Helicity, Topology and Kelvin Waves in reconnecting quantum knots

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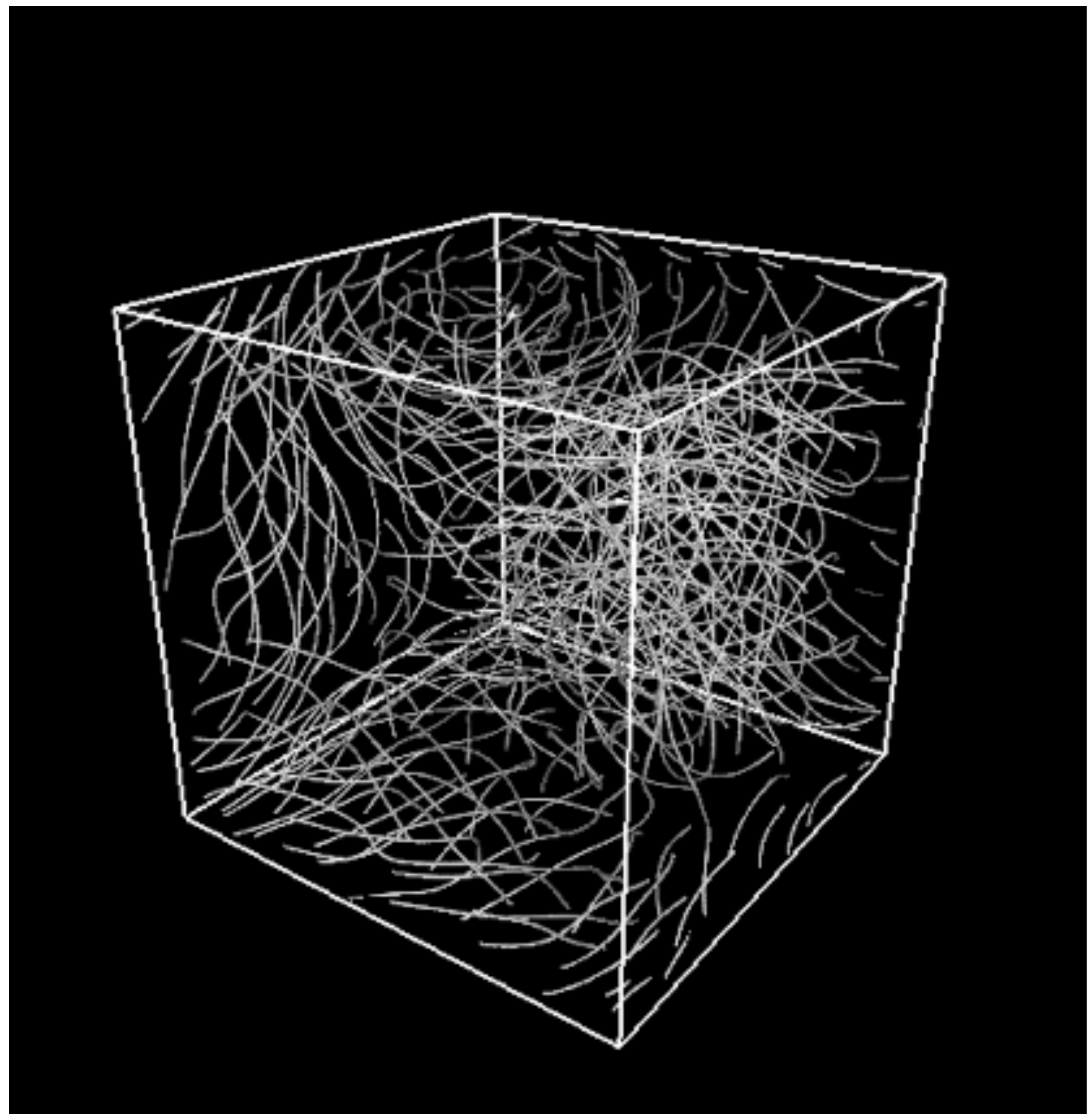

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New Challenges in Mathematical Modelling and Numerical Simulation of Superfluids,
Workshop CIRM Marseille,
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2 Main results for Gross-Pitaevskii Superfluids:

- Detecting Kelvin Waves using spatiotemporal spectrum
- Helicity and Kelvin Waves in reconnecting quantum knots

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Detecting Kelvin Waves using spatiotemporal spectrum

- Main results:

- Space-time resolved spectra allow to find needles in haystacks : Kelvin waves in spatial spectrum
- A practical method to quantify their presence

GPE, Madelung and quantum vortices

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + g|\psi|^2\psi,$$

$$\psi(\mathbf{r},t) = \sqrt{\frac{\rho(\mathbf{r},t)}{m}}e^{im\phi(\mathbf{r},t)/\hbar},$$
$$\mathbf{v} = \boldsymbol{\nabla}\phi,$$

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0,$$
$$\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\frac{g}{m^2}\boldsymbol{\nabla}\rho + \frac{\hbar^2}{2m^2}\boldsymbol{\nabla}\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right).$$

$$\Gamma = \oint_C \mathbf{v}(\ell)\,d\ell = 4\pi\alpha, \qquad \alpha = \hbar/(2m).$$

$$\boldsymbol{\omega}(\mathbf{r}) = \Gamma\int \mathrm{d}s\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}s}\delta^{(3)}(\mathbf{r}-\mathbf{r}'(s)),$$

Spatiotemporal spectra

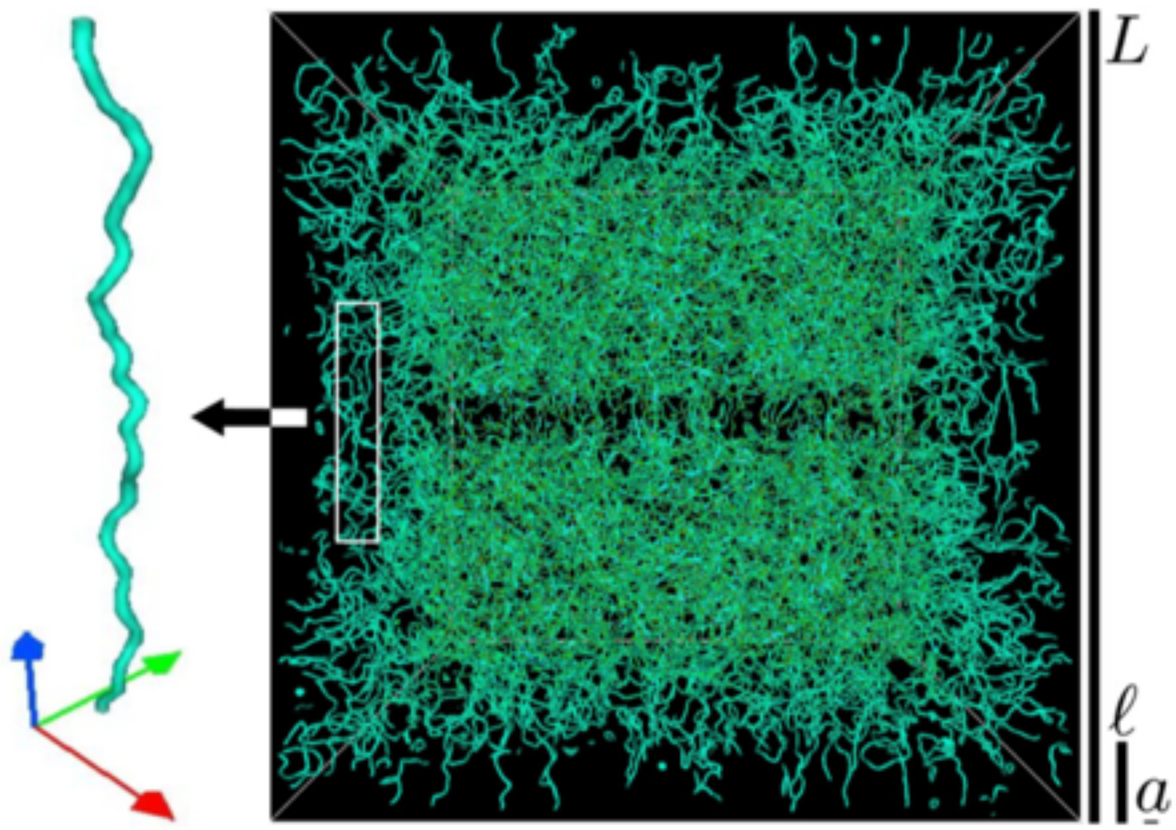
- Finding Kelvin waves in the energy spectra is like looking for needles in a haystack…
- Instantaneous flow visualization is insufficient to identify and extract all the waves in a turbulent flow.
- To quantify their amplitudes as a function of frequency and wave number : calculate space-time resolved spectra.

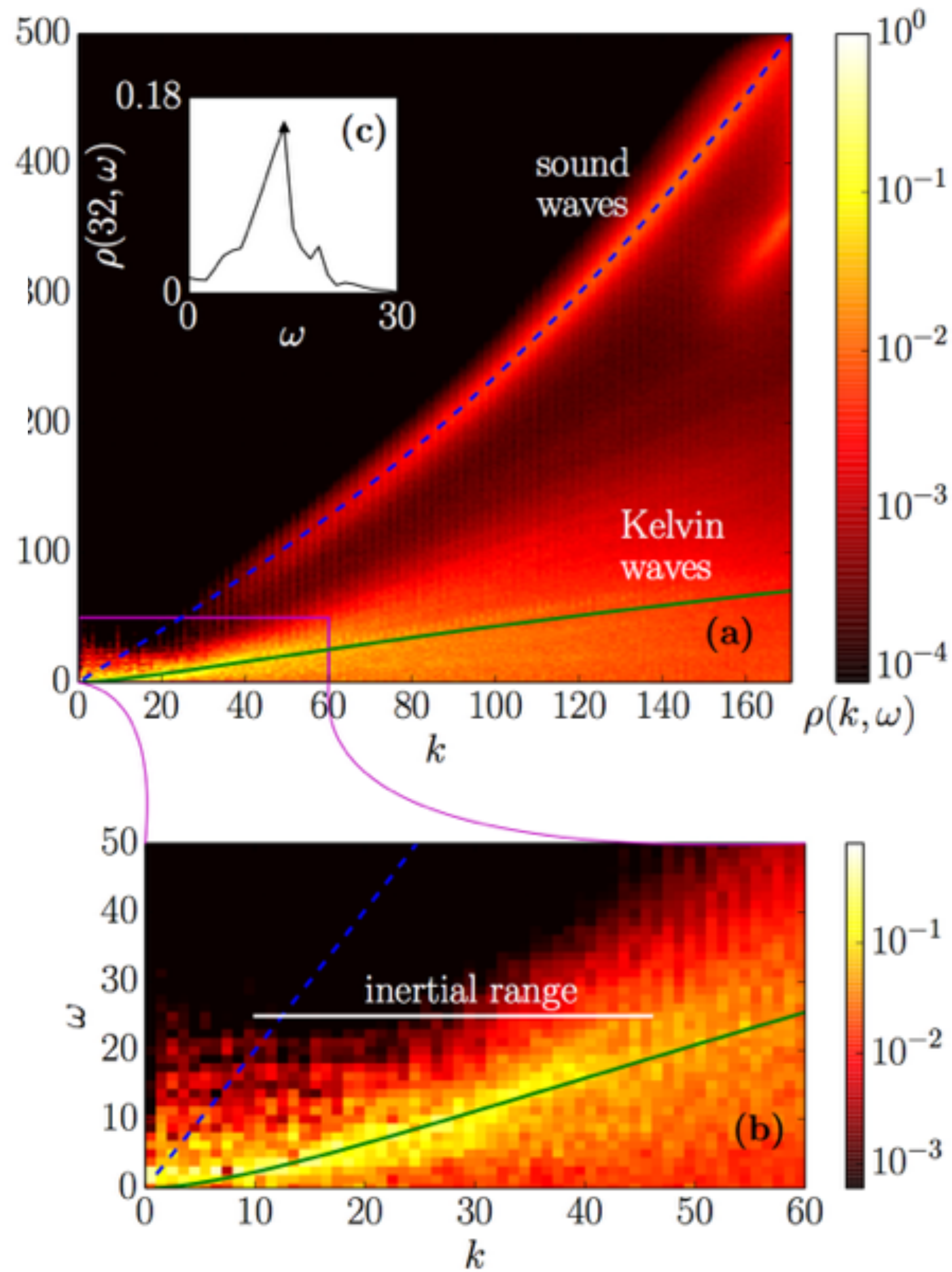
Space-time resolved Mass spectrum, Taylor Green

$$\omega_B(k) = k\sqrt{c^2 + \frac{c^2\xi^2}{2}k^2},$$

$$\xi = \sqrt{\hbar^2/(2m|\psi|^2 g)} \qquad c = \sqrt{g|\psi|^2/m}$$

$$\omega_K(k) = \frac{2c\xi}{\sqrt{2}a^2}\left(1 \pm \sqrt{1 + ka\frac{K_0(ka)}{K_1(ka)}}\right)$$





Helicity and Kelvin Waves in reconnecting quantum knots

- Main results

- Helicity can be directly computed from the GPE 3D complex wave function field using our new regularization method
- Conservation or non-conservation of quantum helicity is an open problem involving not only topological changes, but also excitation (and decay) of Kelvin waves

Helicity in quantum flows

$$\Gamma = \oint_C \mathbf{v}(\ell)\, d\ell = 4\pi\alpha, \qquad \alpha = \hbar/(2m).$$

$$\boldsymbol{\omega}(\mathbf{r}) = \Gamma \int \mathrm{d}s \frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}s} \delta^{(3)}(\mathbf{r} - \mathbf{r}'(s)),$$

$$\mathbf{v} = \frac{\mathcal{P}}{n}, \qquad \mathcal{P}_j = 2\alpha \frac{\overline{\Psi}\partial_j \Psi - \Psi \partial_j \overline{\Psi}}{2i}$$

$$n = \Psi\Psi,$$

$$\mathbf{v} = \frac{\alpha}{i}\left(\frac{\nabla\Psi}{\Psi} - \frac{\nabla\bar{\Psi}}{\bar{\Psi}}\right)$$

Singularity of v

(notice that these definitions are analogous to those derived via the Madelung transformation $\Psi = \sqrt{n}e^{i\phi}$, where the velocity is given by $\mathbf{v} = 2\alpha\boldsymbol{\nabla}\phi$). At a distance $r \to 0$ from a straight vortex line these quantities are known [27] to behave as $n \sim r^2$ and $\mathbf{v} = 2\alpha\mathbf{e}_\theta/r$ where $\mathbf{e}_\theta$ is the azimuthal unit vector and r the radial distance in a cylindrical coordinate system $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ having its origin on the straight vortex line. Thus, the velocity $\mathbf{v}$ has an r^{-1} singularity *perpendicular* to the vortex line.

Need to regularize v

Therefore, as the vorticity (see Eq.(2)) also has a singularity *parallel* to those lines, the standard definition of helicity

$$\mathcal{H} = \int \mathrm{d}\mathbf{r}\, \boldsymbol{\omega}(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r}), \tag{6}$$

is not well behaved, as it involves the product of two singular distributions. The idea of the *regularized* helicity is to replace in Eq. (6) the field $\mathbf{v}$ by a regularized smooth field $\mathbf{v}_{\mathrm{reg}}$ having no divergences perpendicular to the line, and the same regular behavior as $\mathbf{v}$ parallel to the line.

$$\mathbf{v} = \frac{\alpha}{i}\left(\frac{\nabla\Psi}{\Psi} - \frac{\nabla\bar{\Psi}}{\bar{\Psi}}\right)$$

Along the line: 0/0

Idea: use L'Hôpital's rule

Definition of regular v

$$v_{\|} = \frac{2\alpha}{2i} \frac{\mathcal{W}_j \left[(\partial_j \partial_l \Psi)\partial_l(\overline{\Psi})) - (\partial_j \partial_l \overline{\Psi})\partial_l(\Psi))\right]}{\sqrt{\mathcal{W}_l \mathcal{W}_l}(\partial_m \Psi)(\partial_m \overline{\Psi})},$$

where

$$\mathcal{W}_j = \epsilon_{jkl} \partial_k \mathcal{P}_l = \frac{2\alpha}{i} \epsilon_{jkl} \partial_k \overline{\Psi} \partial_l \Psi \tag{7}$$

is a smooth field oriented along the vortex line. Then, we can define the regularized helicity

$$\mathcal{H} = \int \mathrm{d}\mathbf{r}\, \boldsymbol{\omega}(\mathbf{r}) \cdot \mathbf{v}_{\mathrm{reg}}(\mathbf{r}), \tag{8}$$

with $\mathbf{v}_{\mathrm{reg}} = v_{\|} \mathcal{W} / \sqrt{\mathcal{W}_j \mathcal{W}_j}$. We show next how this regularized helicity still holds the geometrical interpretations valid for the standard one.

Relation with writhe. For and isolated structure, helicity can be decomposed into twist (loosely speaking, the total number of helical turns a ribbon does), and writhe (the "coiling" of the structure). Let's start by analyzing the relation between the regularized helicity and the writhe. For a single curve, the writhe Wr is, by definition [28], given by the expression

$$Wr = \frac{1}{4\pi} \frac{\int \int (\mathrm{d}\mathbf{r} \times \mathrm{d}\mathbf{r}_1) \cdot (\mathbf{r} - \mathbf{r}_1)}{|(\mathbf{r} - \mathbf{r}_1)|^3}. \tag{9}$$

It is easy to see that if one uses a velocity field $\mathbf{V}(\mathbf{r})$ given by the Biot-Savart law

$$\mathbf{V}(\mathbf{r}) = \frac{\Gamma}{4\pi} \frac{\int \mathrm{d}\mathbf{r}_1 \times (\mathbf{r} - \mathbf{r}_1)}{|(\mathbf{r} - \mathbf{r}_1)|^3}, \tag{10}$$

where $\mathbf{r}_1$ corresponds to the position of the vortex lines, and the vorticity as defined in Eq. (2), then helicity $\mathcal{H}$ is given by

$$\mathcal{H} = \int \mathbf{V}(\mathbf{r}) \cdot \boldsymbol{\omega}(\mathbf{r}) \mathrm{d}V = \Gamma \int \mathbf{V}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r},$$
$$= \frac{\Gamma^2}{4\pi} \frac{\int \int \mathrm{d}\mathbf{r} \cdot (\mathrm{d}\mathbf{r}_1 \times (\mathbf{r} - \mathbf{r}_1))}{|(\mathbf{r} - \mathbf{r}_1)|^3}.$$

From the identity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ one finds that in this simple case (for a *single* line)

$$\mathcal{H} = \Gamma^2 Wr.$$

Regularized helicity defined as the twist of constant phase ribbon. First we recall that the twist Tw of a ribbon (defined by *both* a curve $\mathbf{r}(\mathbf{s})$, and a vector $\mathbf{U}(s)$ perpendicular to the curve) is defined by the integral over the curve

$$Tw = \frac{1}{2\pi}\int\left(\frac{d\mathbf{U}}{\mathrm{d}s}\times\mathbf{U}\right)\cdot\frac{d\mathbf{r}}{\mathrm{d}s}\mathrm{d}s. \tag{11}$$

One can further show that [6]

$$Tw = N + \frac{1}{2\pi}\int\tau(s)\mathrm{d}s, \tag{12}$$

where τ is the torsion, and N the number of turns round the curve of $\mathbf{U}$ in the Frenet-Serret frame (see *Methods*). The regularized helicity can be presented in a purely geometrical way. Under the GPE, constant phase surfaces will intersect on the vortex lines. Now consider a line at a close distance of the vortex line and lying on a constant phase surface (note that we could construct an equivalent line in the classical Biot-Savart case by requiring the line to be perpendicular to the velocity field). The vortex line and the constant phase line defines a ribbon. Now, using Eqs. (2), (7) and (11) we can see that

$$\mathcal{H} = \Gamma^2\, Tw.$$

Constant phase surfaces : 2 linked rings and trefoil knot

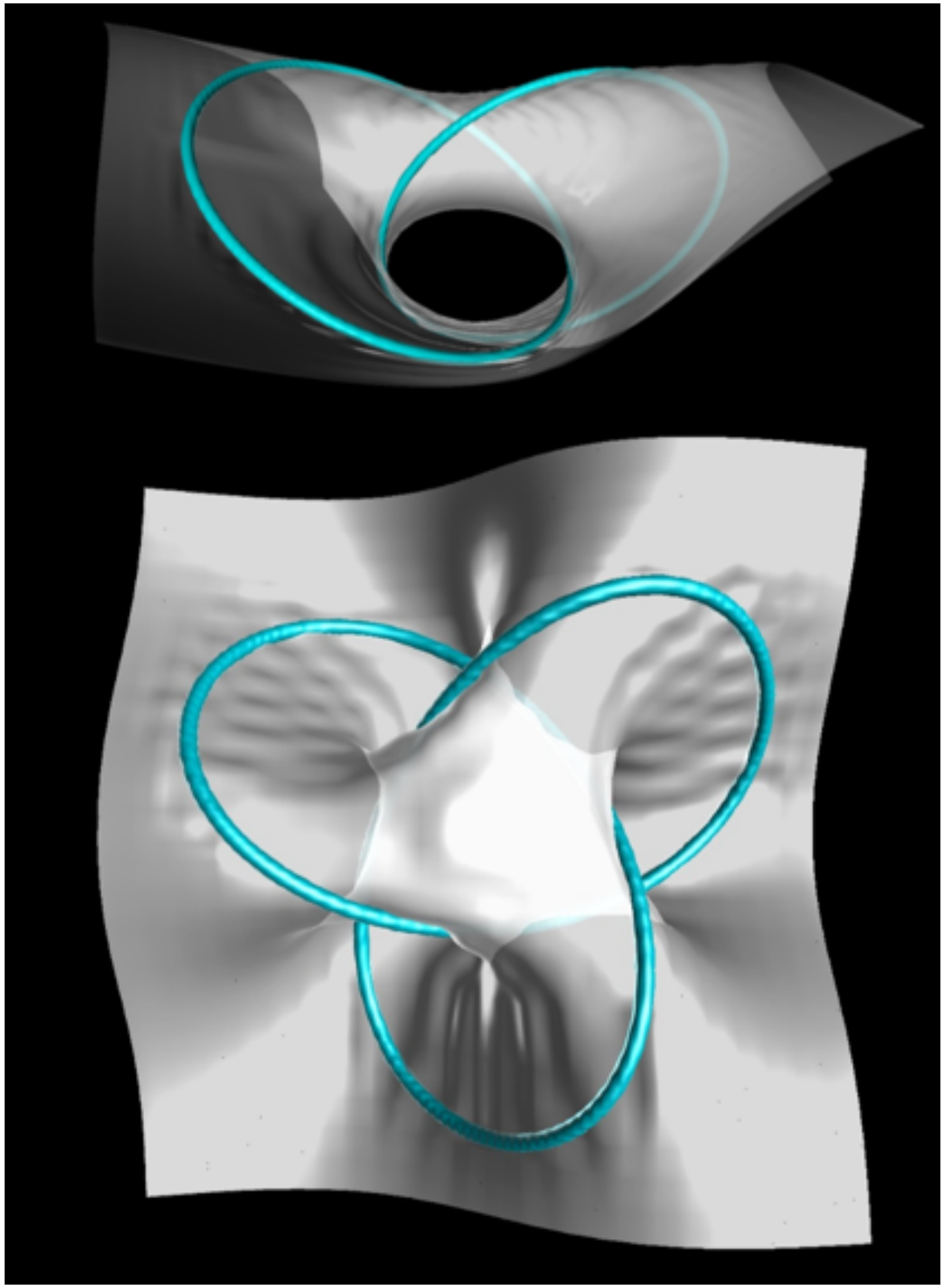
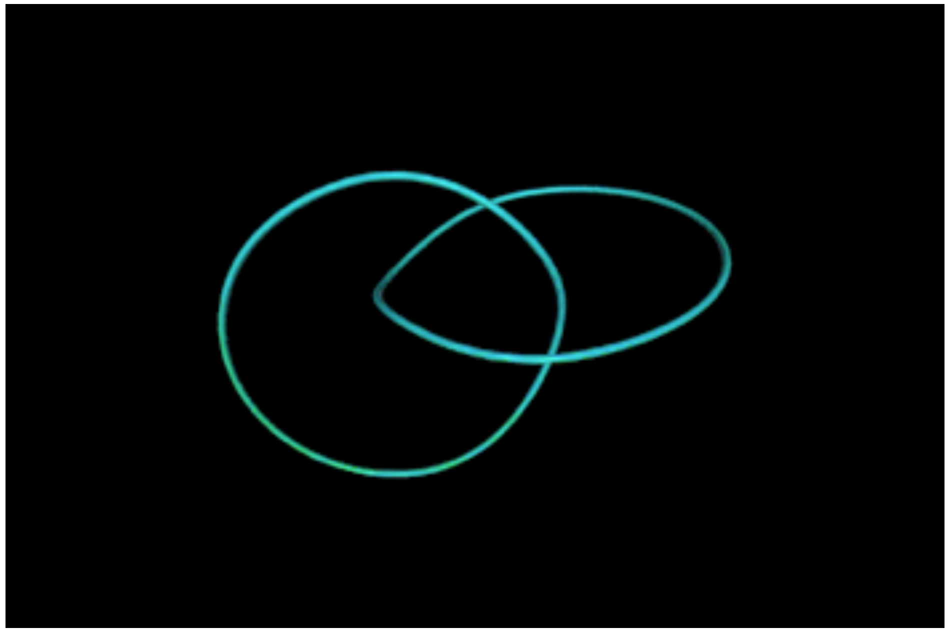
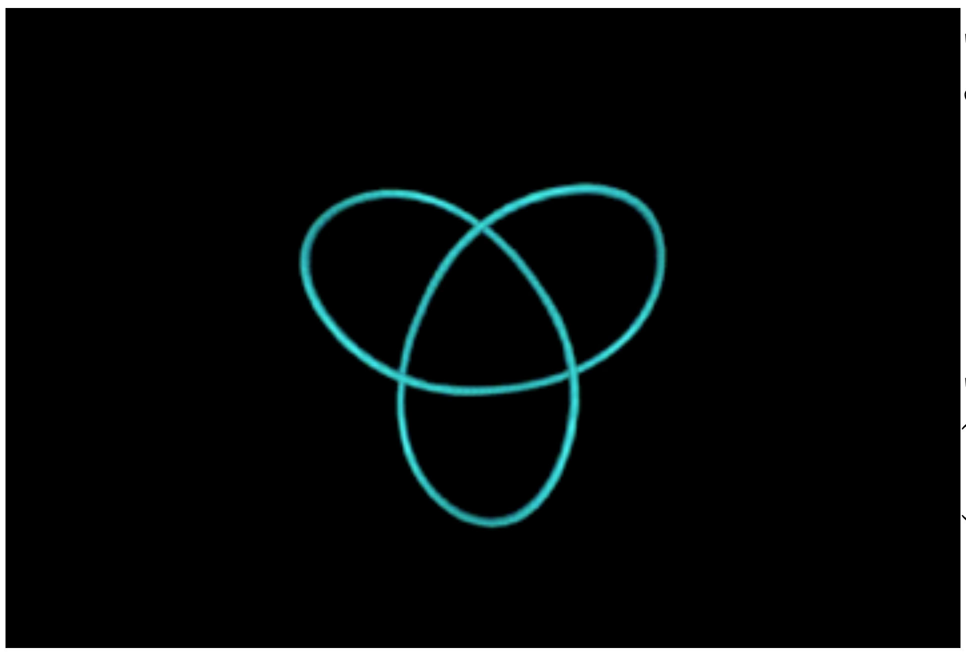


FIG. 1. Renderings of the surface of zero phase for two knots in a quantum fluid. Top: two linked rings, note the surface has one hole. Bottom: trefoil knot, with three holes. The number of holes is associated to the number of turns the vector that lies on the surface perpendicular to the vortex lines does as it moves along the curve.

helicity versus time





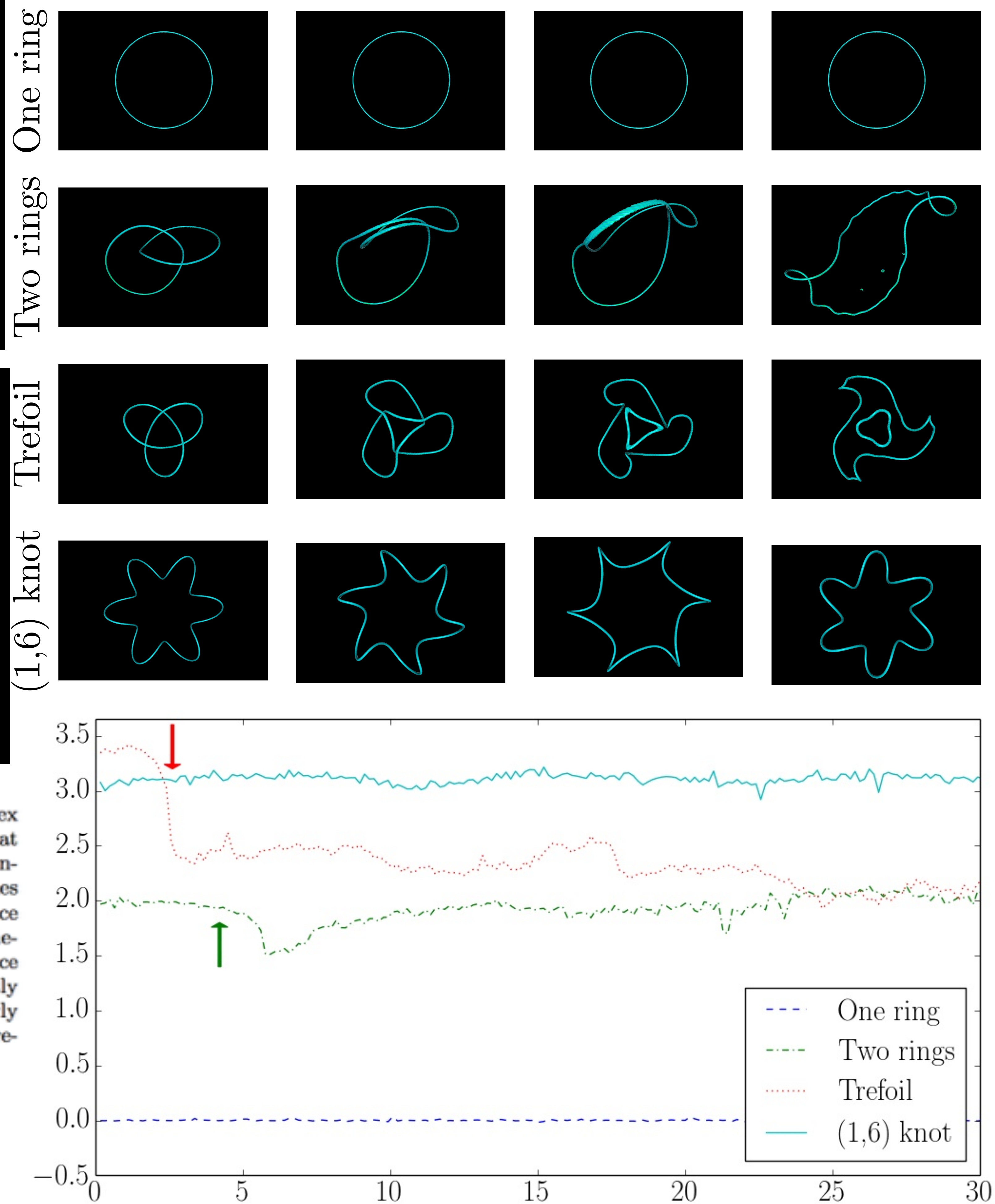
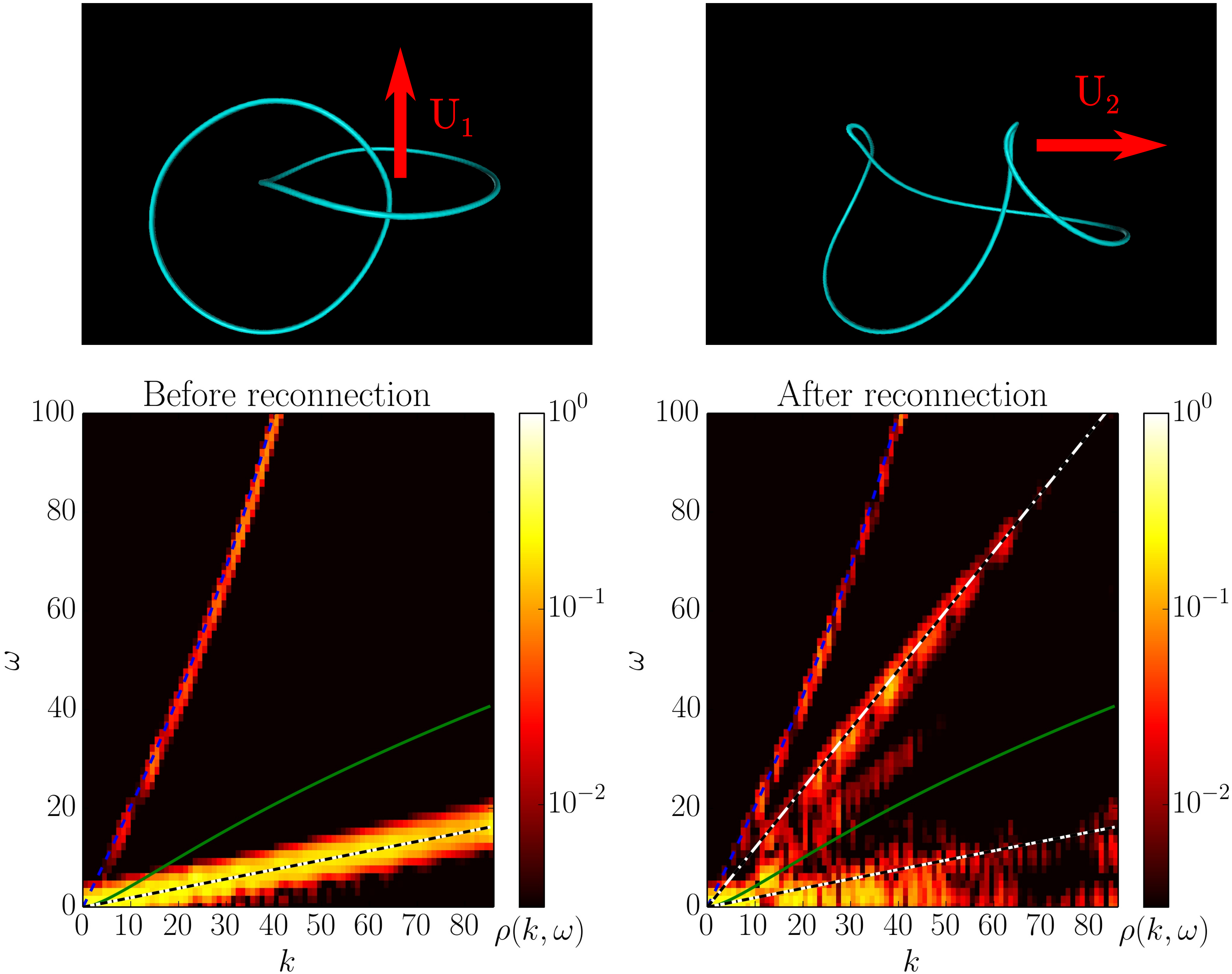
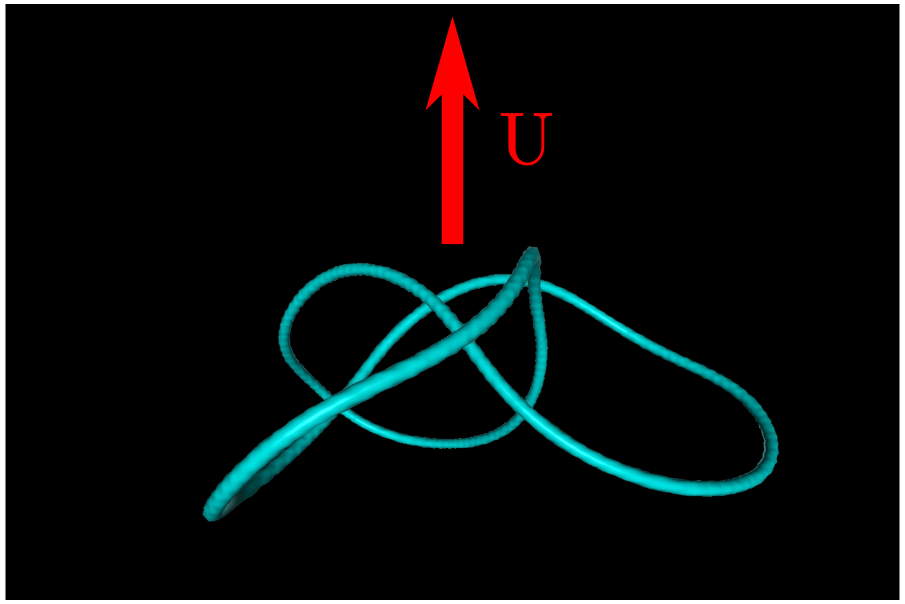


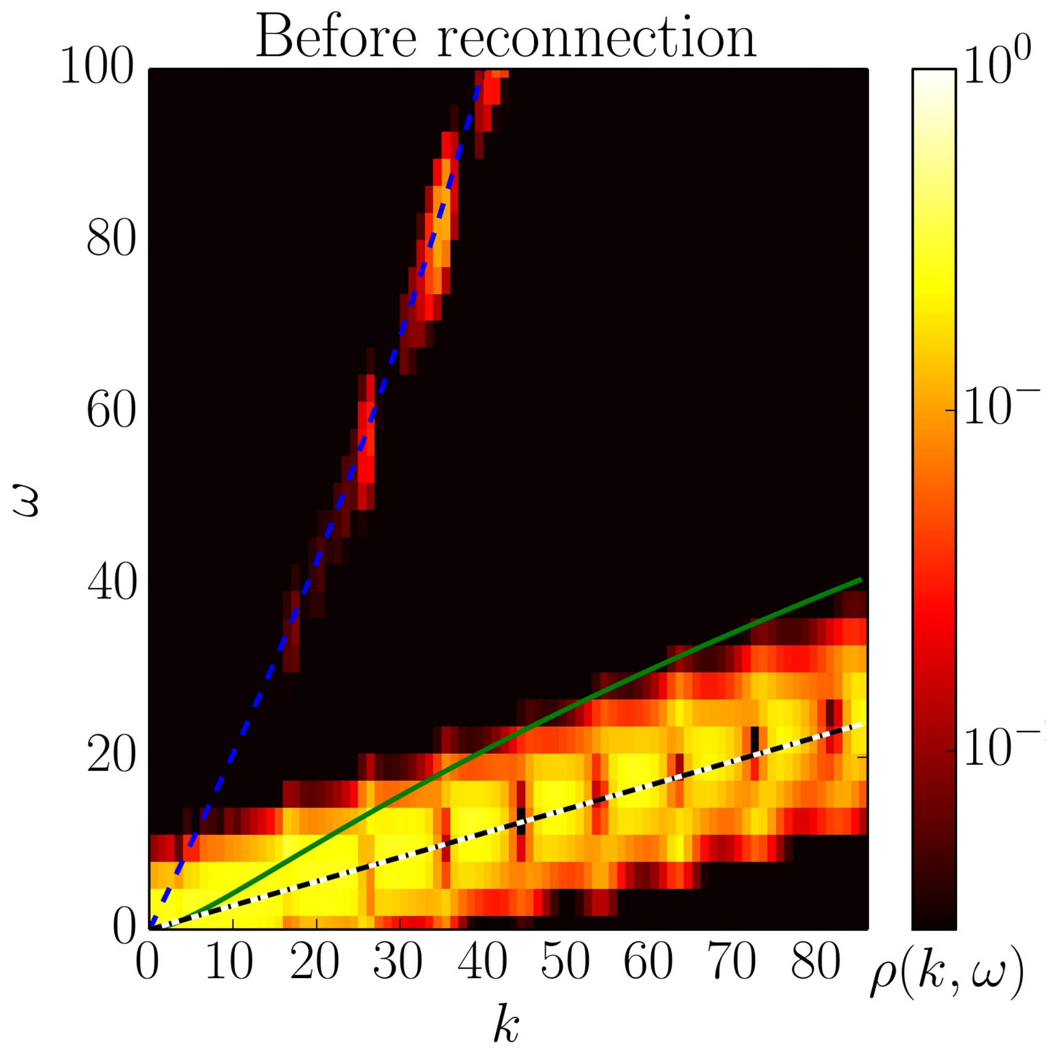
FIG. 2. Time evolution of the helicity for four quantum vortex configurations. At the top, snapshots of the configurations at different times are shown. The single ring only moves at constant speed. The two rings and the trefoil reconnect at times marked by the vertical arrows. When reconnection takes place between two anti-parallel vortex lines (as in the two rings), helicity does not change. In the trefoil reconnection takes place simultaneously at three points and helicity changes abruptly at the time indicated by the red arrow; later it decays slowly to its final value. The (1,6)-torus knot deforms without reconnecting, and its helicity does not change.

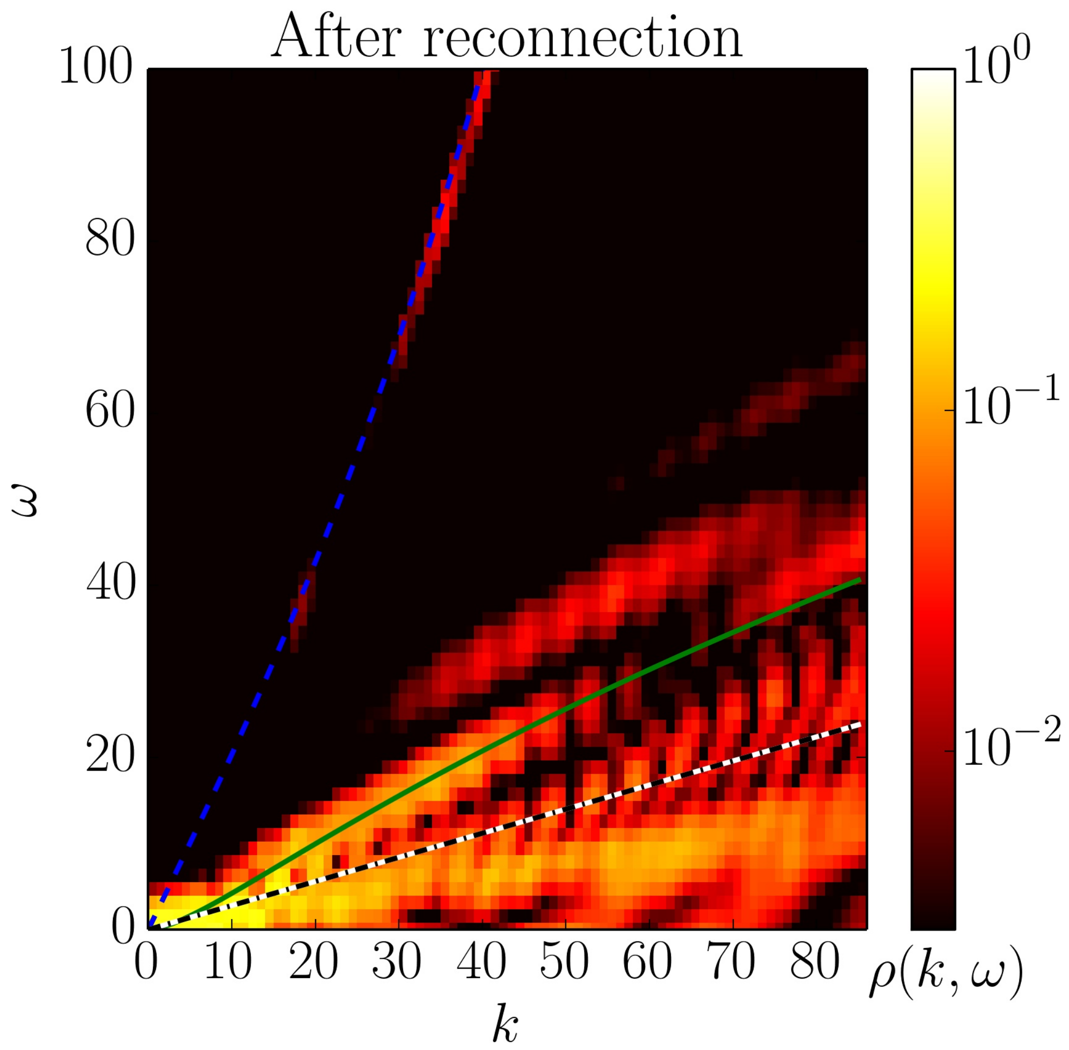
2 linked rings



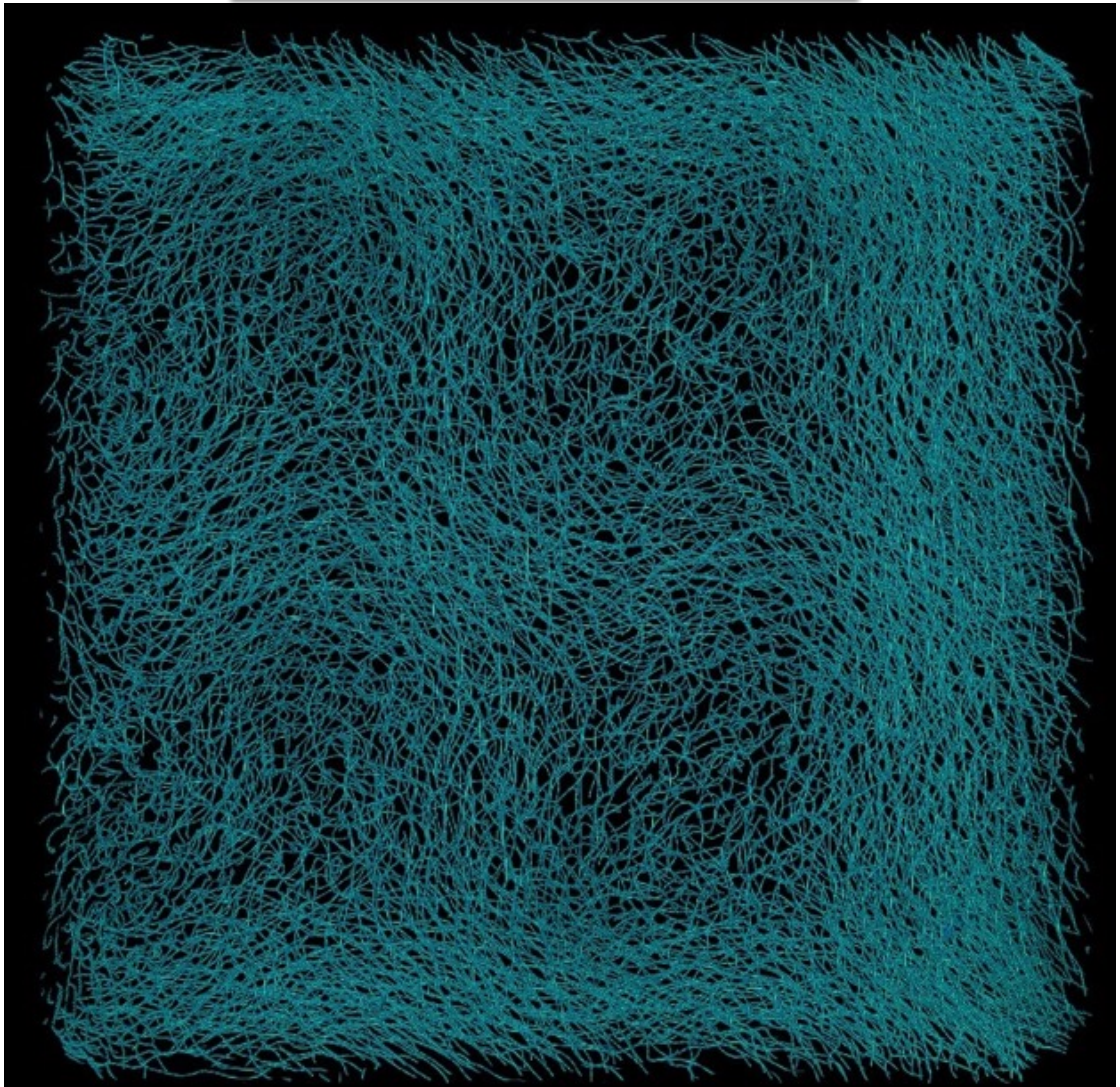
Trefoil knot





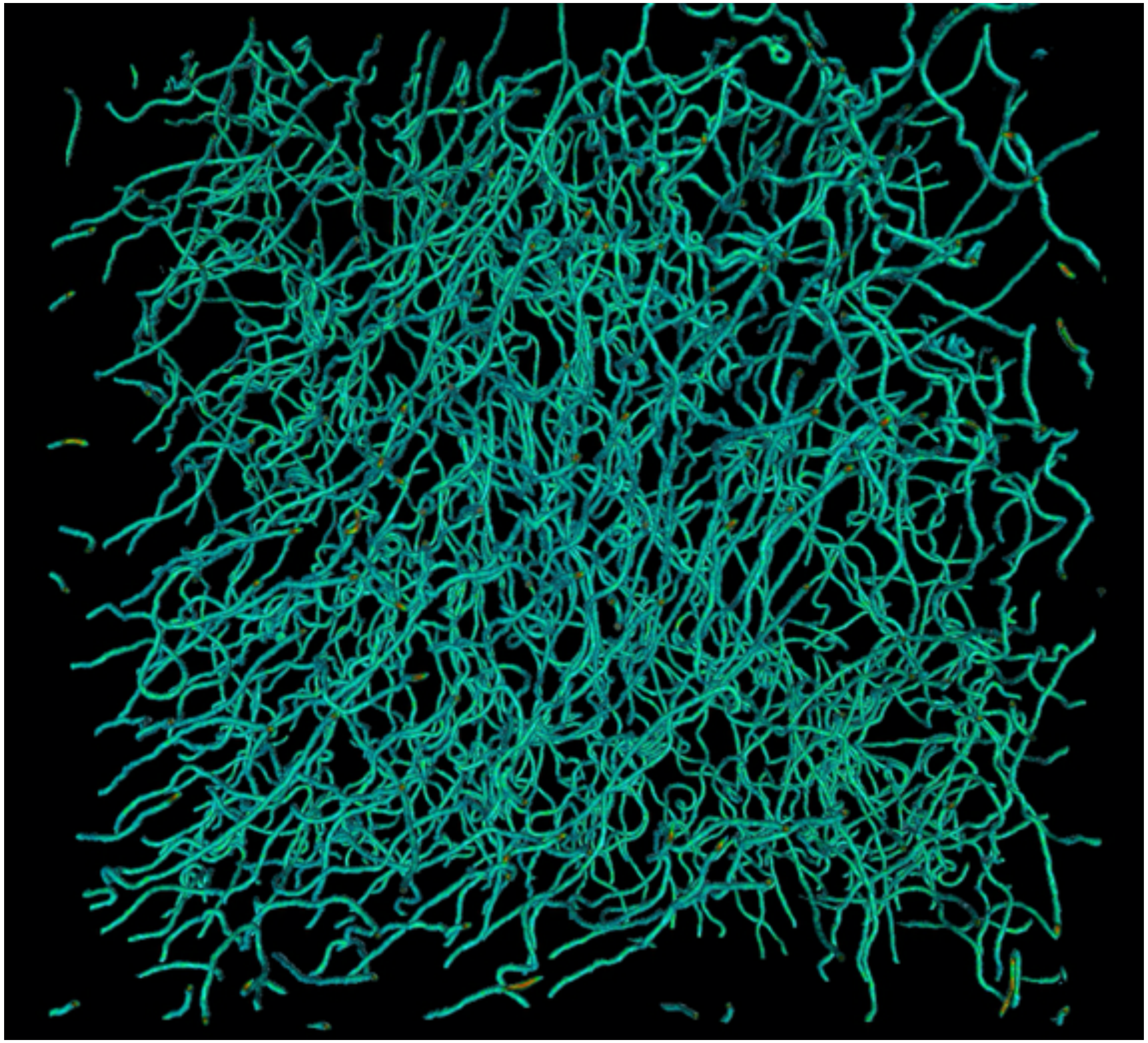


The helicity of this ABC superflow is 450 000 quanta



Conclusion

- Space-time resolved spectra allow to detect and quantify Kelvin Waves
- Regularized helicity is directly computable from 3D complex wave function field, which is very useful for e.g. the study large-scale helical ABC quantum flows
- Conservation or non-conservation of quantum helicity is an open problem involving not only topological changes but also excitation (and decay) of Kelvin waves
- Much remains to be understood!



Thank you!