# Harmonic measure of balls in critical Galton–Watson trees

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Random Trees and Maps:
Probabilistic and Combinatorial Aspects
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# 1. Introduction

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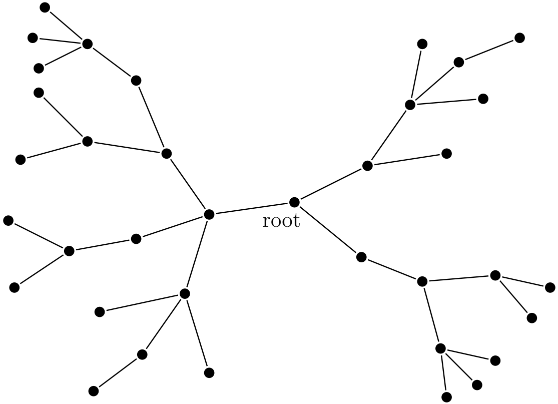

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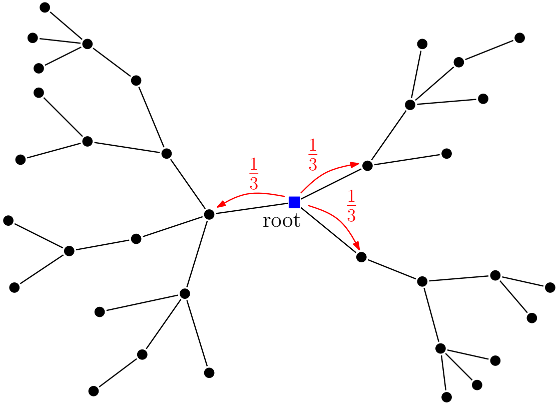

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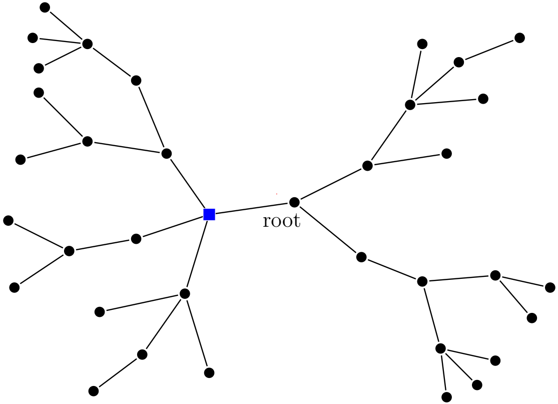

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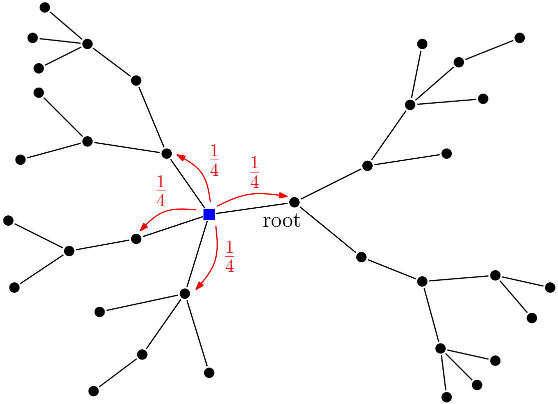

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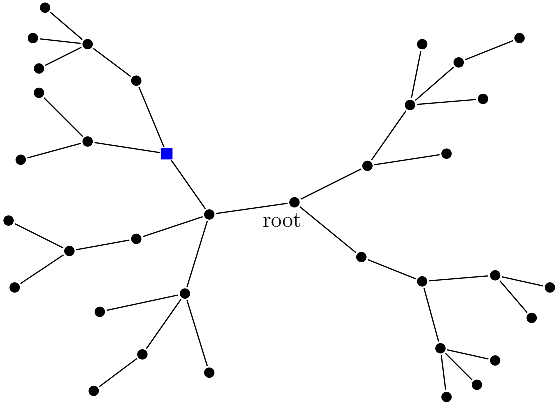

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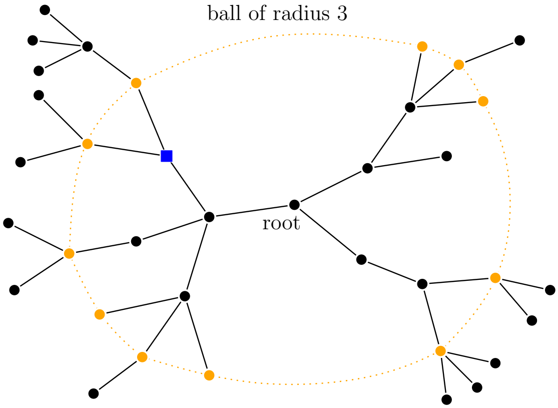

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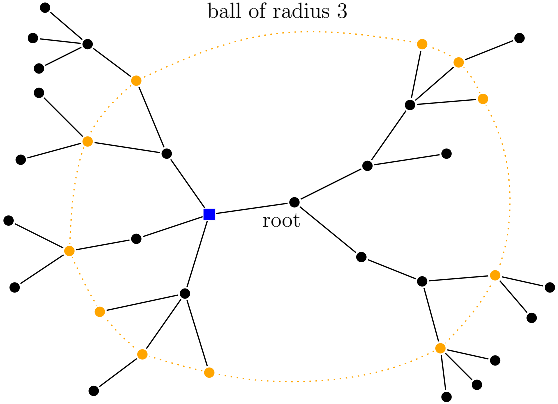

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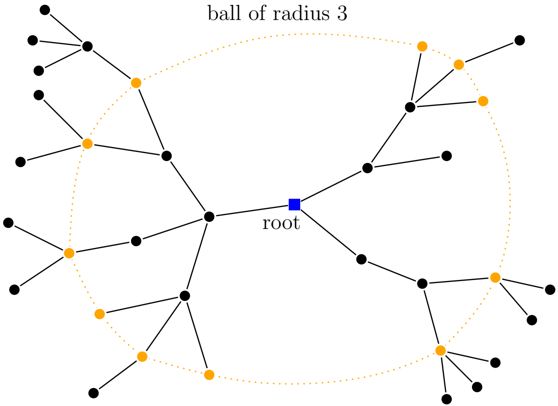

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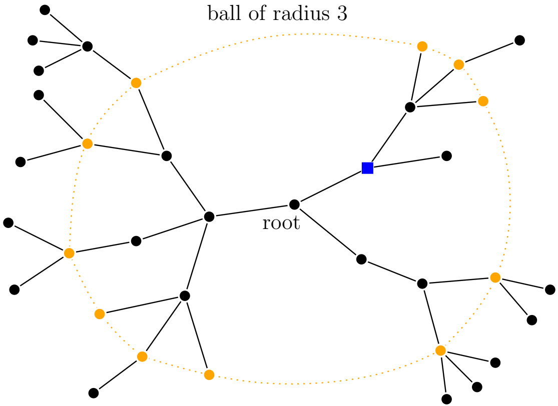

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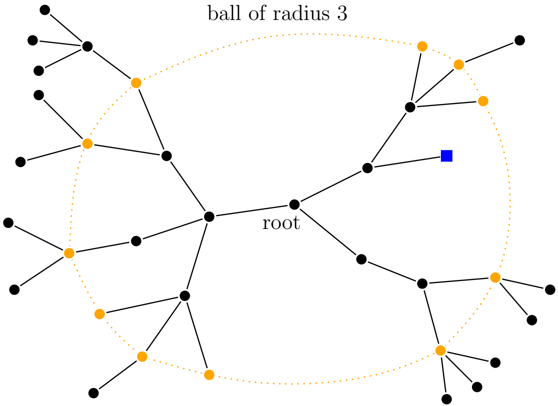

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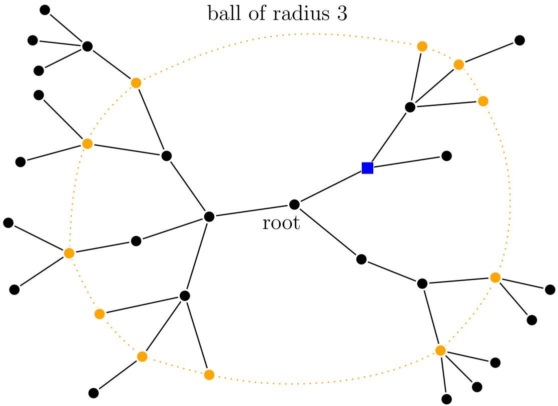

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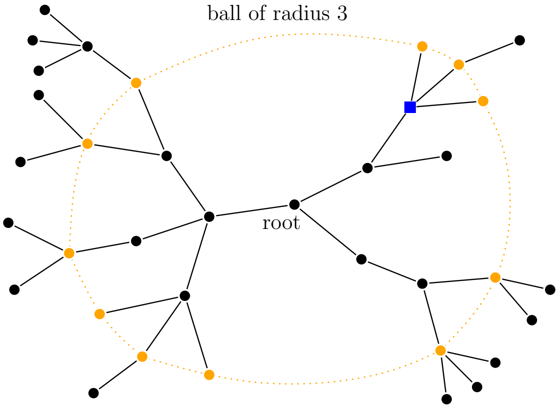

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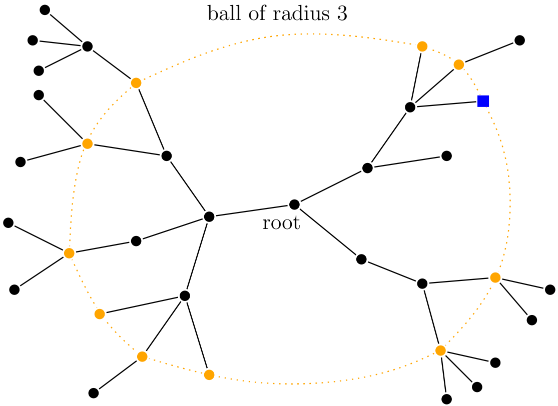

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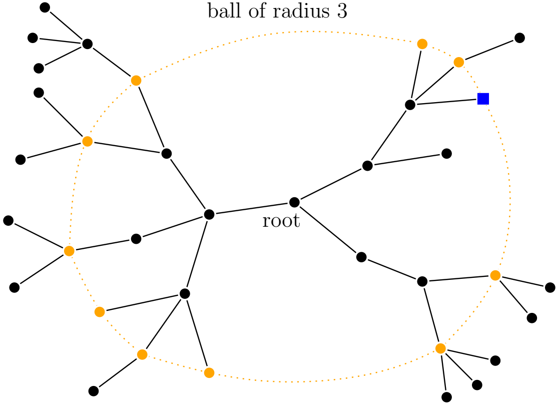


The harmonic measure of the ball is the distribution of the exit point from the ball for SRW started at the root.

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Motivated by a question of T. Jonsson, N. Curien and J.-F. Le Gall (2013) have shown this "dimension drop" phenomenon in the context of discrete combinatorial trees.

# Galton–Watson trees

Let $\theta$ be a probability measure on $\{0, 1, 2, \ldots\}$ such that $\theta(1) < 1$ and

- $\sum_{k=0}^{\infty} k\,\theta(k) = 1$ (criticality),
- $\theta$ is in the domain of attraction of a stable distribution of index $\alpha \in (1, 2]$, i.e.

$$\sum_{k\geq 0} \theta(k)\, r^k = r + (1-r)^{\alpha} L(1-r) \qquad \forall r \in [0,1),$$

where the function $L(x)$ is slowly varying as $x \to 0+$.
(If $\theta$ has finite variance, then this condition holds with $\alpha = 2$.)

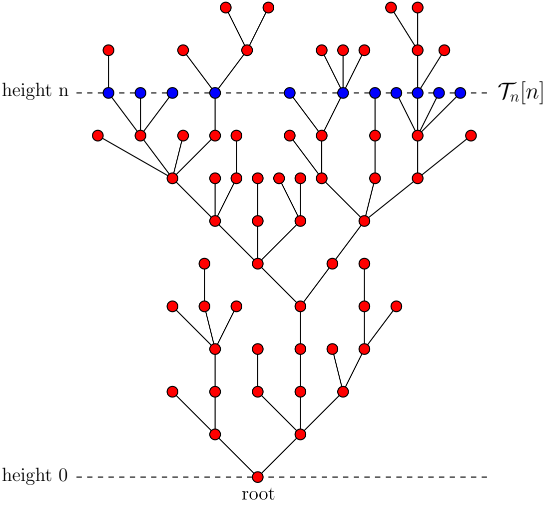
The Galton–Watson tree with offspring distribution $\theta$ (in short $\theta$-GW tree) is the genealogical tree of a population starting with one ancestor, where each individual has $k$ children with probability $\theta(k)$. This tree is finite a.s.

Let $\mathcal{T}_n$ be a $\theta$-GW tree conditioned to have height at least $n$, and

$$\mathcal{T}_n[n] := \{\text{vertices of } \mathcal{T}_n \text{ at height } n\}.$$

# Conditional Galton–Watson trees $\mathcal{T}_n$

$\mathcal{T}_n = \theta$-Galton–Watson tree conditioned to have height at least $n$.




We know that $\#\mathcal{T}_n[n] \approx n^{\frac{1}{\alpha-1}}$.

(When $\theta$ has finite variance, $\frac{1}{n}\#\mathcal{T}_n[n] \xrightarrow[n\to\infty]{(d)} \mathrm{Exp}(2/\mathrm{var}\,\theta)$).

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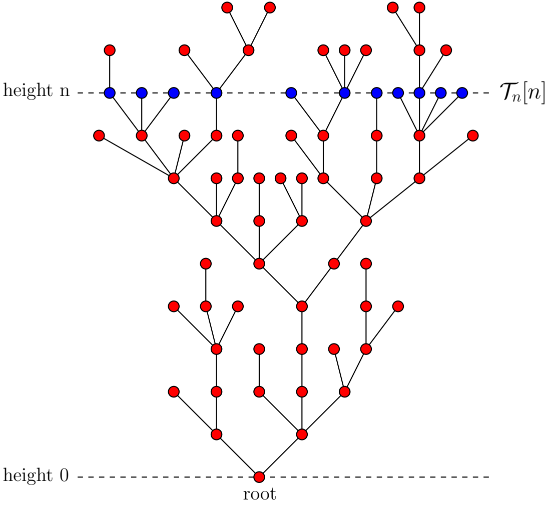


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$\mu_n =$ distribution of the first hitting point of $\mathcal{T}_n[n]$ by SRW on $\mathcal{T}_n$ started from the root.

# Harmonic measure $\mu_n$

Were the harmonic measure uniformly spread over the boundary of the ball of radius $n$, one would expect $\mu_n(v) \approx n^{-\frac{1}{\alpha-1}}$ for $v \in \mathcal{T}_n[n]$.

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**Theorem (A)**

*There exists a constant $\beta_\alpha \in (0, \frac{1}{\alpha-1})$, which only depends on $\alpha$, such that, for every $\varepsilon > 0$,*

$$\mu_n(\{v \in \mathcal{T}_n[n] \colon n^{-\beta_\alpha - \varepsilon} \leq \mu_n(v) \leq n^{-\beta_\alpha + \varepsilon}\}) \xrightarrow[n\to\infty]{\text{(P)}} 1.$$

*Moreover, $\beta_\alpha$ remains bounded when $\alpha \downarrow 1$.*

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Consequences:

- For any $\delta > 0$, there exists with probability $\to 1$ a subset $A \subset \mathcal{T}_n[n]$ s.t.

$$\#A \le n^{\beta_\alpha + \varepsilon} \quad \text{and} \quad \mu_n(A) \ge 1 - \delta.$$

- Conversely,

$$\sup_{A \colon \#A \le n^{\beta_\alpha - \varepsilon}} \mu_n(A) \xrightarrow[n\to\infty]{\text{(P)}} 0.$$

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Remarks:

- The preceding theorem was first shown by N. Curien and J.-F. Le Gall (2013) in the case where $\theta$ has finite variance.
- R. Lyons, R. Pemantle and Y. Peres (1995,1996): harmonic measure at infinity for supercritical GW trees.

# Typical behavior of the harmonic measure $\mu_n$

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**Theorem (B)**

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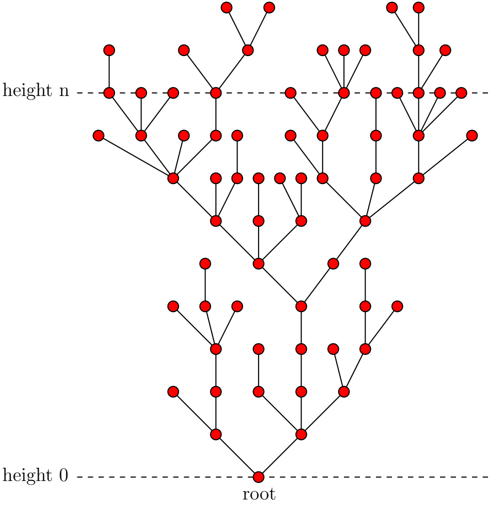
*Moreover,* $\lambda_\alpha$ *is decreasing for all* $\alpha \in (1,2]$*, and we have*

$$0 < \liminf_{\alpha\downarrow 1}(\alpha-1)\lambda_\alpha \leq \limsup_{\alpha\downarrow 1}(\alpha-1)\lambda_\alpha < \infty.$$

# 2. The continuous setting

**Key idea**: consider reduced trees.

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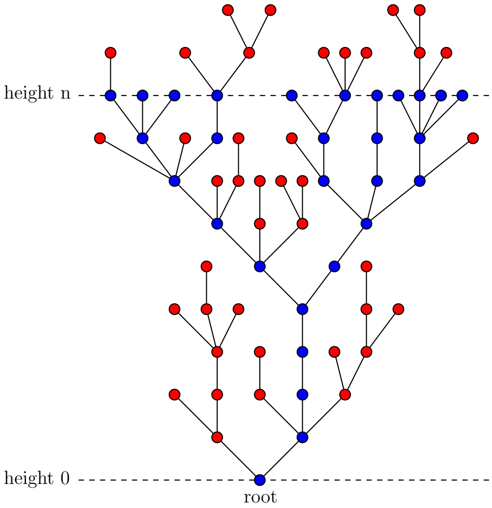


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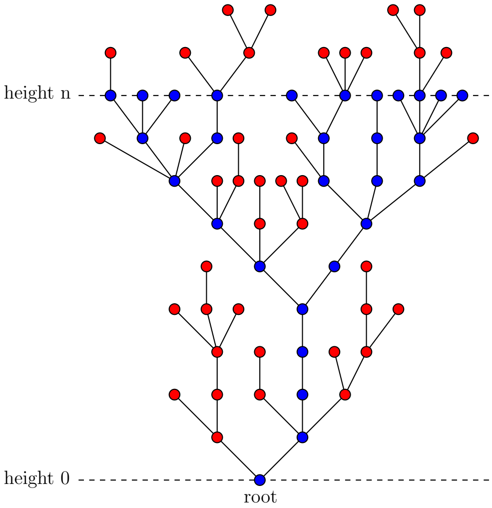


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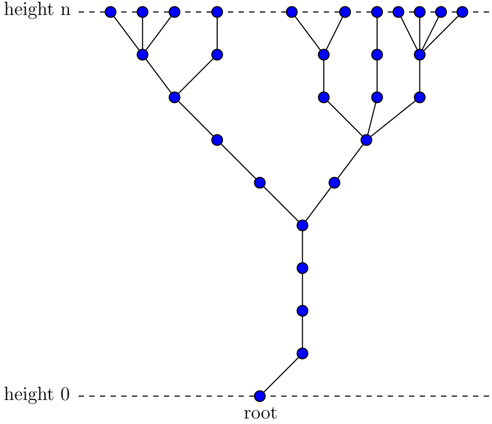


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Notation:
$d_{\mathrm{gr}}$ graph distance on $\mathcal{T}_n^*$

# Convergence of reduced trees

Asymptotics for reduced critical GW trees: Zubkov (1975), Fleischmann & Siegmund-Schultze (1977), Vatutin (1977), Yakymiv (1980)

$(\mathcal{T}_n^*, \frac{1}{n} d_{\mathrm{gr}}) \xrightarrow[n\to\infty]{\mathrm{(d)}} (\Delta^{(\alpha)}, \mathbf{d})$ in the Gromov–Hausdorff sense.

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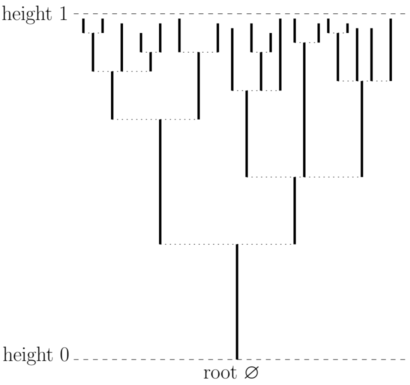


The offspring number is distributed according to $\rho_\alpha$, whose generating function is

$$\sum_{k\geq 0} \rho_\alpha(k) r^k = \frac{(1-r)^\alpha - 1 + \alpha r}{\alpha - 1}.$$

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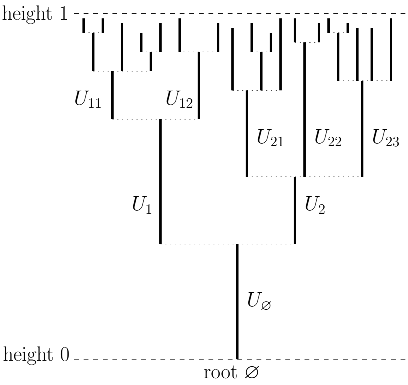


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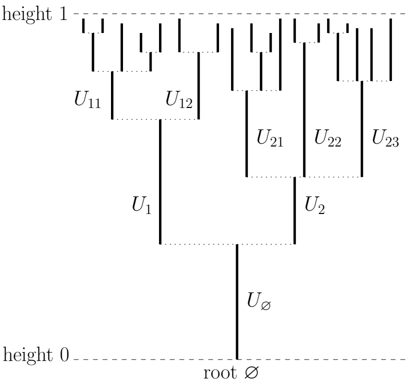


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$\mathbf{d}$ is the natural tree metric.

$\partial\Delta^{(\alpha)} := \{x \in \Delta^{(\alpha)} : \mathbf{d}(\varnothing, x) = 1\}$.

# Harmonic measure on $\partial\Delta^{(\alpha)}$

Let $(B_t)_{t\geq 0}$ be Brownian motion on $\Delta^{(\alpha)}$, started from the root, defined up to the hitting time $T := \inf\{t \geq 0 \colon B_t \in \partial\Delta^{(\alpha)}\}$.

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**Note**: $\dim \partial\Delta^{(\alpha)} = \frac{1}{\alpha-1}$ a.s. (dimension drop as in Makarov's theorem)

# Continuous-time Galton–Watson tree

Applying to the precompact tree $\Delta_0^{(\alpha)}$ the height transform

$$\Psi(r) = -\log(1-r), \qquad r \in [0,1),$$

we get a continuous-time GW tree $\mathbb{T}^{(\alpha)}$ = genealogical tree of a population where

- independently of each other, every individual has a random lifetime $\mathrm{Exp}(1)$;
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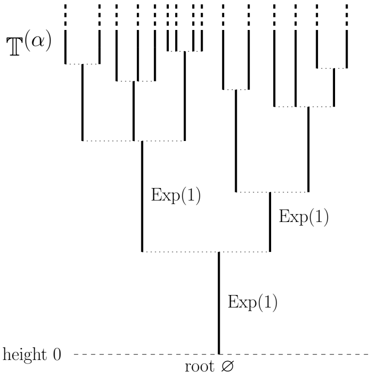

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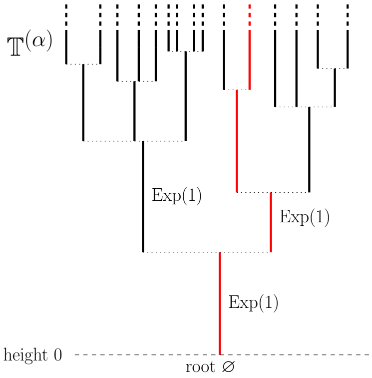


$\partial\mathbb{T}^{(\alpha)} = \{$infinite geodesics from the root (*geodesic rays*)$\}$.

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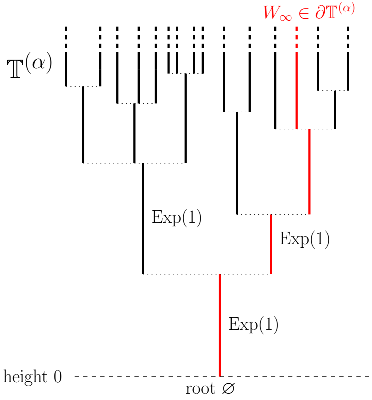


$\partial\mathbb{T}^{(\alpha)} = \{$infinite geodesics from the root (*geodesic rays*)$\}$.

$B$ Brownian motion on $\Delta^{(\alpha)} \xrightarrow{\Psi}$

$W$ BM on $\mathbb{T}^{(\alpha)}$ with drift 1/2 upwards.

We define $W_\infty$ the exit ray of $W$ as the unique ray visited by $W$ at arbitrarily large times.

# Asymptotics for the law of the exit ray

Write

$$\nu_\alpha = \text{ law of } W_\infty \text{ (probability measure on } \partial\mathbb{T}^{(\alpha)}).$$

**Fact**: $\nu_\alpha = \mu_\alpha$ si we identify $\partial\mathbb{T}^{(\alpha)}$ and $\partial\Delta^{(\alpha)}$ via $\Psi$.

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For $y \in \partial\mathbb{T}^{(\alpha)}$ and $r > 0$, let

$$\mathcal{B}(y, r) = \{\text{geodesic rays that coincide with } y \text{ up to height } r\}.$$

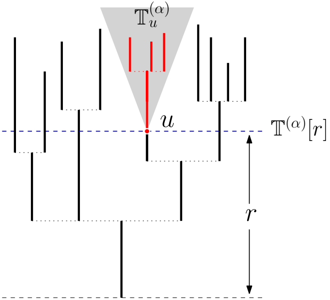
An equivalent form of Theorem (C) is

$$\boxed{\text{A.s. } \nu_\alpha(\mathrm{d}y)\text{-a.e.} \quad \lim_{r\to\infty} \frac{1}{r} \log \nu_\alpha(\mathcal{B}(y, r)) = -\beta_\alpha.}$$

# Uniform measures on $\partial\mathbb{T}^{(\alpha)}$ and $\partial\Delta^{(\alpha)}$

**Notation** :
For all $u \in \mathbb{T}^{(\alpha)}$, $H(u)$ = height of $u$.
$\forall r > 0$, $\mathbb{T}^{(\alpha)}[r] = \{u \in \mathbb{T}^{(\alpha)} : H(u) = r\}$.

The martingale limit
$\mathcal{W}^{(\alpha)} := \lim_{r\to\infty} e^{-\frac{r}{\alpha-1}} \#\mathbb{T}^{(\alpha)}[r] > 0$ a.s.



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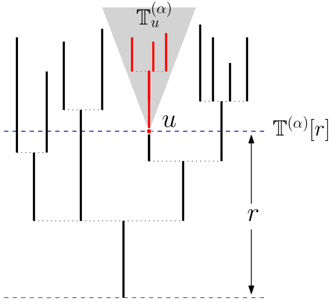
Also set $\mathcal{W}_u^{(\alpha)} := \lim_{r\to\infty} e^{-\frac{r}{\alpha-1}} \#\mathbb{T}_u^{(\alpha)}[r]$.



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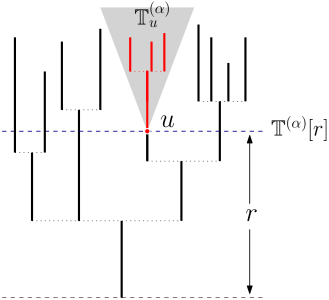
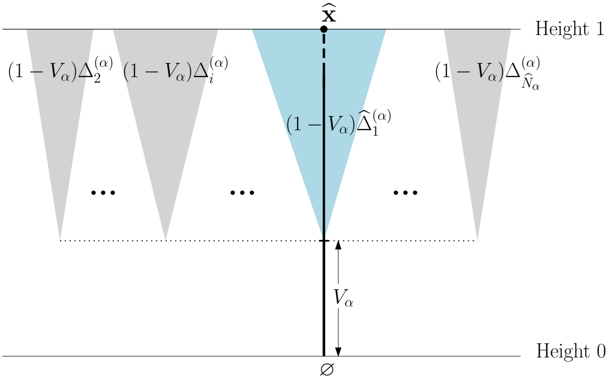



The uniform measure $\bar{\omega}_\alpha$ on $\partial\mathbb{T}^{(\alpha)}$ is the unique probability measure on $\partial\mathbb{T}^{(\alpha)}$ s.t. for every $u \in \mathbb{T}^{(\alpha)}$ and for every $y \in \partial\mathbb{T}^{(\alpha)}$ passing through $u$,

$$\bar{\omega}_\alpha(\mathcal{B}(y, H(u))) = e^{-\frac{H(u)}{\alpha-1}} \frac{\mathcal{W}_u^{(\alpha)}}{\mathcal{W}^{(\alpha)}}.$$

The uniform measure $\omega_\alpha$ on $\partial\Delta^{(\alpha)}$ is the probability measure on $\partial\Delta^{(\alpha)}$ induced by $\bar{\omega}_\alpha$ via $\Psi$.

# Size-biased reduced tree $\widehat{\Delta}^{(\alpha)}$




- $V_\alpha$ has density $\frac{\alpha}{\alpha-1}(1-x)^{\frac{1}{\alpha-1}}$ over $[0,1]$,
- $\widehat{\Delta}_1^{(\alpha)}$ is an independent copy of $\widehat{\Delta}^{(\alpha)}$, $(\Delta_i^{(\alpha)})_{i\geq 2}$ are i.i.d. copies of $\Delta^{(\alpha)}$,
- $\widehat{N}_\alpha$ has the size-biased distribution of $\rho_\alpha$.

$\widehat{\Delta}^{(\alpha)}$ follows the law of $\Delta^{(\alpha)}$ biased by $\mathcal{W}^{(\alpha)}$, with $\widehat{\mathbf{x}}$ distributed according to $\omega_\alpha$.

# Typical behavior of the harmonic measure $\mu_\alpha$

**Theorem (D)**

*For every $\alpha \in (1,2]$, with the same constant $\lambda_\alpha$ as in Theorem (B), we have a.s. $\omega_\alpha(\mathrm{d}x)$-a.e.*

$$\lim_{r\downarrow 0} \frac{\log \mu_\alpha(\mathcal{B}_{\mathbf{d}}(x,r))}{\log r} = \lambda_\alpha,$$

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End of the talk: give explicit formulae for $\beta_\alpha$ and $\lambda_\alpha$.

# 3. Conductance of random trees

View $\Delta^{(\alpha)}$ as an electric network of resistors with unit resistance per unit length.
Let $\mathcal{C}^{(\alpha)} \in [1, \infty)$ be the effective conductance between the root and $\partial\Delta^{(\alpha)}$.

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Similarly, let $\widehat{\mathcal{C}}^{(\alpha)} \in [1, \infty)$ be the effective conductance of $\widehat{\Delta}^{(\alpha)}$ between its root and height 1. (**NB**: $\widehat{\mathcal{C}}^{(\alpha)}$ follows the law of $\mathcal{C}^{(\alpha)}$ biased by $\mathcal{W}^{(\alpha)}$.)

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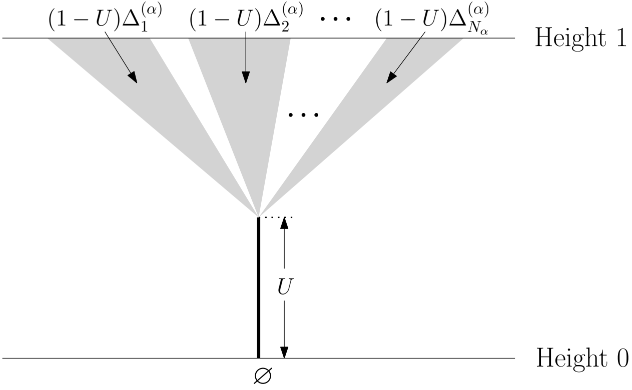
Recall the recursive structure of $\Delta^{(\alpha)}$ :



# The law of the conductances

The law of $\mathcal{C}^{(\alpha)}$ is characterized by the recursive equation in distribution

$$\mathcal{C}^{(\alpha)} \overset{\text{(d)}}{=\!=} \left( U + \frac{1-U}{\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_{N_\alpha}^{(\alpha)}} \right)^{-1},$$

where

- $U$ is uniform over $[0,1]$,
- $(\mathcal{C}_i^{(\alpha)})_{i\geq 1}$ are i.i.d. copies of $\mathcal{C}^{(\alpha)}$,
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Similarly, the law of $\widehat{\mathcal{C}}^{(\alpha)}$ is characterized by the recursive equation in distribution

$$\widehat{\mathcal{C}}^{(\alpha)} \overset{(\mathrm{d})}{=\!=} \left( V_\alpha + \frac{1-V_\alpha}{\widehat{\mathcal{C}}^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_{\widehat{N}_\alpha}^{(\alpha)}} \right)^{-1},$$

- $V_\alpha$ has density $\frac{\alpha}{\alpha-1}(1-x)^{\frac{1}{\alpha-1}}$ over $[0,1]$,
- $\widehat{N}_\alpha$ has the size-biased distribution of $\rho_\alpha$.

# Formulae for $\beta_\alpha$ and $\lambda_\alpha$

**Theorem (E)**

- *The constant* $\beta_\alpha \in (0, \frac{1}{\alpha-1})$ *is given by*

$$\beta_\alpha = \frac{1}{2}\left(\frac{E[\mathcal{C}_1^{(\alpha)}]^2}{E\left[\frac{\mathcal{C}_1^{(\alpha)}\mathcal{C}_2^{(\alpha)}}{\mathcal{C}_1^{(\alpha)}+\mathcal{C}_2^{(\alpha)}-1}\right]} - 1\right),$$

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- *The constant* $\lambda_\alpha \in (\frac{1}{\alpha-1}, \infty)$ *is given by*

$$\lambda_\alpha = E[\widehat{\mathcal{C}}^{(\alpha)}] - 1 = E[\mathcal{W}^{(\alpha)}\mathcal{C}^{(\alpha)}] - 1.$$

# 4. Questions

**State of art** : $\exists\, 0 < \beta_\alpha < \frac{1}{\alpha-1} < \lambda_\alpha$ satisfying that

$$\#\{v \in \mathcal{T}_n[n] \colon \mu_n(v) \approx n^{-\beta_\alpha}\} \approx n^{\beta_\alpha} ,$$
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**Open problems** :

- Full multifractal spectrum of harmonic measure ?
  For every $\eta > 0$, can we find $c_\alpha(\eta) \le \frac{1}{\alpha-1} \wedge \eta$ such that

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- Does the Hausdorff dimension $\beta_\alpha$ increase as $\alpha \downarrow 1$?