Geometry of large random non-crossing trees

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Pictures here and maths

there and there

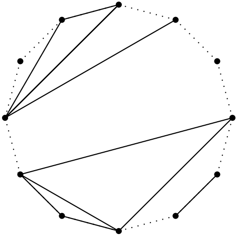


Figure: A non-crossing graph of size 12.

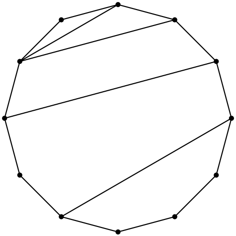


Figure: A dissection

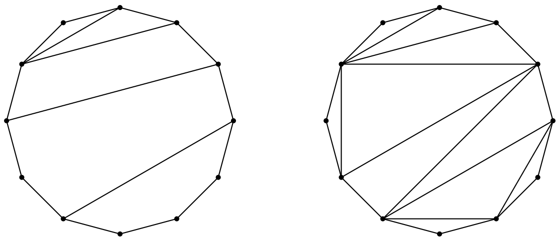


Figure: A dissection, a triangulation

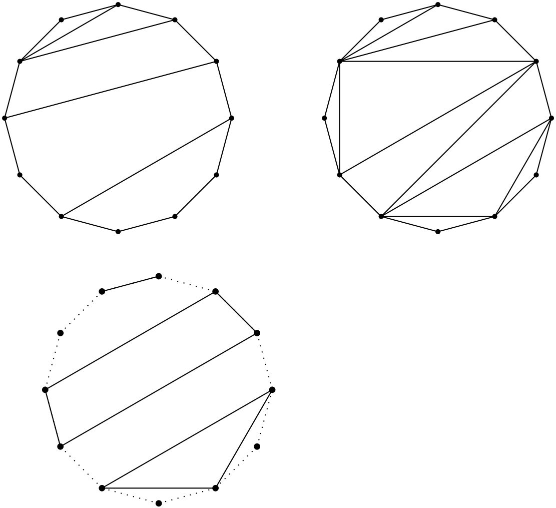


Figure: A dissection, a triangulation, a NC partition

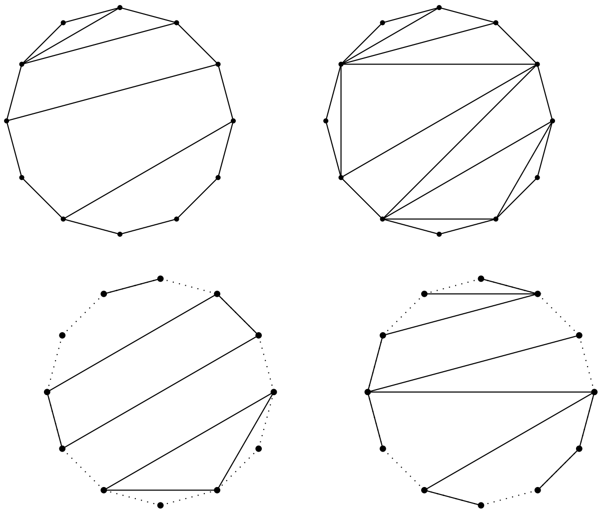


Figure: A dissection, a triangulation, a NC partition and a NC tree of size 12

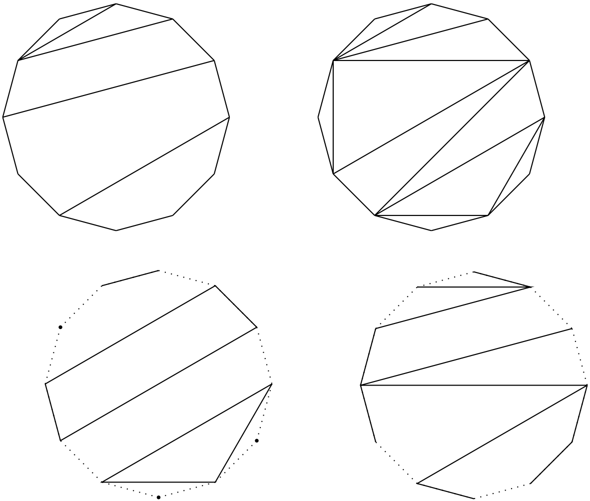


Figure: A dissection, a triangulation, a NC partition and a NC tree of size 12 seen as laminations.

The Brownian triangulation (Aldous ’94, Le Gall & Paulin ’08)

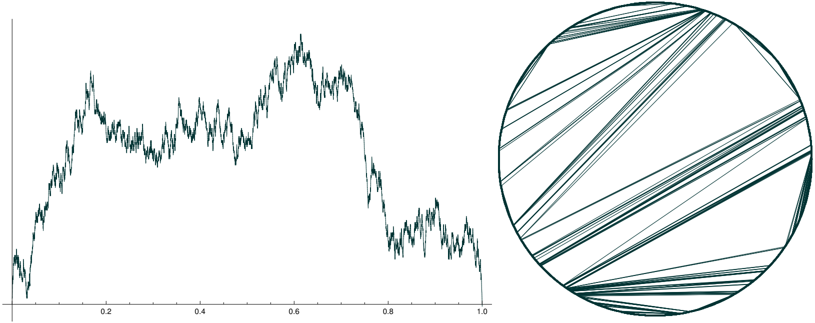


Figure: The Brownian tree & its associated triangulation.

The Brownian triangulation (Aldous '94, Le Gall & Paulin '08)

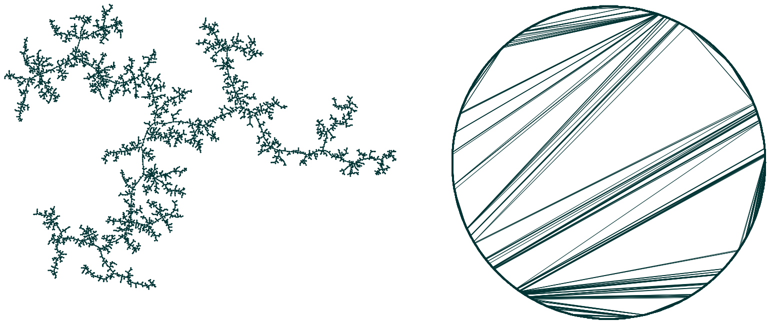


Figure: The Brownian tree & the Brownian triangulation.

Theorem (Kortchemski & ☺ '16). Fix $\alpha \in (1,2]$. There exists a random triangulation of $\overline{\mathbf{D}}$, denoted by Θ_α, with Hausdorff dimension $1+\frac{1}{\alpha}$, such that the following holds. For every sequence w for which there exists $a, b > 0$ such that the sequence

$$\mu(k) = a \cdot (k+1) \cdot w(k+1) \cdot b^k \qquad (k \geq 0)$$

is a probability measure, with expectation 1 and in the domain of attraction of a stable law with index α, if θ_n is sampled according to $\mathbf{P}_n^w$, then the convergence in distribution

$$\theta_n \quad \xrightarrow[n\to\infty]{(d)} \quad \Theta_\alpha$$

holds for the Hausdorff distance.

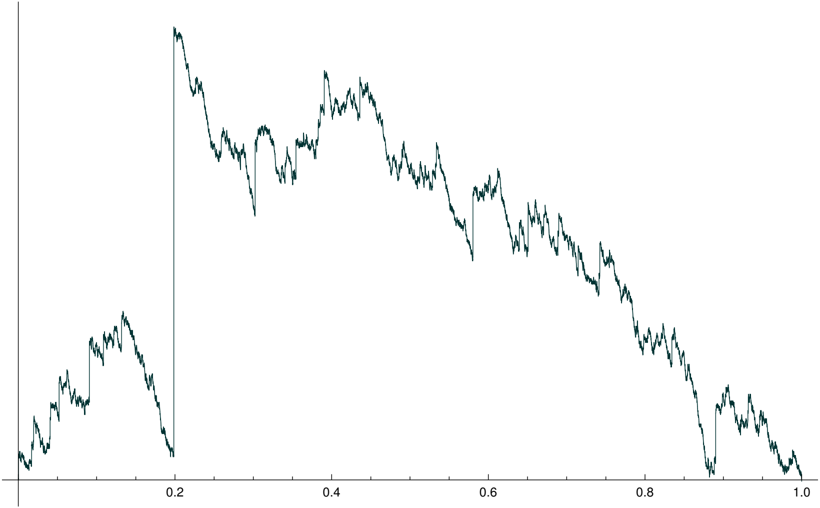


Figure: Simulation of X_α for $\alpha = 1.2$.

The stable laminations (Kortchemski '14)

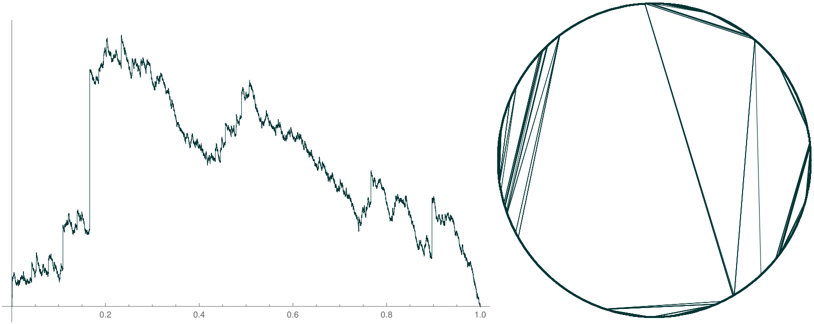


Figure: Simulations of X_α and L_α for $\alpha = 1{,}3$.

$$L_\alpha = \bigcup_{s,t} \left[e^{-2i\pi s}, e^{-2i\pi t} \right], \qquad t = \inf\{u > s : X_\alpha(u) \leq X_\alpha(s-)\}.$$

Triangulating stable laminations (Kortchemski & ☺ ’16)

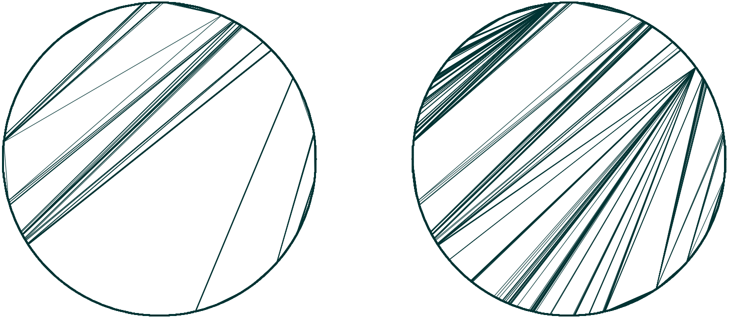


Figure: The lamination L_α and the associated triangulation Θ_α for $\alpha = 1{,}2$.

NC trees & plane trees

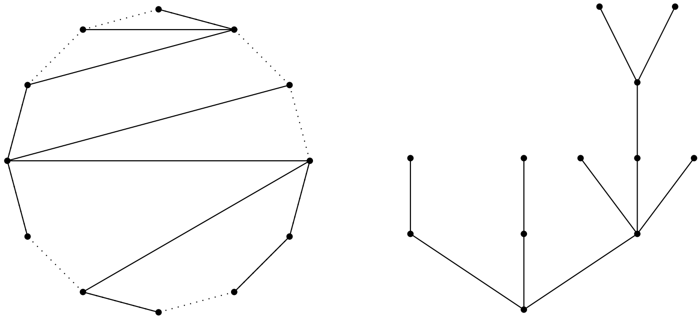
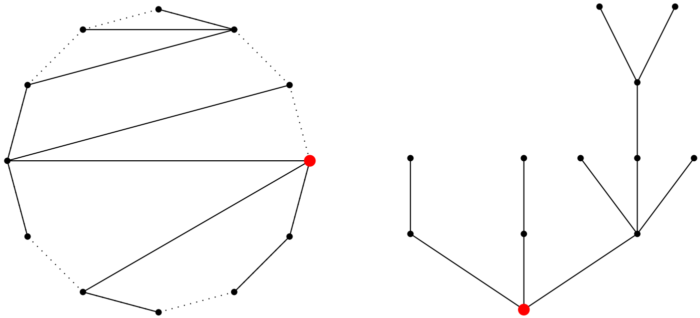


Figure: A NC tree θ and its underlying plane tree $T = S(\theta)$.

NC trees & plane trees



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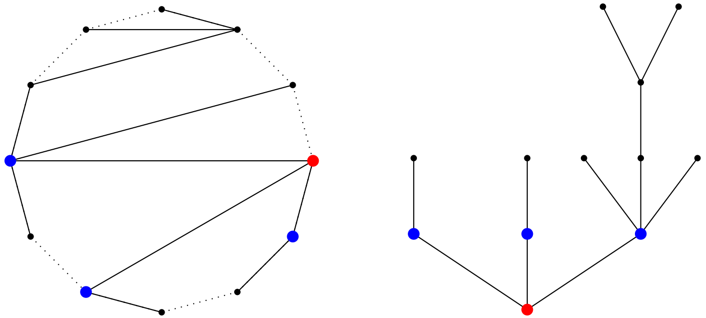

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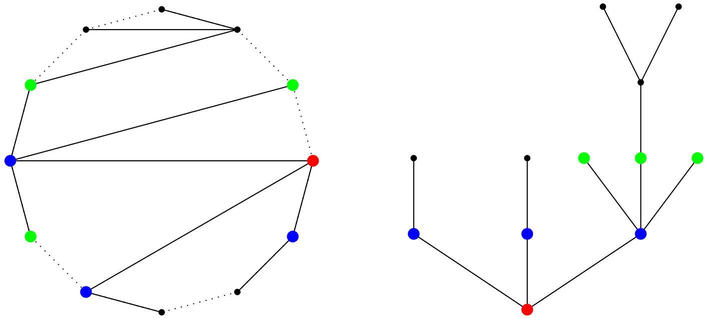

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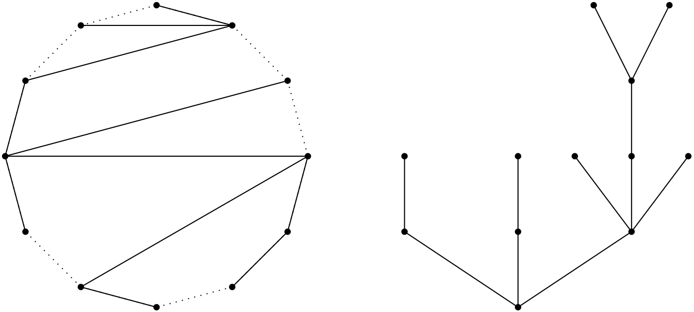

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NC trees & plane trees

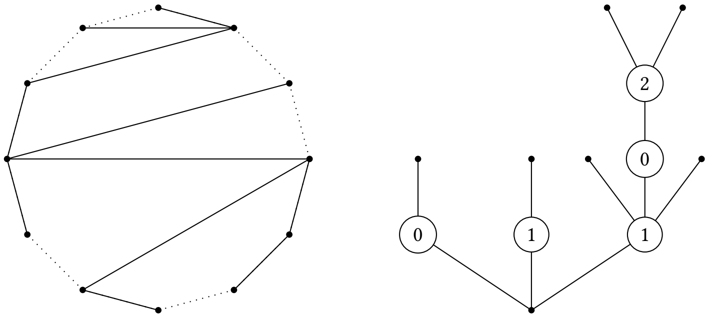

Figure: Bijection between NC trees and labelled plane trees.

Theorem. Assume that

$$\rho \quad := \quad \left(\limsup_{k\to\infty} w(k)^{1/k}\right)^{-1} > 0.$$

Fix $b \in (0, \rho)$ and define two probability measures

$$\begin{cases} \mu(k) = a \cdot (k+1) \cdot w(k+1) \cdot b^k & (k \geq 0), \\ \mu_\varnothing(k) = c \cdot w(k) \cdot b^k & (k \geq 1). \end{cases}$$

If θ_n is sampled according to $\mathbf{P}_n^w$, then $T_n = S(\theta_n)$ is distributed according to $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$. Moreover, conditional on $T_n = S(\theta_n)$, θ_n is uniformly distributed in the set $\{\theta : S(\theta) = T_n\}$.

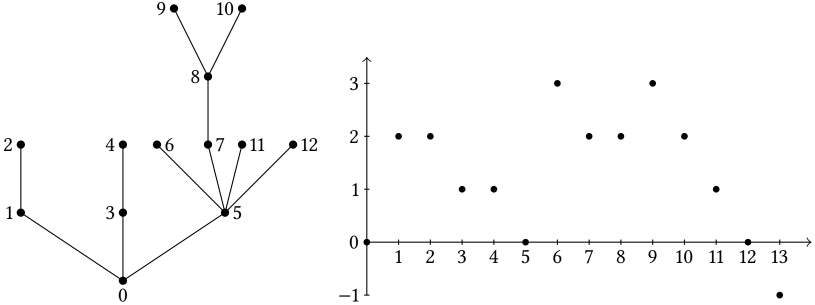

Figure: A plane tree T and its Łukasievicz path $W(T)$.

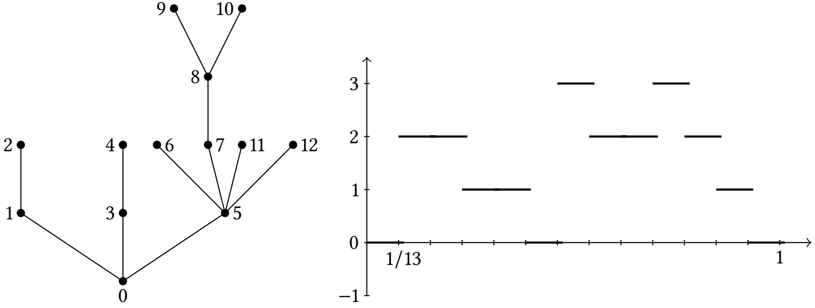

Figure: A plane tree T and its Łukasiewicz path $W(T)$.

Theorem. Let $\mu_\varnothing$ be a probability measure on $\mathbf{N}$ with finite mean and μ a probability measure on $\mathbf{Z}_+$ with mean 1 and in the domain of attraction of a stable law with index $\alpha \in (1,2]$. For every $n \geq 1$, sample T_n according to $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$. We have

$$\frac{1}{B_n} \cdot W(T_n) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad X_\alpha$$

for some sequence $B_n \to \infty$ ($B_n \approx n^{1/\alpha}$). In fact,

$$\left(\frac{1}{B_n} W(T_n), \frac{B_n}{n} H(T_n), \frac{B_n}{n} C(T_n)\right) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad (X_\alpha, H_\alpha, H_\alpha).$$

Proposition. Fix $\alpha \in (1, 2]$ and assume that $(T_n; n \geq 1)$ are random plane trees satisfying:

1. there exists $B_n \to \infty$ such that $W(T_n)/B_n \to X_\alpha$ in distribution,
2. $\max_{u \in T_n} |u|/n \to 0$ in probability.

Given T_n, sample θ_n uniformly at random in $\{\theta : S(\theta) = T_n\}$, then $\theta_n \to \Theta_\alpha$ in distribution.
In the case $\alpha = 2$, **all sequences** $(\theta_n; n \geq 1)$ with $S(\theta_n) = T_n$ converge towards $\Theta_2 = \mathcal{B}$.

Iteration of stable laminations (Kortchemski & ☺ '16)

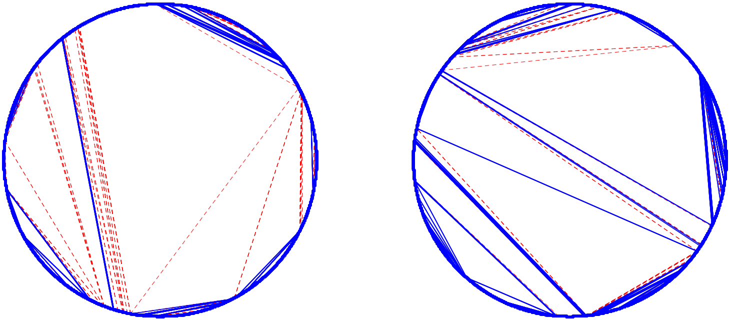

Figure: From left to right: $\beta = 1{,}4$ laminations iterated inside an $\alpha = 1{,}1$ lamination, and $\beta = 1{,}1$ laminations iterated inside an $\alpha = 1{,}4$ lamination. The chords of the β-stable laminations are in dashed red.