Random Trees and Maps:
Probabilistic and Combinatorial Aspects

CIRM: 6 – 10 June 2016

	Monday	Tuesday	Wednesday	Thursday	Friday
9:00 am	9:00–10:00	9:00–10:00	9:00–10:00	9:00–10:00	9:00–10:00
9:30 am	J.–F. LE GALL	T. JONSSON	M. BOUSQUET–MELOU	J. BERESTYCKI	S. STEFANSSON
10:00 am	BREAK	BREAK	BREAK	BREAK	BREAK
10:30 am	10:30–11:30	10:30–11:30	10:30–11:30	10:30–11:30	10:30–11:30
11:00 am	T. DUQUESNE	A. GREVEN	M. DRMOTA	J. BOUTTIER	M. ALBENQUE
11:30 am	11:30–12:15	11:30–12:15	11:30–12:15	11:30–12:15	11:30–12:15
12:00 am	R. STEPHENSON	M. WANG	M. WINKEL	L. CHEN	J.–F. DELMAS
12:30 am	12:30: LUNCH	12:30: LUNCH	12:30: LUNCH	12:30: LUNCH	12:30: LUNCH
1:00 pm					
1:30 pm					
2:00 pm					
2:30 pm					
3:00 pm					
3:30 pm					
4:00 pm	4:00–5:00	4:00–5:00		4:00–5:00	
4:30 pm	S. EVANS	J. BERTOIN		O. ANGEL	
5:00 pm	BREAK	BREAK		BREAK	
5:30 pm	5:30–6:15	5:30–6:15		5:30–6:15	
6:00 pm	S. LIN	T. BUDD		C. MARZOUK	
6:30 pm	6:15–7:00	6:15–7:00		6:15–7:00	
7:00 pm	C. LABBE	Y. KOVCHEGOV		E. BAUR	
7:30 pm	7:30: DINNER	7:30: DINNER	7:30: DINNER	7:30: DINNER	
8:00 pm					
8:30 pm					
9:00 pm					

Marie Albenque: *Convergence of odd angulations*

Miermont and Le Gall proved that rescaled p-angulations converge towards the Brownian map, when p is even or $p = 3$. I will focus in my talk on the case where p is odd. The proof of the convergence in this case boils down essentially to generalize Marckert and Miermonts results about the convergence of labeled trees towards the Brownian snake. I'll give the main ideas of the proof of this generalization. This is a joint work with Louigi Addario-Berry.

Omer Angel: *Bootstrap percolation on $G(n, p)$*

On a graph G, we consider the bootstrap model: some vertices are infected and any vertex with 2 infected vertices becomes infected. We identify the location of the threshold for the event that the Erdos-Renyi graph $G(n, p)$ can be fully infected by a seed of only two infected vertices. Joint work with Brett Kolesnik.

Erich Baur: *Yule processes with strong mutations, and their application to percolation on recursive trees*

We consider a Yule process until the total population reaches size $n \gg 1$, and assume that neutral mutations occur with high probability $1 - p_n$, where $p_n \ll 1$. We are concerned with subpopulations formed by the individuals of the same genetic type. We explain how to obtain limit laws for the number of subpopulations exceeding a given size and point at an application to subcritical Bernoulli bond percolation on random recursive trees. Based on joint work with Jean Bertoin.

Julien Berestycki: *Vanishing corrections for the position of an FKPP front*

The celebrated Fisher-Kolmogorov-Petrovsky-Piscounof equation (FKPP) in one dimension for $h : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is:

$$\partial_t h = \partial_x^2 h + h - h^2, \qquad h(x, 0) = h_0(x).$$

This equation is a natural description of a reaction-diffusion model (Fisher 1937, Kolmogorov et al. 1937, Aronson 1978). It is also related to branching Brownian motion: for the Heaviside initial condition $h_0(x) = \mathbf{1}_{x<0}$, $h(x, t)$ is the probability that the rightmost particle at time t in a branching Brownian motion (BBM) is to the right of x.

One of the beauty of this equation is that for initial conditions that decrease sufficiently fast, a front develops, i.e. there exists a centring term $m(t)$ and an asymptotic shape $\omega(x)$ such that

$$\lim_{t\to\infty} h\big(m(t) + x, t\big) = \omega(x) \in (0, 1).$$

Since the original paper of Kolmogorov et al., the position of the front $m(t)$ has been studied intensely, in particular by Bramson. In this talk, I will present some recent results concerning a prediction of Ebert and van Saarloos about the vanishing corrections of this position.

Based on a joint work with E. Brunet.

Jean Bertoin: *Martingales in self-similar growth-fragmentations and their applications*

This talk is based on a work in progress jointly with Timothy Budd (Copenhagen), Nicolas Curien (Orsay) and Igor Kortchemski (Ecole Polytechnique)

Consider a self-similar Markov process X on $[0, \infty)$ which converges at infinity a.s. We interpret $X(t)$ as the size of a typical cell at time t, and each negative jump as a birth event. More precisely, if $\Delta X(s) = -y < 0$, then s is the birth at time of a daughter cell with size y which then evolves independently and according to the same dynamics. In turn, daughter cells give birth to granddaughter cells each time they make a negative jump, and so on.

The genealogical structure of the cell population can be described in terms of a branching random walk, and this gives rise to remarkable martingales. We analyze traces of these martingales in physical time, and point at some applications for self-similar growth-fragmentation processes and for planar random maps.

Mireille Bousquet-Mélou: *Bipolar orientations in planar maps*

Our understanding of planar maps has evolved a lot since the early enumerative results of Tutte, obtained via a recursive approach in the sixties. Thirty years later, the simplicity of his formulae was at last understood at a combinatorial level, and the underlying bijections then paved the way to the study of large random maps, seen as metric spaces.

For maps equipped with an additional structure, many questions remain open. In many cases, these maps have been enumerated, but not by purely combinatorial methods, and their asymptotic behaviour remains mysterious.

In this talk, we will consider planar maps equipped with a bipolar orientation, and show that they have a very rich combinatorial structure, related, among other topic, to lattice walks confined to cones. (Joint work with Nicolas Bonichon, Éric Fusy and Kilian Raschel)

Jérémie Bouttier: *Nesting statistics in the O(n) loop model on random planar maps*

The O(n) model can be formulated in terms of loops living on the lattice, with n the fugacity per loop. In two dimensions, it is known to possess a rich critical behavior, involving critical exponents varying continuously with n. In this talk, we will consider the case where the model is "coupled to 2D quantum gravity", namely it is defined on a random map.

It has been known since the 90's that the partition function of the model can be expressed as a matrix integral, which can be evaluated exactly in the planar limit. A few years ago, together with G. Borot and E. Guitter, we revisited the problem by a combinatorial approach, which allows to relate it to the so-called Boltzmann random maps, which have no loops but faces of arbitrary (and controlled) face degrees. In particular we established that the critical points of the O(n) model are closely related to the "stable maps" introduced by Le Gall and Miermont.

After reviewing these results, I will move on to a more recent work done in collaboration with G. Borot and B. Duplantier, where we study the nesting statistics of loops. More precisely we consider loop configurations with two marked points and study the distribution of the number of loops separating them. The associated generating function can be computed exactly and, by taking asymptotics, we show that the number of separating loops grows logarithmically with the size of the maps at a (non generic) critical point, with an explicit large deviation function. Using a continuous generalization of the KPZ relation, our results are in full agreement with those of Miller, Watson and Wilson concerning nestings in Conformal Loop Ensembles.

Timothy Budd: *Geometry of random planar maps with high degrees*

Many classes of random planar maps are known to possess scaling limits (in the Gromov-Hausdorff sense) described by a universal random continuum metric space known as the Brownian map. One way to escape this universality class is to consider Boltzmann planar maps with carefully tuned weights on the faces. Le Gall and Miermont have shown that the scaling limits of these may fall into a larger class of continuum random metric spaces, often referred to as the stable maps. In this talk I will consider the geometry of the dual (in the planar map sense) of these Boltzmann planar maps, which shows quite different scaling behavior. In particular, I will discuss the growth of the perimeter and volume of certain geodesic balls of increasing radius based on the examination of a peeling process. For a particular range of parameters, in the so-called the dilute phase, precise scaling limits may be obtained for these processes. A more detailed scaling limit can be obtained by tracking the lengths of all cycles at constant height, the limit of which is naturally described by the growth-fragmentation processes that will be discussed in Jean Bertoin's talk. Based on work with Jean Bertoin, Nicolas Curien, and Igor Kortchemski.

Linxiao Chen: *Random planar quadrangulations coupled to the $O(n)$ loop model*

In this work, we focus on the sizes and the nesting structure of the loops in the O(n) loop model on a random quadrangulation. We encode these data by a tree, in which each vertex represents a loop and is labeled by the size of that loop. The main result is the convergence of this labeled tree, after renormalization of the labels, to an explicit multiplicative cascade. Based on a joint work in progress with Nicolas Curien and Pascal Maillard.

Jean-François Delmas: *Genealogical tree for stationnary quadratic branching process*

We consider the quadratic stationary continuous state branching process (CB) as a population model with random size. We first give a construction of a reverse Brownian continuum tree, which is used to explain a result from Bi and Delmas on the time reversed process for the number of ancestors of the population (at any time). Then, we use this construction to study the asymptotic of the total length of the genealogical tree when the number of sampled individuals of the current population goes to infinity. This problem is similar to the one studied by Pfaffelhuber and Wakolbinger in a constant size population model (related to the Kingman's coalescent process). We also provide an elementary way to simulate the genealogical tree for n sampled individuals of the current population. (This is done simultaneously for all n.) This is a Joint work with R. Abraham.

Michael Drmota: *Vertex Degrees in Planar Maps*

We consider the family of rooted planar maps M_Ω where the vertex degrees belong to a (possibly infinite) set of positive integers Ω. Using a classical bijection with mobiles and some refined analytic tools in order to deal with the systems of equations that arise, we recover a universal asymptotic behavior of planar maps. Furthermore we establish that the number of vertices of a given degree satisfies a multi (or even infinitely)-dimensional central limit theorem. We also discuss some possible extension to maps of higher genus. This is joint work with Gwendal Collet and Lukas Klausner.

Thomas Duquesne: *Decomposition of Lévy trees along their diameter*

This is a joint work with Minmin Wang.

We consider the diameter of Lévy trees that are random compact metric spaces obtained as the scaling limits of Galton-Watson trees. Lévy trees have been introduced by Le Gall and Le Jan [3] and they generalise Aldous' Continuum Random Tree [2] that corresponds to the Brownian case. We first characterize the law of the diameter of Lévy trees and we prove that it is realized by a unique pair of points. We prove that the law of Lévy trees conditioned to have a fixed diameter $r \in (0, \infty)$ is obtained by glueing at their respective roots two independent size-biased Lévy trees conditioned to have height $r/2$ and then by uniformly re-rooting the resulting tree; we also describe by a Poisson point measure the law of the subtrees that are grafted on the diameter. This decomposition relies on a similar one for Lévy trees along the geodesic realizing their height that has been obtained by Abraham & Delmas [1]. The law obtained by glueing two trees with height $r/2$ can be viewed as a natural law for *unrooted* Lévy trees: this can be justified thanks to a limit theorem for unrooted unlabelled planar trees conditioned by their total height that has been obtained in the recent preprint Wang [5]. As an application of this decomposition of Lévy trees according to their diameter, we characterize the joint law of the height and the diameter of stable Lévy trees conditioned by their total mass; we also provide asymptotic expansions of the law of the height and of the diameter of such normalized stable trees, which generalizes the identity due to Szekeres [4] in the Brownian case. Note that Szekeres' result has been proved in a simple way and extended by Wang [6].

References

[1] R. Abraham, J-F. Delmas *Williams' decomposition of the Lévy continuum random tree and simultaneous extinction probability for population with neutral mutations*, Stochastic Process. Appl. **119** (2009), 1124–1143.

[2] D. Aldous, *The continuum random tree. I*, Annals of Probability **19** (1991), 1-28.

[3] J-F. Le Gall, Y. Le Jan, *Branching processes in Lévy processes*, Annals of Probability **26** (1998), 213–252.

[4] G. Szekeres, *Distribution of labelled trees by diameter*, In *Combinatorial Mathematics, X (Adelaide, 1982* Lecture Notes in Math. Springer Berlin **1036** (1983), 392–397.

[5] M. Wang, *Scaling limits for a family of unrooted trees*, preprint available on *arXiv*:1604.08287 [math.PR]

[6] M. Wang, *Height and diameter of Brownian tree*, In *Electronic Communications in Probability*, vol. 20, no 88, pp. 1-15, 2015.

Steve Evans: *Iterated shuffle products, random ballot sequences, random trees, and Doob-Martin boundaries*

The shuffle of two words from some alphabet is the random word produced by interleaving the two words uniformly at random (maintaining the relative order of the letters in the two words). One can build a Markov chain by at each stage shuffling together the current word and some fixed word. The simplest nontrivial example is when the alphabet consists of the letters a and b and the word that is shuffled in at each stage is ab. In that case the chain produces random ballot sequences and hence, by a well-known bijection, random trees. Doob-Martin boundary theory provides insight into the large-scale behavior of such a chain by delineating the ways in which it can "go to infinity" via characterizing all the chains that have the same backward transition dynamics.

Andreas Greven: *Evolving random genealogies: infinite divisibility and branching property*

We model the genealogies of populations by marked ultrametric measure spaces and introduce an algebraic structure to device in this framework concepts of infinite divisibility of genealogical trees and the description of dynamics of genealogies satisfying the branching property. This abstract structure includes a truncation operator based on this operation we have a collection of delphic semigroup of concatinations. This leads to genealogical Lévy Khintchine formulas and simple to verify generator properties guaranteeing the branching property. This can be applied to the tree-valued (better ultrametric measure space valued) Feller diffusions using a Feynman-Kac duality. We shortly discuss some issues concerning the relation between Markov branching trees and infinite divisibility in this framework. The work is motivated by a study of the genealogies in logistic branching models, attempting to extract the change in genealogies when leaving the classical branching world. (Ongoing work with P.Glde and T.Rippl)

Thordur Jonsson: *Infinite volume limit of the splitting vertex trees*

I discuss a model of randomly growing rooted planar trees called the vertex splitting model. When the degrees of vertices are bounded we prove under some technical conditions that the probability measure on trees of size N converges when N goes to infinity to a measure on infinite trees and the limiting measure is independent of the initial tree. We show that with respect to this measure almost surely there is a unique spine and methods are developed to calculate the size distribution of the outgrowths from the spine in special cases.

Yevgeniy Kovchegov: *Tree self-similarity based on Horton ordering and Tokunaga indexing*

The topic for my talk concerns the recent progress made on the tree self-similarity based on Horton ordering and Tokunaga indexing that was recently proven for a variety of stochastic processes that includes the Kingman coalescent trees, level set trees for space-homogeneous stochastic processes, Brownian motion and iid time series, and conjectured for the trees generated by the multiplicative coalescent processes and fractional Brownian motion. In addition, I will describe the properties of a new class of processes, the Hierarchical Branching Processes. Joint work with Ilya Zaliapin (University of Nevada Reno).

Cyril Labbé: *Weakly asymmetric bridges and the KPZ equation*

Consider a system of N particles on a linear lattice of $2N$ sites, evolving according to the simple exclusion process (with zero-flux boundary condition). I will present a classification of the static and dynamic behaviour of this model according to the asymmetry imposed on the jump rates. In particular, there is a precise regime of asymmetry for which the fluctuations around the hydrodynamic limit converge to the solution of the KPZ equation on the line, but truncated to a finite interval of time.

Jean-François Le Gall: *Excursion theory for the Brownian snake, and applications to the Brownian map*

The Brownian snake can be viewed as Brownian motion indexed by a Brownian tree T. Writing Z(a) for the value of this Brownian motion at the vertex a of the tree T, we are interested in excursions of Z above its minimum process, corresponding to connected components of the set of vertices a such that Z(a) is (strictly) greater than the minimum of Z along the geodesic segment between a and the root. The law of one of these connected components, and of the values of Z on it, can be described by an excursion measure for which many explicit distributions can be computed. With each connected component (or excursion) one can associate a random variable measuring the size of its boundary. One then proves that the excursions above the minimum are independent conditionally given their boundary sizes. On the other hand, the set of all vertices a such that Z(a) is equal to the minimum of Z along the geodesic segment between a and the root can be represented by a stable Lévy tree with index 3/2, in such a way that each branching point of the Lévy tree corresponds to an excursion whose boundary size is the weight of the branching point. In the Brownian map setting, this Lévy tree allows one to reconstruct the metric net introduced by Miller and Sheffield, and the Brownian map itself is constructed by glueing independent "Brownian disks" (each corresponding to an excursion) in the holes of the metric net. This is based in part on a joint work with Céline Abraham.

Shen Lin: *Harmonic measure of balls in critical Galton-Watson trees*

Let T_n be a critical Galton-Watson tree conditioned to have height greater than n. We are interested in the harmonic measure of the ball of radius n centered at the root of T_n, which can be interpreted as the exit distribution of the ball by simple random walk on the tree. Under the assumption that the critical offspring distribution belongs to the domain of attraction of a stable distribution with index $\alpha \in (1, 2]$, we show that, although the boundary of the ball has size of order $n^{\frac{1}{\alpha-1}}$, most of the harmonic measure is supported on a boundary set of size approximately equal to n^{β_α} with $\beta_\alpha < \frac{1}{\alpha-1}$, and the mass of the harmonic measure carried by a random vertex uniformly chosen from the boundary is roughly equal to $n^{-\lambda_\alpha}$ with $\lambda_\alpha > \frac{1}{\alpha}$. Both constants, β_α and λ_α, depend only on the parameter α. They can be expressed in terms of the asymptotic distribution of the conductance of the random tree T_n.

Cyril Marzouk: *The geometry of large random non-crossing trees*

(joint work with Igor Kortchemski)

A non-crossing tree of size n is a tree embedded in the unit disk of the complex plane, with n vertices placed at the n-th roots of unity. Viewing each edge as a line segment, we study the asymptotic behaviour as the size tends to infinity of such random trees in the space of compact subsets of the unit disk, equipped with the Hausdorff distance. Curien & Kortchemski, 2014, have shown that, when chosen uniformly at random, these trees converge towards Aldous Brownian triangulation, a random compact subset of the unit disk made of non-intersecting chords and which is coded by the Brownian excursion. We consider more general Boltzmann-type distributions and obtain a one-parameter family of distributional limits, which are coded by excursions of stable Lévy processes.

Sigurdur Stefansson: *Scaling limits of random planar maps towards the Brownian tree*

Random planar maps are defined by assigning non-negative weights to each face of a planar map and the weight of a face depends only on its degree. I will recall the Bouttier-Di Francesco-Guitter bijection between the planar maps and a class of labelled trees called mobiles. By throwing away labels one can, via another bijection, relate the mobiles to the model of so-called simply generated trees which are understood in detail. For certain choices of weights a unique large face, having degree proportional to the total number of edges in the maps, appears with high probability when the maps are large. This corresponds to what has been referred to as condensation in simply generated trees where a vertex having degree proportional to the size of the trees appears. In this case the planar maps, with a properly rescaled graph metric, are shown to converge in distribution towards Aldous Brownian tree in the Gromov-Hausdorff topology.

Robin Stephenson: *Convergence of bivariate Markov chains to multi-type self-similar processes, and applications to scaling limits of some random trees*

We describe the scaling limits of bivariate Markov chains (X, J) on $\mathbb{Z}_+ \times \{1, \ldots, \kappa\}$ where X can be viewed as aposition marginal and $\{1, \ldots, \kappa\}$ is a set of κ types. The chain starts from an initial value $(n, i) \in \mathbb{Z}_+ \times \{1, \ldots, \kappa\}$, with i fixed and $n \to \infty$, and typically we will assume that X is nonincreasing, and that its macroscopic jumps are rare, i.e. arrive with a probability proportional to a negative power of the current state. We then observe different asymptotic regimes according to whether the rate of type change is proportional to, faster than, or slower than the macroscopic jump rate. In these different situations, we obtain in the scaling limit Lamperti transforms of Markov additive processes, that sometimes reduce to standard positive self–similar Markov processes.

We will then give a preview of our upcoming main application of this result, which is obtaining the scaling limit of general type of random trees called (multi-type) Markov branching trees, notably including conditioned Galton-Watson trees and growing trees constructed by a random algorithm.

Minmin Wang: *Lévy trees and continuum limits of inhomogeneous continuum random graphs*

We consider the model of Poisson random graphs in its critical regime and show that, under optimal conditions on the vertex weights, the sequence of connected components of the graphs converge in distribution in Gromov–Hausdorff–Prokhorov sense towards a sequence of compact measure metric spaces. This improves notably recent results of Bhamidi, van der Hofstad and Sen. Our method relies on a new representation of the Poisson random graph in terms of multitype Galton-Watson forests. In consequence, our limit metric space is viewed as a subset of certain Lévy forests with some pairs of points identified. Relying on well-known results on Lévy trees, we also study some probabilistic and fractal properties of the continuum graphs.

Matthias Winkel: *Interval-partition-valued diffusions*

We construct two diffusions (path-continuous strong Markov processes) on a space of interval partitions and study total length evolutions, evolutions of individual intervals, stationarity properties and immigration. These diffusions are related to zero sets of Brownian motion and Brownian bridge (and hence spinal interval partitions of the Brownian CRT), squared Bessel processes and their excursions, and Petrov's Poisson-Dirichlet diffusions on the space of decreasing sequences. Specifically, we construct a sigma-finite excursion measure for squared Bessel processes of dimension -1, under which the excursion lengths have the same distribution as the jumps of a spectrally positive stable(3/2) process, which is at the centre of our construction. This is joint work in progress with Noah Forman, Soumik Pal and Douglas Rizzolo.