Weakly asymmetric bridges and the KPZ equation

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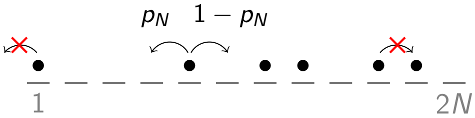
The model

N particles on $2N$ sites



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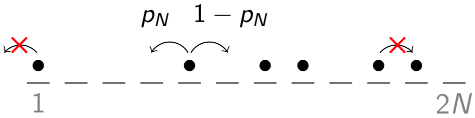

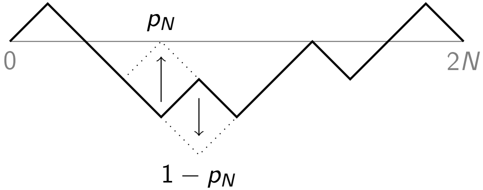

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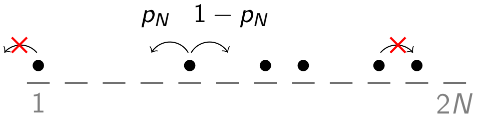

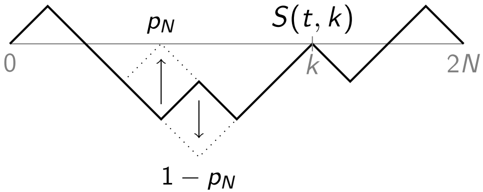

A simple fact

The unique invariant, reversible probability measure is

$$\mu_N(S) = \frac{1}{Z_N}\left(\frac{p_N}{1-p_N}\right)^{\frac{1}{2}A(S)} ,$$

where we define the (signed) area under the interface S

$$A(S) = \sum_{k=1}^{2N} S(k) .$$

Objective of this work

$\rightarrow$ Understand the behaviour of the interface according to the asymmetry $p_N - (1 - p_N)$.

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Choice of parametrisation

$$\frac{p_N}{1 - p_N} = e^{\frac{4\sigma}{(2N)^\alpha}} \,, \qquad \sigma, \alpha > 0 \,.$$

This means:

$$p_N = \frac{1}{2} + \frac{\sigma}{(2N)^\alpha} + \ldots \,, \qquad 1 - p_N = \frac{1}{2} - \frac{\sigma}{(2N)^\alpha} + \ldots \,.$$

Scaling limit of the invariant measure

We present a Central Limit Theorem for the interface under μ_N.

To that end, we rescale the interface in the following generic way:

$$u^N(x) := \frac{S(x\,N^{\cdots}) - \Sigma_\alpha^N(x)}{N^{\cdots}} \ ,$$

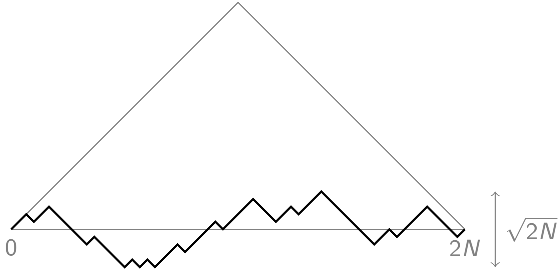
where Σ_α^N is the mean under μ_N.

Scaling limit of the invariant measure: $\alpha > 3/2$

Theorem (L.)

For all $\alpha \in (3/2, \infty)$,

$$\left(\frac{S(x2N)}{\sqrt{2N}}, x \in [0,1] \right) \Rightarrow \text{Brownian Bridge}$$

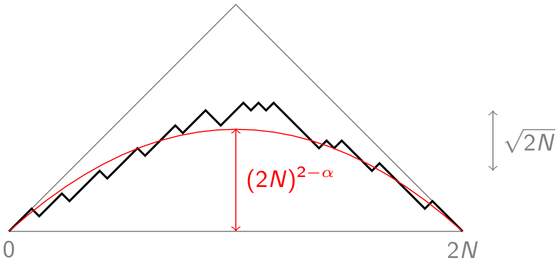


Scaling limit of the invariant measure: $\alpha \in (1, 3/2]$

Theorem (L.)

For all $\alpha \in (1, 3/2]$,

$$\left(\frac{S(x2N) - \sigma x(1-x)(2N)^{2-\alpha}}{\sqrt{2N}} \, , x \in [0,1] \right) \Rightarrow \text{ Brownian Bridge .}$$



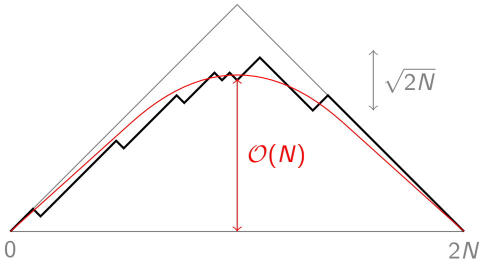
Scaling limit of the invariant measure: $\alpha = 1$

Theorem (L.)

For all $\alpha = 1$,

$$\left(\frac{S(x2N) - \Sigma_1^N(x)}{\sqrt{2N}} , x \in [0,1] \right) \Rightarrow \text{ Time changed Brownian Bridge }.$$

Sim. to Dobrushin-Hryniv (PTRF 96), Derrida-Enaud-Landim-Olla (JSP 05).



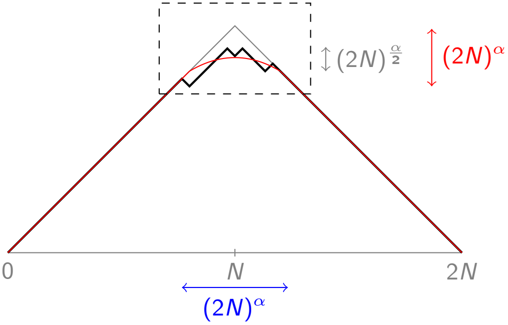
$$\Sigma_1^N(x) = 2N \int_0^x L'(\sigma(1-2y))dy + \mathcal{O}(1) \ , \quad x \in [0,1] \ .$$

Scaling limit of the invariant measure: $\alpha < 1$

Theorem (L.)

For all $\alpha < 1$,

$$\left(\frac{S(N + x(2N)^\alpha) - \Sigma^N_\alpha(x)}{(2N)^{\frac{\alpha}{2}}} , x \in \mathbf{R} \right) \Rightarrow \text{ Time changed Brownian Bridge .}$$



$$\Sigma^N_\alpha(x) = N + (2N)^\alpha \left(x + \int_{-x}^{\infty} (L'(2\sigma y) - 1) dy \right) + \mathcal{O}(1) , \quad x \in \mathbf{R} .$$

Dynamic at equilibrium

What does the dynamical interface $(S(t,k), t \geq 0, k \in [0,2N])$ look like when it starts from μ_N ?

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- Keep the same scalings in space and height as in the previous results.
- Speed up time appropriately.

Dynamic at equilibrium: $\alpha > 1$

$$u^N(t,x) := \frac{S(t(2N)^2, x2N) - \sigma(2N)^{2-\alpha}x(1-x)}{\sqrt{2N}}\ , \quad x \in [0,1]\ .$$

Theorem (L.)

Start from μ_N. Then, $u^N \Rightarrow u$ where

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_x^2 u + \xi\ , & x \in [0,1] \\ u(t,0) = u(t,1) = 0\ , & \end{cases}$$

and ξ space-time white noise.

Dynamic at equilibrium: $\alpha = 1$

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and ξ space-time white noise, and $\Sigma_1(\cdot) = \lim_N \frac{\Sigma_1^N(\cdot)}{2N}$.

Similar to De Masi-Presutti-Scacciatelli (Ann. IHP 89), Dittrich-Gärtner (MN 91).

Dynamic at equilibrium: $\alpha < 1$

$$u^N(t,x) := \frac{S(t(2N)^{2\alpha}, N + x(2N)^{\alpha}) - \Sigma_\alpha^N(x)}{(2N)^{\frac{\alpha}{2}}}\ , \quad x \in \mathbf{R}\ .$$

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We look at:

- the hydrodynamic limit,
- the fluctuations.

Hydrodynamic limit

Set

$$m^N(t,x) = \begin{cases} \frac{1}{(2N)^{2-\alpha}} S\big(t(2N)^2, x(2N)\big) \ , & \alpha \in [1, 3/2) \ , \\ \frac{1}{2N} \ S\big(t(2N)^{1+\alpha}, x(2N)\big) \ , & \alpha < 1 \ . \end{cases}$$

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Theorem (L.)

Start from a flat initial condition. Then $m^N \Rightarrow m$ where

$$\partial_t m = \begin{cases} \frac{1}{2}\partial_x^2 m + \sigma , & \alpha \in (1, 3/2) , \\ \frac{1}{2}\partial_x^2 m + \sigma\big(1 - (\partial_x m)^2\big) , & \alpha = 1 , \\ \sigma\big(1 - (\partial_x m)^2\big) , & \alpha < 1 , \end{cases}$$

with Dirichlet boundary conditions $m(t,0) = m(t,1) = 0$.

Similar results: Gärtner (SPA 88), De Masi-Presutti-Scacciatelli (Ann. IHP 89), Kipnis-Olla-Varadhan (CPAM 89), Rezakhanlou (CMP 91), Bahadoran (CMP 12) ...

More on the case $\alpha < 1$

$$\begin{cases} \partial_t m = \sigma\big(1 - (\partial_x m)^2\big) \ , \\ m(t=0,\cdot) \equiv 0 \ , \\ m(t,0) = m(t,1) = 0 \ . \end{cases}$$

More on the case $\alpha < 1$

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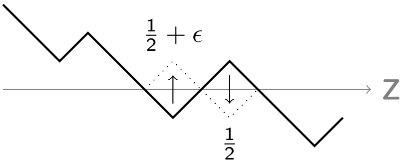
Inviscid Burgers

We actually prove convergence of the density of particles towards the (entropy!) solution of the inviscid Burgers equation:

$$\begin{cases} \partial_t \varrho = \sigma \partial_x \big(\varrho(1 - \varrho)\big) \ , \\ \varrho(t = 0, \cdot) = 1/2 \ , \\ \varrho(t, 0) = 1 \ , \quad \varrho(t, 1) = 0 \ . \end{cases}$$

Solution theory by Bardos-Le Roux-Nédélec (CPDE 79).
CV result similar to Rezakhanlou (CMP 91), Bahadoran (CMP 12).

A famous result of Bertini and Giacomin on KPZ



Theorem (Bertini-Giacomin CMP 97)

Set $h^\epsilon(t,x) = \epsilon\left(S\left(\frac{t}{\epsilon^4}, \frac{x}{\epsilon^2}\right) - \frac{t}{\epsilon^3}\right)$. Then, $h^\epsilon \Rightarrow h$ where

$$\text{(KPZ)} \quad \partial_t h = \frac{1}{2}\partial_x^2 h - \frac{1}{2}\left(\partial_x h\right)^2 + \xi\;, \quad x \in \mathbf{R}\;.$$

KPZ fluctuations

Let's apply Bertini-Giacomin's scaling in our case:

- Height scaling: $\epsilon \leftrightarrow \frac{1}{(2N)^\alpha}$.
  Heights are smaller than N. So we need to take $\alpha \leq 1$.

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- Space scaling: $\epsilon^{-2} \leftrightarrow (2N)^{2\alpha}$.
  Lattice of order N. So we need to take $\alpha \leq 1/2$.

- Time scaling: $\epsilon^{-4} \leftrightarrow (2N)^{4\alpha}$.
  Hydro. limit reaches equilibrium in finite time in the time scale $(2N)^{1+\alpha}$. So we need to take $4\alpha \leq 1+\alpha$, or equivalently $\alpha \leq 1/3$.

KPZ fluctuations

For $\alpha \leq 1/3$, set

$$h^N(t,x) = \frac{1}{(2N)^\alpha}\left(S\left(t(2N)^{4\alpha}, N + x(2N)^{2\alpha}\right) - \sigma t(2N)^{3\alpha}\right).$$

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Theorem (L.)

We have $h^N \Rightarrow h$ in $\mathbb{D}\big([0,T), \mathcal{C}(\mathbf{R})\big)$ where h is the solution of (KPZ) on the line and

$$T = \begin{cases} +\infty & \alpha < 1/3 \, , \\ \frac{1}{2\sigma} & \alpha = 1/3 \, . \end{cases}$$

Notice that the fluctuations vanish *suddenly* at time T in the case $\alpha = 1/3$.

Thank you for your attention.