Ground states and dynamics for SO coupled Bose-Einstein Condensation

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Marseille, 2016

Spin-orbit coupling

- Interaction of a particle's spin with its motion
- fine structure of Hydrogen
- Electron: orbital angular momentum (generates magnetic field), interacts with the electron spin magnetic moment (internal Zeeman effect)
- Crucial for quantum-Hall effects, topological insulators

Spin-orbit coupling in two component BEC

- Major experimental breakthrough in 2011, Lin et al. have created a SO coupled BEC, ^{85}Rb: $|\uparrow\rangle = |F=1,\, m_f=0\rangle$ and $|\downarrow\rangle = |F=1,\, m_f=-1\rangle$.
- SO coupling in cold atoms have been hot topics in recent years

Mathematical Model for SO-coupled BEC

- Coupled Gross-Pitaevskii equations (re-scaled):

$\Psi := (\psi_1(\mathbf{x}, t), \psi_2(\mathbf{x}, t))^T$, $\mathbf{x} \in \mathbb{R}^d$ in d dimensional spaces

$$i\partial_t\psi_1 = \left[-\frac{1}{2}\nabla^2 + V_1 + ik_0\partial_x + \frac{\delta}{2} + (\beta_{11}|\psi_1|^2 + \beta_{12}|\psi_2|^2)\right]\psi_1 + \frac{\Omega}{2}\psi_2,$$

$$i\partial_t\psi_2 = \left[-\frac{1}{2}\nabla^2 + V_2 - ik_0\partial_x - \frac{\delta}{2} + (\beta_{21}|\psi_1|^2 + \beta_{22}|\psi_2|^2)\right]\psi_2 + \frac{\Omega}{2}\psi_1,$$

- Trapping potential: $V_j(\mathbf{x}) = \frac{1}{2}(\gamma_x^2 x^2 + \gamma_y^2 y^2 + \gamma_z^2 z^2)$ $(j = 1, 2)$ for 3D case
- Interaction constants: β_{jl} between j-th and l-th component (positive for repulsive and negative for attractive)
- k_0: wave number of Raman lasers
- Ω: Rabi frequency (internal Josephson junction)
- δ: detuning constant for Raman transition

Conserved quantities

- Mass:

$$N(t) := \|\Psi(\cdot,t)\|^2 = \int_{\mathbb{R}^d} [|\psi_1(\mathbf{x},t)|^2 + |\psi_2(\mathbf{x},t)|^2] d\mathbf{x} \equiv N(0) = 1,$$

- Energy per particle

$$E(\Psi) = \int_{\mathbb{R}^d} \Bigg[\sum_{j=1}^{2} \left(\frac{1}{2}|\nabla\psi_j|^2 + V_j(\mathbf{x})|\psi_j|^2 \right) + \frac{\delta}{2}\left(|\psi_1|^2 - |\psi_2|^2\right) + \Omega\, \mathrm{Re}(\psi_1\overline{\psi}_2) + ik_0\left(\overline{\psi}_1\partial_x\psi_1 - \overline{\psi}_2\partial_x\psi_2\right) + \frac{\beta_{11}}{2}|\psi_1|^4 + \frac{\beta_{22}}{2}|\psi_2|^4 + \beta_{12}|\psi_1|^2|\psi_2|^2 \Bigg] d\mathbf{x}.$$

- Ground state patterns and dynamics properties

An equivalent form of CGPEs

- Introducing new variable

$$\psi_1(\mathbf{x},t)=\tilde{\psi}_1(\mathbf{x},t)e^{i(\omega t+k_0x)},\quad \psi_2(\mathbf{x},t)=\tilde{\psi}_2(\mathbf{x},t)e^{i(\omega t-k_0x)},$$

with $\omega=\frac{-k_0^2}{2}$

- CGPEs II

$$i\partial_t\tilde{\psi}_1=\left[-\frac{1}{2}\nabla^2+V_1+\frac{\delta}{2}+\beta_{11}|\tilde{\psi}_1|^2+\beta_{12}|\tilde{\psi}_2|^2\right]\tilde{\psi}_1+\frac{\Omega}{2}e^{-i2k_0x}\tilde{\psi}_2,$$
$$i\partial_t\tilde{\psi}_2=\left[-\frac{1}{2}\nabla^2+V_2-\frac{\delta}{2}+\beta_{21}|\tilde{\psi}_1|^2+\beta_{22}|\tilde{\psi}_2|^2\right]\tilde{\psi}_2+\frac{\Omega}{2}e^{i2k_0x}\tilde{\psi}_1.$$

- Advantage: better suited for $|k_0|\gg 1$

Conserved quantities for CGPE II

- $N(t) = \|\tilde{\Psi}(\cdot, t)\|^2 \equiv \|\tilde{\Psi}(\mathbf{x}, 0)\|^2 = 1$
- energy per particle

$$\begin{aligned}\tilde{E}(\tilde{\Psi}) \;\; = \;\; & \int_{\mathbb{R}^d} \Bigg[\sum_{j=1}^{2} \left(\frac{1}{2}|\nabla \tilde{\psi}_j|^2 + V_j(\mathbf{x})|\tilde{\psi}_j|^2 \right) + \frac{\delta}{2}\left(|\tilde{\psi}_1|^2 - |\tilde{\psi}_2|^2 \right) \\ & + \Omega \,\mathrm{Re}(e^{i2k_0x}\tilde{\psi}_1\overline{\tilde{\psi}_2}) + \frac{\beta_{11}}{2}|\tilde{\psi}_1|^4 \\ & + \frac{\beta_{22}}{2}|\tilde{\psi}_2|^4 + \beta_{12}|\tilde{\psi}_1|^2|\tilde{\psi}_2|^2 \Bigg] d\mathbf{x}.\end{aligned}$$

Ground States

- Nonconvex minimization problem

$$E_g := E\left(\Phi_g\right) = \min_{\Phi \in S} E\left(\Phi\right),$$

and

$$S := \left\{\Phi = (\phi_1, \phi_2)^T \in H^1(\mathbb{R}^d)^2 \mid \|\Phi\|^2 = 1, E(\Phi) < \infty\right\}$$

- Nonlinear Eigenvalue problem (Euler-Lagrange eq.)

$$\mu\phi_1 = \left[-\frac{1}{2}\nabla^2 + V_1(\mathbf{x}) + ik_0\partial_x + \frac{\delta}{2} + (\beta_{11}|\phi_1|^2 + \beta_{12}|\phi_2|^2)\right]\phi_1 + \frac{\Omega}{2}\phi_2,$$

$$\mu\phi_2 = \left[-\frac{1}{2}\nabla^2 + V_2(\mathbf{x}) - ik_0\partial_x - \frac{\delta}{2} + (\beta_{12}|\phi_1|^2 + \beta_{22}|\phi_2|^2)\right]\phi_2 + \frac{\Omega}{2}\phi_1,$$

- Chemical potential μ:

$$\mu = \mu = E(\Phi) + \int_{\mathbb{R}^d}\left(\frac{\beta_{11}}{2}|\phi_1|^4 + \frac{\beta_{22}}{2}|\phi_2|^4 + \beta_{12}|\phi_1|^2|\phi_2|^2\right) d\mathbf{x}.$$

Ground sate $k_0 = 0$

When $k_0 = 0$, no SO coupling

Theorem

Under condition $\lim_{|x|\to\infty} V(x) = \infty$, $\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{pmatrix}$ *is positive definite. There exists minimizers, i.e., the ground state* (ψ_1^g, ψ_2^g) *exists, and* $(|\psi_1^g|, |\psi_2^g|)$ *is unique. Moreover,* $(\psi_1^g, \psi_2^g) = (e^{i\theta_1}|\psi_1^g|, e^{i\theta_2|}\psi_2^g|)$, *where*

- *if* $\Omega > 0$, $\theta_1 - \theta_2 = \pm\pi$
- *if* $\Omega < 0$, $\theta_1 - \theta_2 = 0$

Limiting behavior

Theorem

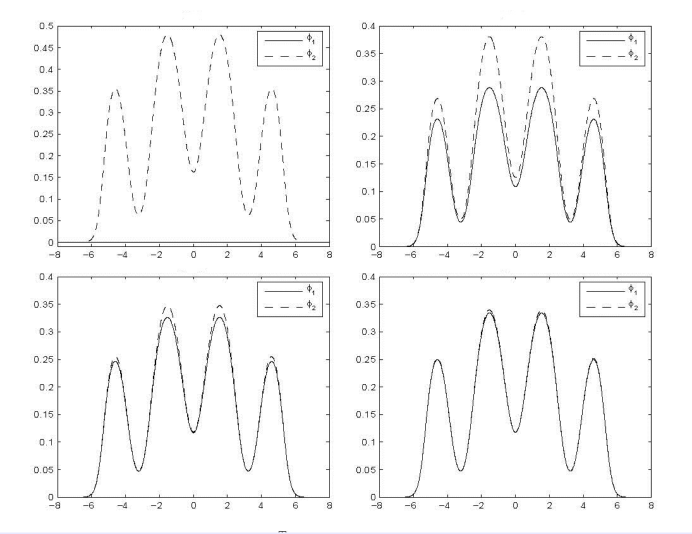
Let (ϕ_1^g, ϕ_2^g) *be the ground state of CGPEs. As* $\Omega \to -\infty$*, we have*

$$\phi_1^g - \phi_2^g \to 0, \quad j = 1, 2.$$

Theorem

Let (ϕ_1^g, ϕ_2^g) *be the ground state of CGPEs. As* $\delta \to -\infty$*, we have*

$$\phi_2^g \to 0.$$



0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−8 −6 −4 −2 0 2 4 6 8
ϕ_1
ϕ_2

Phase separation

Property Let $\beta_{12} \to +\infty$, the phase of two components of the ground state $\Phi_g = (\phi_1^g, \phi_2^g)^T$ will be segregated, i.e. Φ_g will converge to a state such that $\phi_1^g \cdot \phi_2^g = 0$.

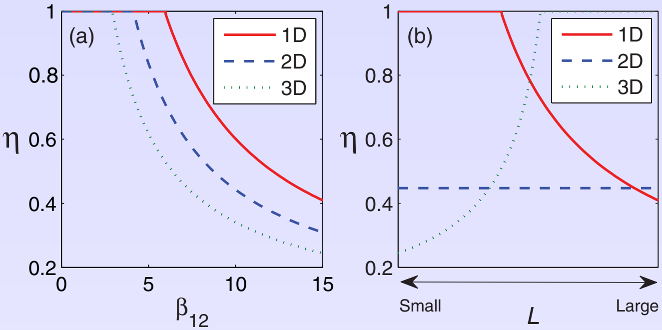
Phase separation when $k_0, \Omega, \delta = 0$

- Repulsive interactions only:

$$E(\phi_1, \phi_2) = \int \frac{1}{2}|\nabla \phi_1|^2 + \frac{1}{2}|\nabla \phi_2|^2 + \frac{\beta_{11}}{2}|\phi_1|^4 + \frac{\beta_{22}}{2}|\phi_2|^4 + \beta_{12}|\phi_1|^2|\phi_2|^2$$

- Homogeneous case: $\beta_{11}\beta_{22} \geq \beta_{12}^2$ mixed; otherwise separated
- Nonhomogeneous case?

- $\beta_{11} = \beta_{22}$, box potential (width L)
- mixing factor: $\eta = 2\int \phi_1\phi_2$



- Exist $\beta_c > \beta$, when $\beta_{12} \leq \beta_c$, $\eta = 1$
- proof by Fundamental gap+elliptic estimates

Fundamental gap

- consider linear case $-\Delta + V(\mathbf{x})$, $\mathbf{x} \in U \subset \mathbb{R}^d$ (U compact convex)[1] with Dirichlet boundary conditions
- eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, eigenfunctions $\{\phi_k\}_{k=0}^{\infty}$

$$\Delta\phi_k - V(\mathbf{x})\phi_k + \lambda_k\phi_k = 0, \quad \phi_k|_{\partial U} = 0$$

- fundamental gap $:= \lambda_1 - \lambda_0$
- Gap conjecture: Let U be a bounded convex domain with diameter D, $V(\mathbf{x})$ be convex, then the fundamental gap

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}$$

[1] B. Andrews AND J. Clutterbuck, JAMS, 2011

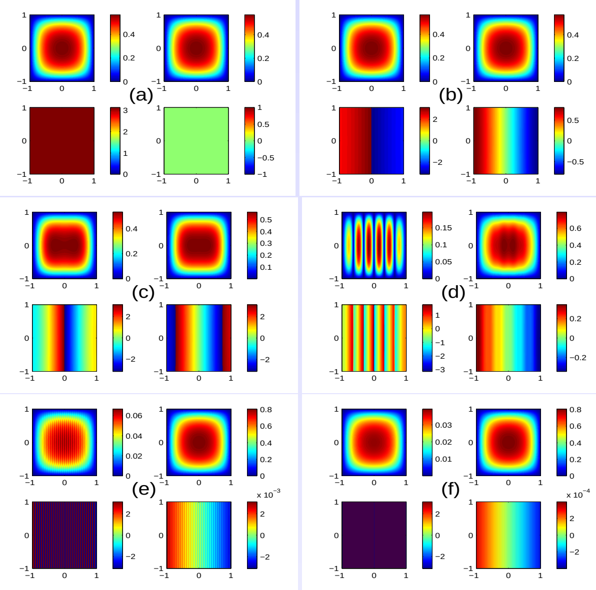
Ground state phases

- If $|\Omega|/|k_0|^2 \gg 1$, $|\Omega| \to \infty$, the ground state $\Phi_g = (\phi_1^g, \phi_2^g)^T \approx (|\phi_1^g|, \mathrm{sgn}(-\Omega)|\phi_2^g|)^T$ (constant phase), i.e. k_0 effect will vanish.
- If $|\Omega|/|k_0| \ll 1$, $|k_0| \to \infty$, the ground state $\Phi_g = (\phi_1^g, \phi_2^g)^T \approx (e^{-ik_0x}|\tilde{\phi}_1^{g,0}|, e^{ik_0x}|\tilde{\phi}_2^{g,0}|)^T$ (plane wave phase), i.e., Ω effect will vanish.
- If $|k_0| \ll |\Omega| \ll |k_0|^2$ and $|k_0| \to \infty$, density modulation.

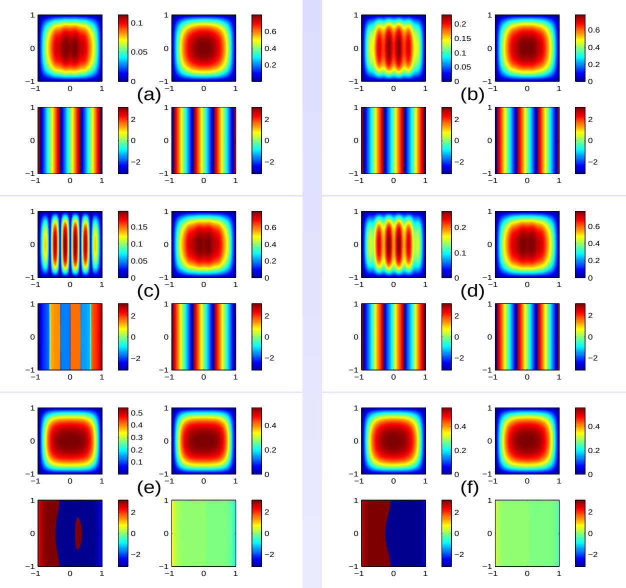
Numerical examples

- We consider $d = 2$ with box potential, $\delta = 0$, $\beta_{11} : \beta_{12} : \beta_{22} = 1 : 0.9 : 0.9$ with $\beta_{11} = 10$.
- For $\Omega = 0$, the first component $\phi_1 = 0$.

Large k_0 limit



Large Ω limit



Dynamical properties

Center-of-mass motion

- Potentials $V_1 = V_2$ are harmonic potentials.
  $V_1 = \frac{1}{2}(\gamma_x^2 x^2 + \gamma_y^2 y^2 + \gamma_z^2 z^2)$ in 3D; $V_1 = \frac{1}{2}(\gamma_x^2 x^2 + \gamma_y^2 y^2)$ in 2D; $V_1 = \frac{1}{2}\gamma_x^2 x^2$ in 3D.
- Center-of-mass (COM) of the BEC:

$$\mathbf{x}_c(t) = \int_{\mathbb{R}^d} \mathbf{x} \sum_{j=1}^{2} |\psi_j(\mathbf{x}, t)|^2 \, d\mathbf{x}, \qquad t \geq 0,$$

  and the momentum as

$$\mathbf{P}(t) = \int_{\mathbb{R}^d} \sum_{j=1}^{2} \mathrm{Im}(\overline{\psi_j(\mathbf{x}, t)} \nabla \psi_j(\mathbf{x}, t)) \, d\mathbf{x}, \qquad t \geq 0,$$

- Mass difference between two components

$$\delta_N(t) := N_1(t) - N_2(t) = \int_{\mathbb{R}^d} \left[|\psi_1(\mathbf{x}, t)|^2 - |\psi_2(\mathbf{x}, t)|^2 \right] \, d\mathbf{x}.$$

Center-of-mass motion

- For the x-component $x_c(t)$ of the center-of-mass $\mathbf{x}_c(t)$ with any initial data $\Psi(\mathbf{x},0) := \Psi_0(\mathbf{x})$ satisfying $\|\Psi_0\| = 1$, we have

$$x_c(t) = x_0 \cos(\gamma_x t) + \frac{P_0^x}{\gamma_x} \sin(\gamma_x t) - k_0 \int_0^t \cos(\gamma_x (t-s)) \delta_N(s)\, ds.$$

  where x_0 initial x-component of center-of-mass and P_0^x initial x-component of momentum.

- In 2D (3D), the y (y,z)-component of the center-of-mass is periodic with period γ_y (γ_y, γ_z).

General initial data

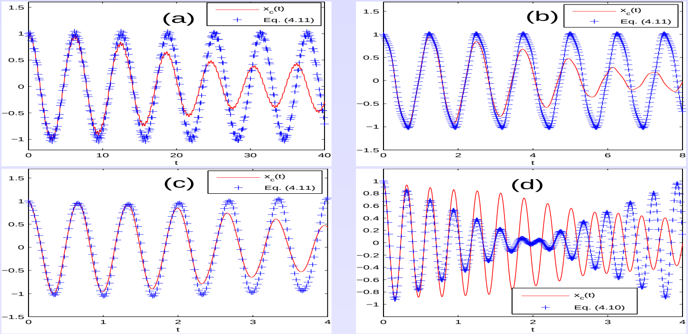


FIG. 4.1. *Time evolution of the center-of-mass $x_c(t)$ for the CGPEs* (1.8) *obtained numerically from its numerical solution (i.e. labeled by '$x_c(t)$' with solid lines) and asymptotically as Eqs.* (4.10) *and* (4.11) *in Theorem 4.2 (i.e. labeled by 'Eq.' with '+ + +') with $\Omega = 20$ and $k_0 = 1$ for different γ_x: (a) $\gamma_x = 1$, (b) $\gamma_x = 5$, (c) $\gamma_x = 3\pi$, and (d) $\gamma_x = 20$.*

Shift of ground state initial condition

- Given the ground state $\Phi_g = (\phi_1^g, \phi_2^g)^T$ for the CGPEs , the initial condition is chosen as

$$\psi_1(\mathbf{x}, 0) = \phi_1^g(\mathbf{x} - \mathbf{x}_0), \quad \psi_2(\mathbf{x}, 0) = \phi_2^g(\mathbf{x} - \mathbf{x}_0), \qquad \mathbf{x} \in \mathbb{R}^d,$$

where $\mathbf{x}_0 = x_0$ in 1D, $\mathbf{x}_0 = (x_0, y_0)^T$ in 2D and $\mathbf{x}_0 = (x_0, y_0, z_0)^T$ in 3D.

Shift of ground state initial data

Theorem

For the initial data chosen as the shift of ground state, we have (i) when $\frac{|k_0|^2}{|\Omega|} \gg 1$*, the dynamics of the center-of-mass* $x_c(t)$ *can be approximated by the ODE*

$$\ddot{x}_c(t) = -\gamma_x^2 x_c(t), \quad x_c(0) = x_0, \quad \dot{x}_c(0) = 0,$$

i.e., $x_c(t) = x_0 \cos(\gamma_x t)$*.*
(ii) when $\frac{|k_0|^2}{|\Omega|} \ll 1$*,* $\beta_{jl} \approx \beta$ *with* β *a fixed constant, the dynamics of the center-of-mass* $x_c(t)$ *can be approximated by*

$$\dot{x}_c(t) = P^x(t) - \frac{k_0[2k_0 P^x(t) - \delta]}{\sqrt{[2k_0 P^x(t) - \delta]^2 + \Omega^2}}, \quad \dot{P}^x(t) = -\gamma_x^2 x_c(t),$$

with $x_c(0) = x_0$ *and* $P^x(0) = k_0 \delta_N(0)$*. In particular, the solution is periodic, and, in general, its frequency is different with the trapping frequency* γ_x*.*

Shift of ground state

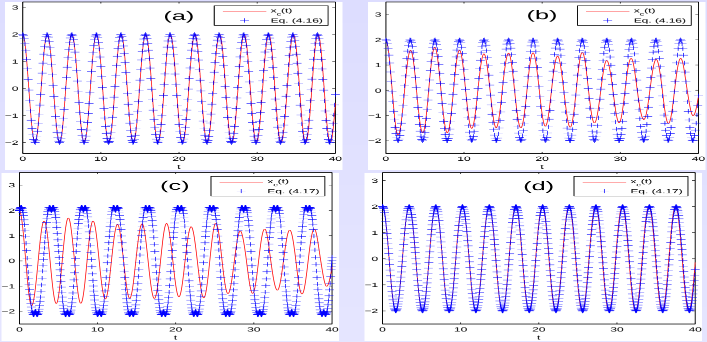


FIG. 4.2. *Time evolution of the center-of-mass $x_c(t)$ for the CGPEs* (1.8) *obtained numerically from its numerical solution (i.e. labeled as $x_c(t)$ with solid lines) and asymptotically as Eqs.* (4.16) *and* (4.17) *in Theorem 4.3 (i.e. labeled as 'Eq.' with '+ + +') for different sets of parameters: (a)* $(\Omega, k_0) = (50, 20)$*, (b) and (c)* $(\Omega, k_0) = (2, 2)$*, and (d)* $(\Omega, k_0) = (50, 2)$.

conclusion

- SO-coupled BEC described by Coupled Gross-Pitaevskii equations
- Ground state properties: competition between SO coupling and Raman transition
- Center-of-mass dynamics: periodic v.s. non-periodic in different cases
- Future: SO coupling effects in other systems (nonlocal interaction, rotating frame), phase separation, domain wall, mass transfer,...

THANK YOU!