{\boldsymbol{h}}|} \int_{I_{\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}}^{\alpha,\beta}} |u(0,\bar{\boldsymbol{y}}) - u(y_1,\bar{\boldsymbol{y}})|^2 dy_1 \\
& + \frac{3}{(\beta-\alpha)|\bar{\boldsymbol{h}}|} \int_{I_{\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}}^{\alpha,\beta}} |u(x_1,\bar{\boldsymbol{x}}) - u(0,\bar{\boldsymbol{x}})|^2 dx_1 \\
& + \frac{3}{(\beta-\alpha)^2|\bar{\boldsymbol{h}}|^2} \iint_{I_{\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}}^{\alpha,\beta} \times I_{\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}}^{\alpha,\beta}} |u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{x}})|^2 dy_1 dx_1 \,.
\end{aligned}
$$

Proving the trace theorem

Integrating over $\Gamma \Rightarrow$ an estimate on the near-by boundary norm:

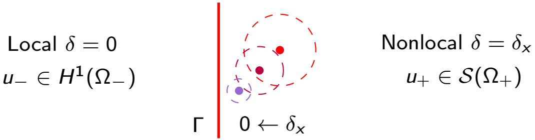
$$\iint_{\Gamma_r^2} \frac{|u(0,\bar{\mathbf{y}}) - u(0,\bar{\mathbf{x}})|^2}{|\bar{\mathbf{y}} - \bar{\mathbf{x}}|^d} d\bar{\mathbf{y}} d\bar{\mathbf{x}} \leq \frac{6}{\beta - \alpha} \iint_{\Gamma_r^2} \int_{I_{\bar{x},\bar{y}}^{\alpha,\beta}} \frac{|u(x_1,\bar{\mathbf{x}}) - u(0,\bar{\mathbf{x}})|^2}{|\bar{\mathbf{y}} - \bar{\mathbf{x}}|^{d+1}} dx_1 d\bar{\mathbf{y}} d\bar{\mathbf{x}}$$

$$+ \frac{3}{(\beta-\alpha)^2} \iint_{\Gamma_r^2} \iint_{I_{\bar{x},\bar{y}}^{\alpha,\beta} \times I_{\bar{x},\bar{y}}^{\alpha,\beta}} \frac{|u(y_1,\bar{\mathbf{y}}) - u(x_1,\bar{\mathbf{x}})|^2}{|\bar{\mathbf{y}} - \bar{\mathbf{x}}|^{d+2}} dy_1 dx_1 d\bar{\mathbf{y}} d\bar{\mathbf{x}} = \mathrm{I} + \mathrm{II}$$

Step 5 Estimating II involves changes to variables/order-of-integration, but more amendable; I requires different and more technical estimates in order to yield a bound like $C(r)|u|_w^2$, which in combination with $(\#)$ gives $(\star\star)$.

Local/nonlocal coupling

The trace theorem leads immediately to well-posed local/nonlocal coupling.

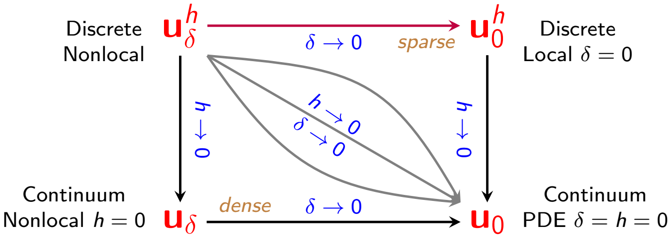


With the change in scales, robust numerical scheme is important.

Wanted: a monolithic discretization that works for both nonlocal models (finite $\delta > 0$) and their localizations ($\delta = 0$ limit).

Asymptotically compatible discretization

- Asymptotically Compatible (Tian-Du): converging to nonlocal solution with a fixed δ as $h \to 0$, and as $\delta \to 0$, $h \to 0$ to the correct local limit.

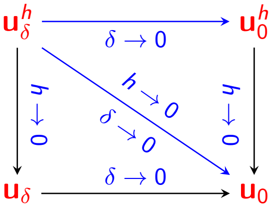


AC scheme: monolithic discretization of heterogeneous (local/nonlocal) models

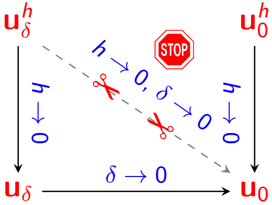
AC: specialized to nonlocal problems

Tian-Du 2014[10] provided an abstract framework and specified conditions for AC schemes. In particular, for nonlocal PD systems in multi-dimensions: AC if containing C^0 pw linear. For pw constants, conditional AC if $h/\delta \to 0$.

AC with C^0 pw linear



Discontinuous pw constant



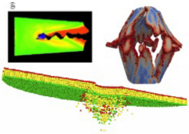
AC schemes are more robust (good for adaptive multiscale computation).

[10] Asymptotically compatible schemes and applications to robust discretization of nonlocal models

Summary

An alternative view:

Nonlocal models may provide effective mathematical descriptions of various phenomena, being Lévy flights of bumblebees, crack paths in materials, ...

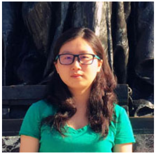


Systematic/axiomatic mathematical analysis of nonlocal models are not only mathematically interesting but also important in various applications.

Heterogeneous localization and AC schemes may provide a possible path to a seamless (robust and adaptive) coupling of local/nonlocal models.

(think nonlocal, act local)

Collaborators



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