Scaling limits of random planar maps towards the Brownian tree

Sigurður Örn Stefánsson,
University of Iceland

Joint work with
Svante Janson,
Uppsala University

Random Trees and Maps:
Probabilistic and Combinatorial Aspects
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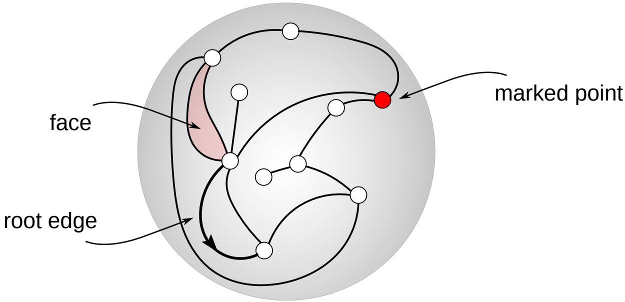
Planar maps

- A planar map is a finite, connected graph drawn on the two sphere such that edges do not cross. Viewed up to orientation preserving homeomorphisms of the sphere.



- The connected components of the complement of the edges are called faces. The degree of a face f is the number of edges adjacent to it (counted with multiplicity) and is denoted by $\deg(f)$.
- Single out a directed root edge and a marked point
  $\rightarrow$ rooted and pointed maps.

Random planar maps

- Let $\mathcal{M}_n$ be the set of rooted and pointed planar maps with n edges which are furthermore bipartite (equivalently all faces have even degrees).
- Let $(q_i)_{i\geq 1}$ be a sequence of non–negative numbers and define the weight of a map $m \in \mathcal{M}_n$ as

$$W(m) = \prod_{f \text{ face in } m} q_{\deg(f)/2}$$

- Define the probability measure μ_n on $\mathcal{M}_n$ by

$$\mu_n(m) = \mathcal{Z}_n^{-1} W(m)$$

  $\mathcal{Z}_n = \sum_{m' \in \mathcal{M}_n} W(m') =$ normalization a.k.a. partition function.
- Let M_n be the random planar map distributed by μ_n. Want to obtain a 'continuum limit' of M_n and describe the different phases which appear by varying the parameters q_i.

Motivation

- For 'nice' weights the scaling limit is the Brownian map, e.g. $q_i = \delta_{i,p}$, $p \geq 2$ (Le Gall 2011), ($p = 2$: Miermont 2011).
- Scaling limits of random planar maps with large faces (Le Gall & Miermont '11).

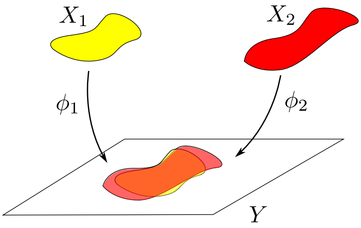
  Weights chosen such that the distribution of the degree of a typical face is in the d.o.a. of a stable distribution with index $\alpha \in (1, 2)$.
- Is this the whole story?

Compact metric spaces and Gromov–Hausdorff metric

$\mathbb{M}$ = set of all compact metric spaces modulo isometries.

Gromov–Hausdorff metric on $\mathbb{M}$ defined as

$$d_{\mathrm{GH}}(X_1, X_2) = \inf \left\{ d_{\mathrm{H}}^{Y}(\phi_1(X_1), \phi_2(X_2)) \ : \ Y, \phi_1, \phi_2 \right\}$$



- Infimum is over all metric spaces Y and all isometric embeddings ϕ_1, ϕ_2 of X_1, X_2 resp. into Y.
- d_{H}^{Y} is the Hausdorff metric on the set of compact subsets in Y

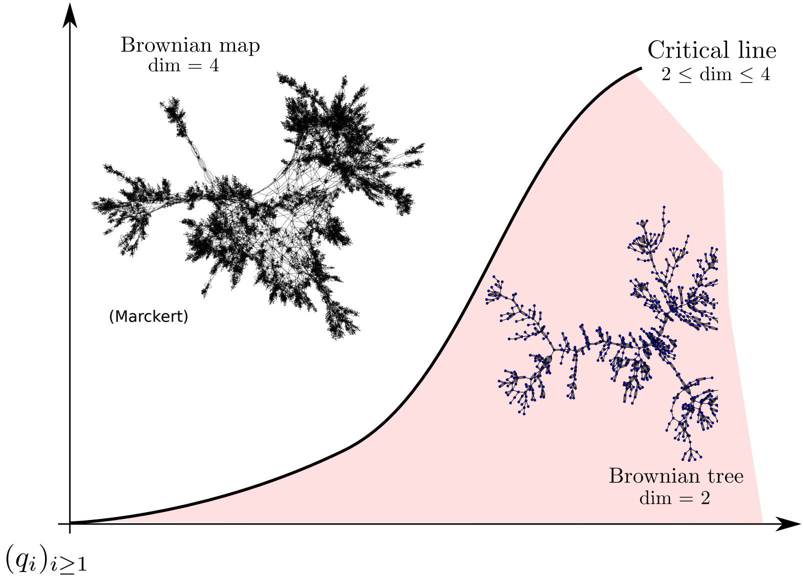
$$d_{\mathrm{H}}^{Y}(U, V) = \inf\{\epsilon > 0 \ : \ V \subseteq U^{\epsilon} \text{ and } U \subseteq V^{\epsilon}\}.$$

- $U^{\epsilon} = \bigcup_{u \in U} B_{\epsilon}(u)$ is the ϵ neighbourhood of U.

Convergence of random planar maps - scaling limit

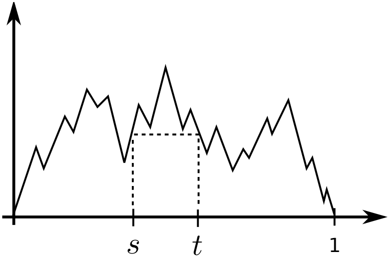
- Denote the graph metric on M_n by d_n
  $\rightarrow d_n(x, y) =$ smallest number of edges in a path connecting x and y.
- View $(M_n, n^{-a} d_n)$ as a random element in $\mathbb{M}$ where $a > 0$.
- Question: For a given weight sequence $(q_i)_i$, does there exist an $a > 0$ such that $(M_n, n^{-a} d_n)$ has an interesting limit in $(\mathbb{M}, d_{\mathrm{GH}})$ as $n \to \infty$?

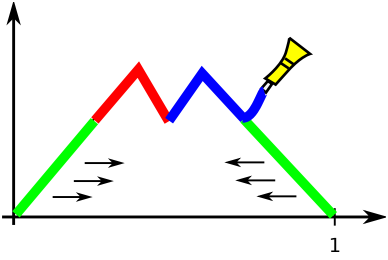
The phase diagram of the random planar maps



Aldous' Brownian tree

Let $\mathbf{e}$ be a standard Brownian excursion on $[0,1]$, i.e. a standard Brownian motion on $[0,1]$ conditioned on being nonnegative on $(0,1)$ and on taking the value 0 at 1.





For $s,t \in [0,1]$, $s<t$, define

$$\delta_{\mathbf{e}}(s,t) = \mathbf{e}(s) + \mathbf{e}(t) - 2 \inf_{s<u<t} \mathbf{e}(u).$$

The Brownian tree is defined as $\mathcal{T}_{\mathbf{e}} = [0,1] \big/ \{\delta_{\mathbf{e}} = 0\}$ and we let δ_e be the induced distance on the quotient.

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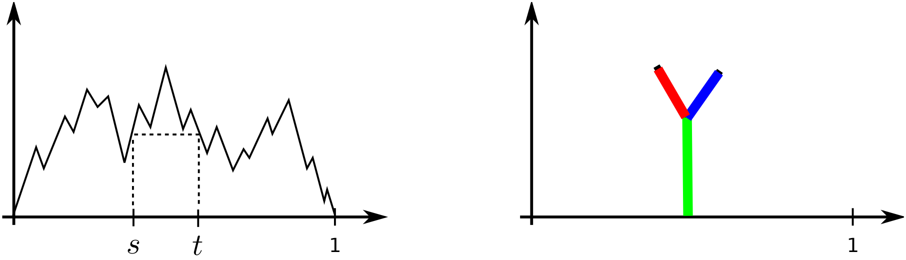

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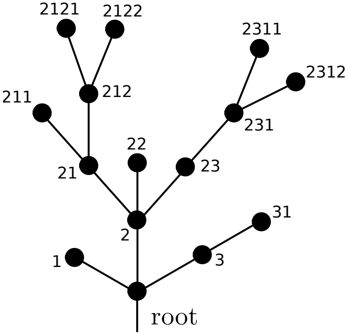
Mobiles and the Bouttier-Di Francesco-Guitter bijection

- Consider a rooted plane tree T_n having n edges. $(T_n \in \mathrm{Trees}_n)$

  Colour the vertices of even generations white and the vertices of odd generations black.

  $V^\circ(T_n)$ set of white vertices.
  $V^\bullet(T_n)$ set of black vertices.

- Assign integer labels $\ell_n(v)$ to the white vertices $v \in V^\circ(T_n)$
  1. The root has label 0.
  2. The labels of white vertices neighbouring any given black vertex can decrease at most by one in clockwise order.
- Call these labelled trees mobiles.



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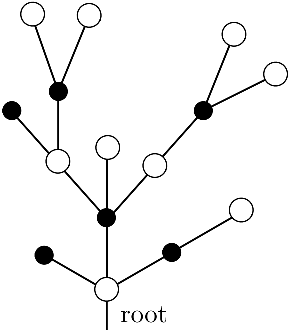

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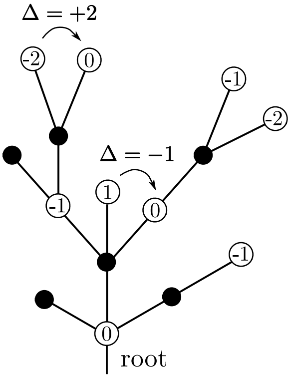

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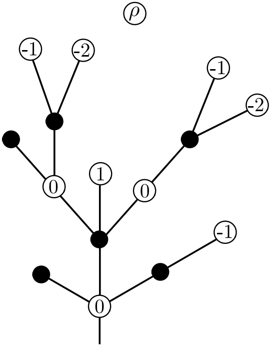

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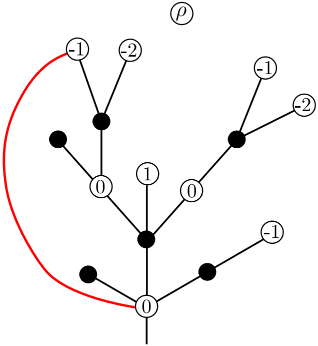

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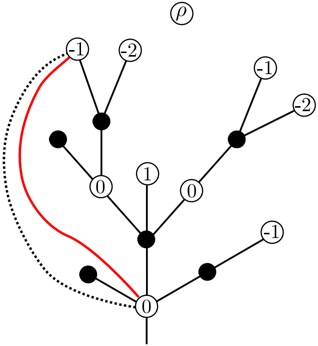

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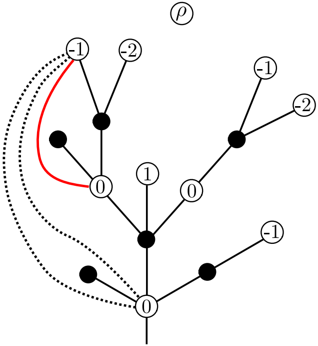

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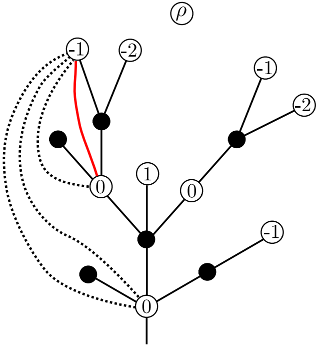

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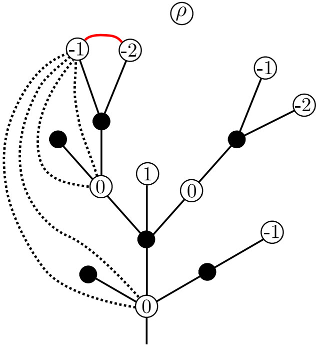

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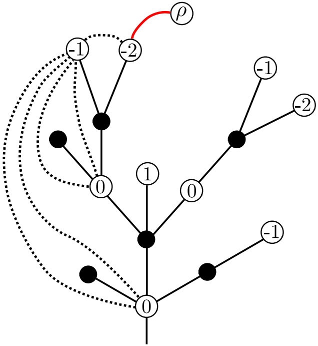

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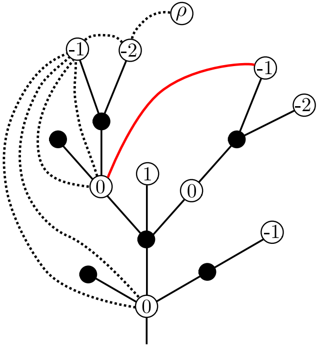

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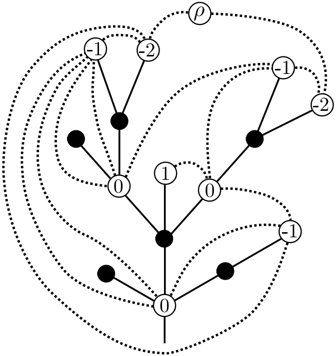

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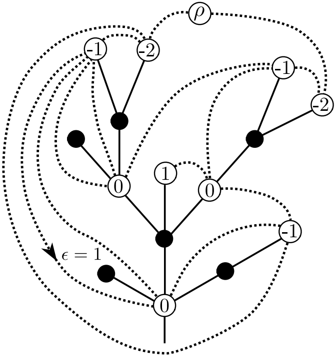

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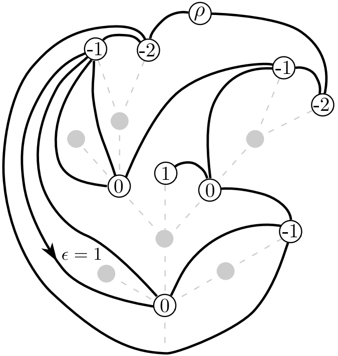

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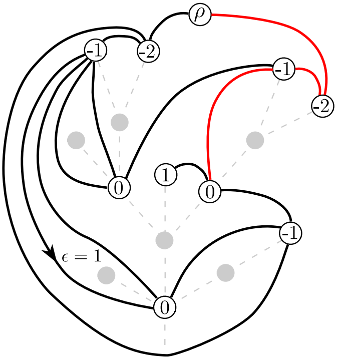

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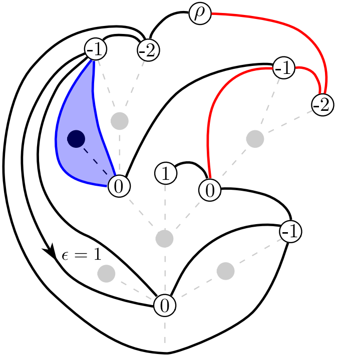

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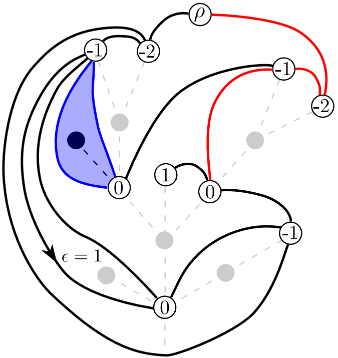

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Random mobiles

Image of μ_n by the BDG bijection:

1. Select a plane tree T_n having n edges with relative probability

$$\tilde{W}(T_n) = \underbrace{\prod_{v \in V^\bullet(T_n)} \binom{2\deg(v)-1}{\deg(v)-1}}_{\sharp \text{ ways to assign valid labels to } T_n} \prod_{v \in V^\bullet(T_n)} q_{\deg(v)} = \prod_{v \in V^\bullet(T_n)} w_{\deg(v)}$$

   where $w_i := q_i \binom{2i-1}{i-1}$.

2. Given T_n, select a labeling $(\ell_n(v))_{v \in V^\circ(T_n)}$ with uniform probability.
3. Select an element ϵ from $\{-1, 1\}$ with uniform probability.

- Note: $\left((\ell_n(v))_{v \in V^\circ(T_n)} \mid T_n\right)$ and ϵ do not depend on the w_i's.
- Therefore, the phase structure of the planar maps distributed by μ_n is determined by the phase structure of trees distributed by

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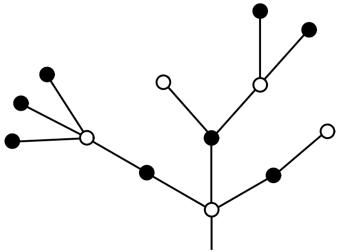
Another bijection

A bijection $\Psi_n : \mathrm{Trees}_n \to \mathrm{Trees}_n$ (Trees_n = set of trees with n edges)



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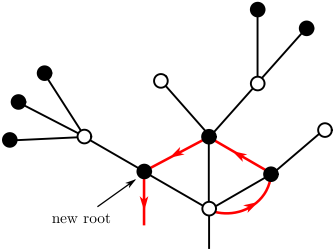

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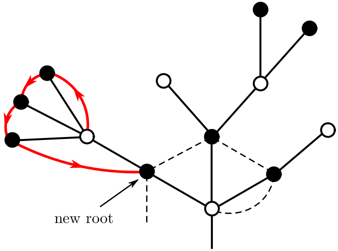

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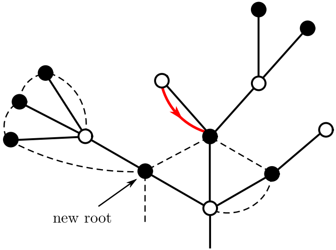

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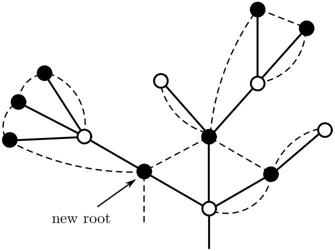

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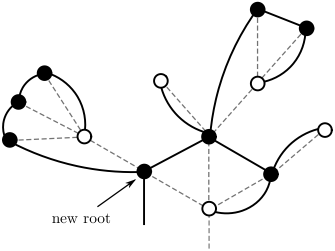

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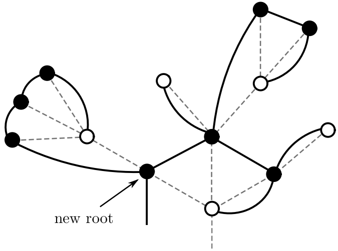

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The measure $\tilde{\nu}_n$ on Trees_n is carried to a new measure ν_n defined by

$$\nu_n(T_n) = 2\mathcal{Z}_n^{-1} \prod_{v \in V(T_n)} w_{\deg(v)-1}$$

where $w_0 := 1$. → simply generated trees (Meir and Moon, 1978).

Simply generated trees

Let $g(x) = \sum_{i=0}^{\infty} w_i x^i$ with radius of convergence ρ.

For any $\tau \le \rho$, define the sequence of probabilities

$$\pi_i = \frac{\tau^i w_i}{g(\tau)}, \quad i \ge 0 \qquad \text{and let} \qquad m = m(\tau) = \sum_i i\pi_i.$$

If $\tau > 0$, π_i and w_i define the same measures ν_n:

$$\prod_{v \in T_n} ab^{\deg(v)-1} = a^{n+1}b^n$$

The 'natural' choice of τ:

1. Let τ be the unique number in $[0, \rho]$ such that $m = \frac{\tau g'(\tau)}{g(\tau)} = 1$ (critical).
2. If no such solution exists let $\tau = \rho$, thus $m < 1$ (sub–critical).

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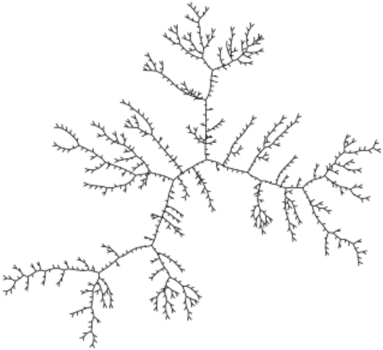
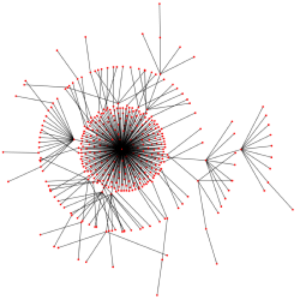
Critical vs. sub–critical



Size conditioned critical GW tree with finite variance.



Size conditioned sub-critical GW tree

Condensation when $m < 1$

Let T_n = random tree distributed by ν_n.

Theorem (Jonsson and S, 2011)

If π_i is sub–critical $(0 < m < 1)$ and $\pi_i \sim Ci^{-\beta}$ with $\beta > 2$ and $C > 0$ then there is a unique vertex in T_n of maximum degree equal to $(1-m)n + o_p(n)$ when $n \to \infty$.

- For large n the tree T_n can roughly be described as consisting of the large vertex of degree approximately $(1-m)n$ from which independent copies of sub–critical π_i–GW trees grow.

Theorem (Janson, Jonsson and S, 2011)

If $\rho = m = 0$ and $w_i = (n!)^\alpha$ then there is a unique vertex in T_n of maximum degree equal to $n + o_p(n)$ when $n \to \infty$.

- The tree T_n is now composed of a root of degree approximately n with very few vertices in lower generations.

Developed further by Janson 2011, Kortchemski, 2012 and Abraham & Delmas, 2013.

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The phases of random planar maps

(1a) π_i is critical and has finite variance.

- $(T_n, n^{-1/2} d_n)$ converges weakly in $\mathbb{M}$ towards Aldous' Brownian tree (Aldous, 1993).
- It is conjectured that $(M_n, n^{-1/4} d_n)$ converges weakly in $\mathbb{M}$ towards the Brownian map. Proved in special (and related) cases (C. Abraham, Addario-Berry, Albenque, Beltran, Bettinelli, Jacob, Le Gall, Miermont).

(1b) $\pi_i \sim c i^{-\alpha-1}$, $\alpha \in (1, 2)$, is critical and has infinite variance.

- $(T_n, n^{-(\alpha-1)/\alpha} d_n)$ converges weakly in $\mathbb{M}$ towards the stable tree with index α (Duquesne and Le Gall, 2002).
- $(M_n, n^{-1/2\alpha} d_n)$ converges weakly in $\mathbb{M}$ (at least along a subsequence) towards something different from the Brownian map (Le Gall and Miermont, 2008).

(2a) $\pi_i \sim C i^{-\beta}$, $\beta > 2$, is sub–critical.

- There is no interesting limit of T_n with rescaled d_n in $\mathbb{M}$ (Kortchemski, 2012).
- What about M_n?

(2b) $\rho = 0$ and $w_i = n!^{\alpha}$?

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Main results

Theorem (Janson and S, 2012)

If

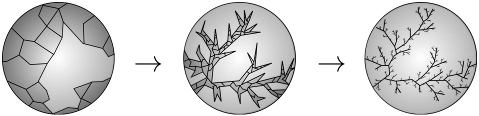
$$\pi_i \sim L(i) i^{-\beta},$$

with $\beta > 2$ and $L(i)$ slowly varying, is sub–critical $(0 < m < 1)$, or

$$w_i \sim (n!)^{\alpha},$$

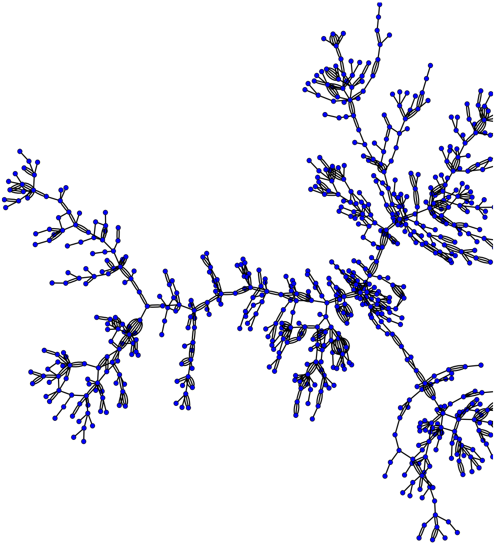
with $\alpha > 0$, (thus $m = \rho = 0$) then

$$(M_n, (2(1-m)n)^{-1/2} d_n) \Rightarrow (\mathcal{T}_{\mathbf{e}}, \delta_{\mathbf{e}}).$$



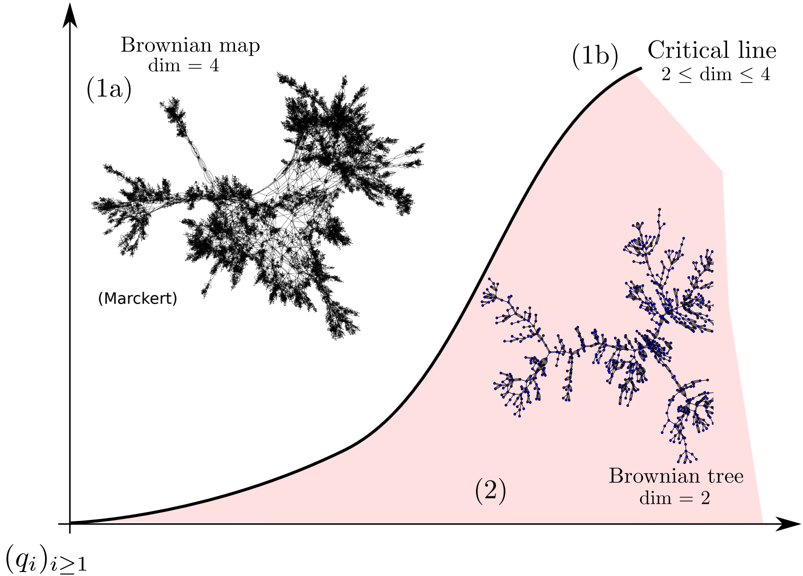
Recall that a black vertex of large degree corresponds to a face of large degree.

A planar map with one large face (and a few small)



$w_i \sim ci^{-3}$ and $m = 0.66$.

The phase diagram of the random planar maps



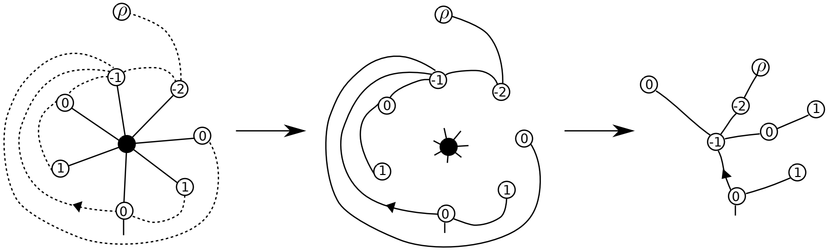
Remark

More examples of non-trees converging towards the Brownian tree:

- Stack triangulations Albenque and Marckert, 2008
- Random planar quadrangulations with a boundary Bettinelli, 2011
- Random dissections Curien, Haas and Kortchemski
- Uniform outerplanar maps Caraceni, 2014
- Random graphs from subcritical classes Panagiotou, Stufler and Weller, 2014
- Random enriched trees, outerplanar maps Stufler, 2015

Idea of proof

Mobiles with a single black vertex of degree $\lfloor(1-m)n\rfloor$, yield planar maps with $\lfloor(1-m)n\rfloor$ edges which are trees:



When the mobile is fixed and the labels chosen uniformly at random from allowed labelings the maps are distributed as the uniform tree.

Call this random map $M_n^\star$ and the graph metric $d_n^\star$. According to Aldous' theorem

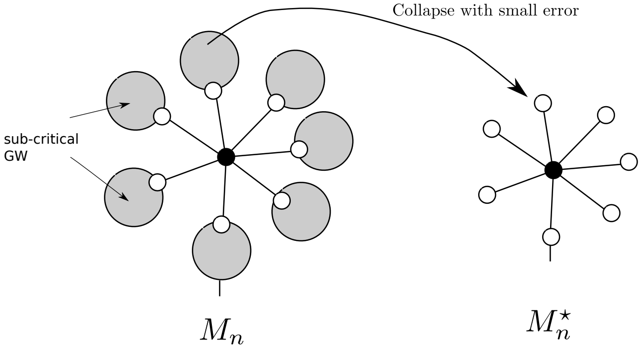
$$(M_n^\star, (2(1-m)n)^{-1/2} d_n^\star) \to (\mathcal{T}_{\mathbf{e}}, \delta_{\mathbf{e}})$$

weakly in $\mathbb{M}$.

Since sub–critical GW processes die out fast the trees growing from the large vertex will be small (they are even smaller when $\rho = 0$) and we can show that

$$d_{\mathrm{GH}}((M_n, d_n), (M_n^\star, d_n^\star)) \to 0$$

in probability which along with the weak convergence of $(M_n^\star, d_n^\star)$ completes the proof.



Thank you!